NEW AND EXTENDED RESULTS IN RENEWAL AND QUEUEING THEORIES

LES RESULTATS NOUVEAUX ET ETENDUS EN

THEORIE DES ENSEMBLES RENOUVELES ET DES

FILES D'ATTENTE

A Thesis Submitted

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by

James Jaehak Kim, B.Sc., rmc
Captain

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ABSTRACT

Kim, James Jaehak, M.Sc., Royal Military College of Canada, March, 2016, New and Extended Results in Renewal and Queueing Theories. Supervised by Dr. M.L. Chaudhry.

This thesis encompasses new and extended results in renewal and queueing theories.

In the renewal theory portion of this thesis, the asymptotic result of renewal mass function and new asymptotic moments are found using the method of generating functions. This method is not only simple but also provides the extra constant terms in the asymptotic second moment which are unavailable in the literature. Higher asymptotic moments and their corresponding extra constant terms can also be found using the method of generating functions. Previous results in the existing literature do not have these extra constant terms. Recent work in renewal theory has the extra constant terms in a non-bulk renewal processes. The purpose of this thesis is to extend that recent work to the bulk-renewal processes in discrete-time.

In the queueing theory portion of this thesis, the imbedded Markov chain technique is used to determine the distributions of the number of uncompleted service stages, the number of customers in the system, and the waiting-time-in-queue. Single-server queues with a fixed number of service stages have been analyzed by many authors, some of whom state that there is no simple way to analyze the queue $GI/E_X/1$. The purpose of this thesis is to review and extend the previous work on $GI/E_r/1$ to the more general model $GI/E_X/1$ in which the number of stages is randomly distributed.

Keywords: Stochastic processes, Markov chains, Renewal theory, Discrete-time, Bulk-renewal processes, Queueing theory, Single-server queues, Service stages.

RÉSUMÉ

Kim, James Jaehak, M.Sc., Collège militaire royal du Canada, mars, 2016, Les Résultats Nouveaux et Etendus en Théorie des Ensembles Renouvelés et des Files D'attente. Dirigé par Dr. M.L. Chaudhry

Cette thèse comprend des résultats nouveaux et étendus en théorie des ensembles renouvelés et des files d'attente.

Dans la portion de la théorie des ensembles renouvelés de cette thèse, la fonction de masse de renouvellement asymptotique et de nouveaux moments asymptotiques sont trouvés en employant la méthode de la fonction génératrice. Des résultats précédents dans la littérature actuelle n'ont pas ces termes constants supplémentaires. Du travail récent dans la théorie des ensembles renouvelés aux termes constants supplémentaires dans un processus de renouvellement non-vrac. L'objectif de cette thèse est d'étendre ce travail récent à un processus de renouvellement vrac en temps discret.

Dans la portion de la théorie des files d'attente de cette thèse, la technique intégrée de la chaîne de Markov est utilisée pour déterminer les distributions du nombre des étapes de service, du nombre de clients dans le système, et le temps d'attente dans la file d'attente. Les files d'attente de serveursuniques avec un nombre fixe d'étapes de service ont été analysées par plusieurs auteurs, dont certains ont déclaré qu'il n'y a aucune manière simple d'analyser la file $GI/E_X/1$. L'objectif de cette note est de revoir et d'étendre le travail précédent de $GI/E_r/1$ à un modèle plus général de $GI/E_X/1$, dans laquelle le nombre d'étapes est distribué de façon aléatoire.

Mots-clés: Le processus stochastique, la chaîne Markov, la théorie des ensembles renouvelés, temps discret, processus de renouvellement vrac, la théorie des files d'attente, les files d'attente de serveurs uniques, les étapes de service.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF ACRONYMS

c.d.f.	Cumulative Distribution Function
d.f.	Distribution Function
d.g.f.	Double Generating Function
g.f.	Generating Function
i.i.d.r.v.	Independent Identically Distributed Random Variable
L-S.T.	Laplace- Stieltjes Transform
L.T.	Laplace Transform
p.d.f.	Probability Density Function
p.g.f.	Probability Generating Function
p.m.f.	Probability Mass Function
r.v.	Random Variable

LIST OF EQUATIONS

(1) $m(v) = \frac{f(v)}{1-f(v)}, (|v| < 1)$

(2) $P(z,v) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} P_n(k)v^k z^n = \frac{1-f(v)}{(1-v)[1-zf(v)]}, \ (|z| < 1, |v| < 1)$

(3) $\sum_{n=0}^{\infty} B_n(k)z^n = \sum_{n=0}^{\infty}[P_X(z)]^n P_n(k), \ (|z| < 1, k \geq 1)$

(4) $B(z,v) = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} B_n(k)z^n v^k = \frac{1-f(v)}{(1-v)[1-P_X(z)f(v)]}, \ (|z|, |v| < 1)$

(5) $M_k^{(1)} = E\left[Y_{N_k}^{(1)}\right] = \left(\frac{\mu_x}{\mu}\right)k + \mu_x\left(\frac{\sigma^2-\mu^2+\mu}{2\mu^2}\right) + o(1)$

(6) $M^{(1)}(v) = \frac{f(v)}{(1-v)[1-f(v)]}\mu_X, \ (|v| < 1)$

(7) $M^{(1)}(v) = \frac{C_{-2}}{(1-v)^2} + \frac{C_{-1}}{(1-v)} + O(1)$

(8) $M_k^{(1)} = (k+1)C_{-2} + C_{-1} + o(1)$

(9) $M_k^{(2)} = E\left[Y_{N_k}^{(2)}\right] = k^2\left(\frac{\mu_x}{\mu}\right)^2 + k\left(\frac{\mu_x^2}{\mu^2} + \frac{\mu_x}{\mu} - \frac{2\mu_x^2}{\mu} + \frac{P_X''(1)}{\mu} + \frac{2\mu_x^2\sigma^2}{\mu^3}\right) + \Big(2\mu_x^2 - \mu_x -$

$P_X''(1) + \frac{8\mu_x^2}{3\mu^2} - \frac{2\mu_x^2\mu_3}{3\mu^3} + \frac{\mu_x}{\mu} - \frac{4\mu_x^2}{\mu} + \frac{P_X''(1)}{\mu} + \frac{P_X''(1)\sigma^2}{2\mu^2} - \frac{2(\sigma\mu_x)^2}{\mu^2} + \frac{\mu_x\sigma^2}{2\mu^2} +$

$\frac{4(\sigma\mu_x)^2}{\mu^3} + \frac{3\mu_x^2\sigma^4}{\mu^4}\Big) + o(1)$

(10) $M^{(2)}(v) = \frac{f(v)}{(1-v)(1-f(v))}\left(\frac{2f(v)\mu_X^2+\mu_X-\mu_X f(v)}{1-f(v)} + P_X''(1)\right), \ (|v| < 1)$

(11) $M^{(2)}(v) = \frac{D_{-3}}{(1-v)^3} + \frac{D_{-2}}{(1-v)^2} + \frac{D_{-1}}{(1-v)} + O(1)$

(12) $M_k^{(2)} = \frac{(k+2)!}{2!k!}D_{-3} + (k+1)D_{-2} + D_{-1} + o(1)$

(13) $t = \Delta k$

(14) $N_{n+1}^- = (N_n^- + k - D_n)^+ = \begin{cases} N_n^- + k - D_n, & N_n^- + k - D_n > 0 \\ 0, & N_n^- + k - D_n \leq 0 \end{cases}$

(15) $N_{n+1}^- = (N_n^- + X_n - D_n)^+ = \begin{cases} N_n^- + X_n - D_n, & N_n^- + X_n - D_n > 0 \\ 0, & N_n^- + X_n - D_n \leq 0 \end{cases}$

(16) $P^{-}(z) = \frac{\sum_{m=0}^{\infty} q_m(1-z^{-m})}{1-S(z)K(z^{-1})},\ (|z| \le 1)$

(17) $P^{-}(z) = \prod_{h=1}^{r}\left(\frac{1-z_h}{1-z_h z}\right),\ (|z| \le 1)$

(18) $\lim_{t\to\infty} \frac{D_k(t)}{t} = \lim_{t\to\infty} \frac{\bar{U}_k(t)}{t}$

(19) $\lim_{t\to\infty} \frac{D_k(t)}{t} = \lim_{t\to\infty} \frac{\sum_{j=0}^{k-1} U_j(t) \sum_{h=k-j}^{r} s_h}{t}$

(20) $p_k = \frac{\lambda}{\mu} \sum_{j=0}^{k-1} p_j^{-} \left(1 - \sum_{h=1}^{k-j-1} s_h\right)$

(21) $\lim_{t\to\infty} \left[\frac{\sum_{j=0}^{\infty} U_j(t)}{t}\right]\left[\frac{\bar{s} D_k(t)}{\sum_{k=1}^{\infty} D_k(t)}\right] = \lim_{t\to\infty} \frac{D_k(t)}{t}$

(22) $p_{k-1}^{+} = \frac{1}{\rho} p_k,\ \ (k \ge 1)$

(23) $W_q^{-}(t) = p_0^{-} + \mu \sum_{i=1}^{\infty} p_i^{-} \int_0^t \frac{(\mu x)^{i-1}}{(i-1)!} e^{-\mu x} dx,\ \ (t \ge 0)$

(24) $\bar{Q}(z,\omega) = \int_0^{\infty} e^{-\omega t} Q(z,t) dt = \frac{1}{\omega} - \lambda \frac{\left(1-\bar{a}(\omega)\right)(1-z)}{\omega^2\left(1-z\bar{a}(\omega)\right)}$

(25) $Q_n = \begin{cases} 1-\rho + P\left(M_a\left(w_q^{-}\right) = 0\right), & (n=0) \\ \int_0^{\infty} P\left(M_a\left(w_q^{-}\right) = n \middle| w_q^{-} = t\right) dW_q^{-}(t), & (n \ge 1) \end{cases}$

(26) $G_{N_q}(z) = 1 - \rho + \int_0^{\infty} Q(z,t)\, dW_q^{-}(t)$

(27) $G_{N_q}(z) = Q_0 + Q_1 z + Q_2 z^2 + Q_3 z^3 + \ldots, (|z| \le 1)$

(28) $R_n = \begin{cases} 1-\rho, & (n=0) \\ \rho - 1 + Q_0, & (n=1) \\ Q_{n-1}, & (n \ge 2) \end{cases}$

(29) $M_{phase} = \sum_{i=1}^{\infty} i\, p_i$

(30) $L_q = \sum_{n=1}^{\infty} n\, Q_n$

(31) $L_s = \sum_{n=1}^{\infty} n\, R_n$

(32) $L_q = L_s - \rho$

(33) $E_{W_q^{-}} = \int_0^{\infty} td\, W_q^{-}(t)$

(34) $E_{W^-} = E_{W_q^-} + \frac{\bar{s}}{\mu}$

(35) $L_q = \lambda E_{W_q^-}$

(36) $L_s = \lambda E_{W^-}$

1 INTRODUCTION

1.1 Problem Description

This thesis addresses two different but related problems in renewal and queueing theories.

1.1.1 Problem description in renewal theory

Renewal theory is the study of renewal processes: Processes that count randomly occurring events (known as renewals) over duration of time in either continuous or discrete-time domain. Renewals can occur individually (single-renewal) or in groups (bulk-renewal). In applying renewal theory to real world problems, the asymptotic results of renewal processes serve as an important tool whenever there is a need to observe the long term behaviour of the number of renewals. Despite their importance, such asymptotic results were historically difficult to determine due to lack of available computing power and techniques to handle lengthy and complex expressions. In recent studies, asymptotic results in the discrete-time single-renewal processes were found, but the extended results in the discrete-time bulk-renewal processes are yet to be determined. In addition, the connection between the asymptotic results in continuous and discrete-time domains is unavailable in literature. Acquiring new asymptotic results in the discrete-time bulk-renewal processes and building the connection with their equivalent results in the continuous-time bulk-renewal processes would further the knowledge and understanding of asymptotic results in renewal theory. Such findings would also enable researchers to apply renewal theory across both analog and digital spectrums (also perform conversion between spectrums) which is considered an important practical application of renewal theory in engineering and telecommunications.

1.1.2 Problem description in queueing theory

A queue forms whenever and wherever demand exceeds supply. It is for this reason that a study of queues naturally emerged as a practical field of study known as queueing theory. Among many different types of queues, queues with multi-staged service (multi-staged queues) are particularly useful in modeling cases where a customer must proceed through several service stages. Queues of this type are evident when observing manufacturing lines, annual medical check-ups, or scheduled inspections of any sort. In the past, extensive studies have been done on models with a server that has a fixed number of service stages. On the contrary, almost no work has been done on models with a server that has a random number of service stages. This is mainly due to difficulties in handling the random nature of service stages and the associated probabilities. In review of previous work done by others, Yao et al. (1984) state that **"there is no simple way to analyze queues with server that has random number of service stages"**. A thorough and a complete analysis of these queues would not only address such a statement, but would also enhance previous knowledge of multi-staged queues and provide queueing theorists and practitioners with a new problem solving tool.

1.2 Thesis Objectives

The objective of this thesis is two-fold: To present new and extended results in renewal and queueing theories.

1.2.1 Objectives in renewal theory

- To derive new asymptotic results in the discrete-time bulk-renewal processes.
- To build a connection between asymptotic results of discrete and continuous time-bulk renewal processes.

- To provide new numerical examples of asymptotic results in the discrete-time bulk-renewal processes.

1.2.2 Objectives in queueing theory

- To extend the queueing model $GI/E_k/1$ to $GI/E_X/1$.
- To solve and derive several relations between different findings in $GI/E_X/1$.
- To provide new numerical examples of $GI/E_X/1$.

2 RENEWAL THEORY

Readers may refer to Appendix A.1 for summaries on probability theory and stochastic processes, which are important topics that lead to renewal theory. The definitions and properties of a discrete r.v. and its moments, generating function (g.f.), probability generating function (p.g.f.), and double generating function (d.g.f.) are provided in Appendix A.3. In addition, all supplementary proofs, derivation, and theorem that are used in discussing renewal theory are provided in Appendix B.

2.1 Literature review

Renewal theory can be divided into continuous and discrete-time renewal theories, both of which are important tools of application when solving problems in areas such as failure and replacement of equipment, traffic-flow, risk-based asset management models and queues (see Van Noortwijk, 2003).

In literature, Cox (1962) and Feller (1968) are among the most prominent of the various authors who discuss the theoretical (and analytical) aspect of renewal theory. Their ideas are reiterated in the works of Heyman and Sobel (1982), Tijms (2003) and Beichelt (2006). The computational aspect of renewal theory had been limited in the past mainly due to lack of computing power, software, and known techniques to perform such computations.

Particular interest in both continuous and discrete-time renewal theories is in finding their asymptotic results. Asymptotic results consist of the asymptotic result of renewal mass function (in discrete-time) or asymptotic result of renewal density (in continuous-time) and asymptotic moments of the number of renewals. Such results are important in practical applications of renewal theory due to their readily interpretable and measurable way of describing the number of renewals in the long run. However, despite their importance, asymptotic results were historically difficult to determine due to lack of computing power and complex derivations that lead to such results (Fisher 2014).

In continuous-time renewal theory, Cox (1962) provides the renewal density, and the first and second moments of the number of renewals in the continuous-time single-renewal processes. For the same processes, he also derives the asymptotic renewal density, as well as the asymptotic first and second moments of the number of renewals using Laurent series. Cox (1962)'s results are largely theoretical and his asymptotic second moment is missing the extra constant terms. Chaudhry (1995) discusses the computational aspect of continuous-time renewal theory, where he considers several different patterns of renewal periods in the continuous-time single-renewal processes. To compute renewal density and the moments of the number of renewals, one has to first take the L.T. of what is being computed. In doing so, Chaudhry (1995) classifies these L.T.'s into three distinct groups (rational, irrational, and those that cannot be represented in a closed-form) and shows how to perform computations for each group. Using the computational technique established by Chaudhry (1995), Chaudhry et al. (2013) provide various numerical examples in the continuous-time single-renewal processes. In their work, Padé approximation (see Appendix A.2.6) is used for the group of L.T.'s that Chaudhry (1995) classifies as irrational. Though the work by Chaudhry (1995), and Chaudhry, Yang and Ong (2013) cover several examples including the asymptotic results, the extra constant terms in their asymptotic second moment is still missing.

In extending previous works on the continuous-time single-renewal processes, Fisher and Chaudhry (2014) provide new results in the continuous-time bulk-renewal processes. In their derivations, the method of L.T. used by Chaudhry (1995) is extended to a bulk-renewal case. Fisher and Chaudhry (2014) also provide the extra constant terms in the asymptotic second moment.

In discrete-time renewal theory, the asymptotic first and second moments in the discrete-time single-renewal processes are available in the study by van der Weide et al. (2007). This result provides extra constant terms in the second moment yet states that it is not clear from Feller (1949) as to how to obtain those terms using g.f.'s. The same problem persists in Feller (1968) and Hunter (1983). Recently, Chaudhry and Fisher (2012) have responded to this problem by providing the asymptotic first and second moments in the discrete-time single-renewal processes using g.f.'s.

2.2 Discrete-time single-renewal processes

The discrete-time single-renewal processes are stochastic processes that count the number of randomly occurring events known as 'renewals' over a discrete period of time. These processes have been studied by several researchers in the past using various techniques. For details, see Feller (1968), Hunter (1983), and recent work by Chaudhry and Fisher (2012). A review of basic concepts in the discrete-time single-renewal processes is required prior to discussing the discrete-time bulk-renewal processes.

2.2.1 Renewal periods

The fundamental building blocks of the discrete-time single-renewal processes are renewal periods, which are time intervals between renewals. In the discrete-time single-renewal processes, renewals occur individually at instances of time $\sigma_1', \sigma_2', \sigma_3', \ldots$, and renewal periods $T_n = \sigma_n' - \sigma_{n-1}'$, $n \geq 1$, with $\sigma_0' = 0$ are independent identically distributed random variables (i.i.d.r.v.'s) that are distributed as T. As well, T has a probability mass function (p.m.f.) $f_k = P(T = k), k \geq 1, f_0 = 0$ and a probability generating function (p.g.f.) $f(v) = \sum_{k=1}^{\infty} f_k v^k$, $(|v| \leq 1)$ with mean $\mu \equiv \mu_1 = E[T] < \infty$, variance $\sigma^2 = E[T^2] - E^2[T] < \infty$, $a_n = \frac{d^n}{dv^n} f(v)|_{v=1}$, $(n \geq 1)$ and n-th moment $\mu_n = E[T^n]$, $(n \geq 1)$. If W_n is the waiting time until the n-th renewal, then $W_n = \sum_{i=1}^{n} T_i$. The p.g.f. of W_n is

$$E[v^{W_n}] = E\left[v^{\sum_{i=1}^{n} T_i}\right], (n > 0, v \geq 1)$$

$$= E[v^{T_1 + T_2 + \ldots + T_n}]$$

Given that T_i's are i.i.d.r.v.'s distributed as T, without loss of generality, the p.g.f of W_n can also be written as

$$E[v^{W_n}] = E[v^{nT}] = \{E[v^T]\}^n = f^n(v)$$

The stochastic processes $\{T_n, n \geq 1\}$ is called recurrent if $f(1) = 1$, and transient if $f(1) < 1$.

2.2.2 Renewal mass function

The renewal mass function is the probability of an event that there is a renewal at time k. It can be described as

$$m_k = P(\text{renewal at } k)$$

where $k \geq 1$. It is important to indicate that the renewal mass function is not a p.m.f. since $\sum_{k=0}^{\infty} m_k \neq 1$. Intuitively, m_k would consist of several different possibilities that lead to a renewal at k. For instance, assuming that there is no renewal prior to k, then m_k would be the same as f_k. Alternately, assuming that there is a renewal prior to k (say at 1), then m_k would be $m_1 f_{k-1}$. Different possibilities that lead to a renewal at k are depicted in Figure 1 below.

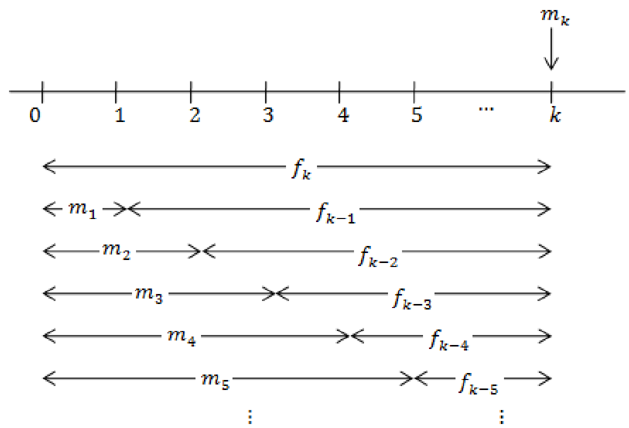

Figure 1: Different possibilities that lead to a renewal at *k*.

By considering all different possibilities that lead to a renewal at k, the renewal mass function can be expressed in terms of what is known as the renewal equation:

$$m_k = f_k + \sum_{j=1}^{k} m_{k-j} f_j$$

with $m_1 = f_1$ and $m_0 = 0$ (implying that there is no renewal at time 0). The left-hand side of the renewal equation is the probability of a renewal taking place at time k. The right-hand side of the equation is either a probability of the first renewal at time k or a previous renewal at time $k - j, (1 \leq j \leq k)$ with probability m_{k-j} and a subsequent renewal after j time units with probability f_j. Let $m(v)$ be the g.f. of m_k, which can be found by taking the g.f. of the renewal equation such that

$$m(v) = \sum_{k=1}^{\infty} m_k v^k = \sum_{k=1}^{\infty} \left(f_k + \sum_{j=1}^{k} m_{k-j} f_j \right) v^k$$

$$= \sum_{k=1}^{\infty} f_k v^k + \sum_{k=1}^{\infty} \sum_{j=1}^{k} m_{k-j} f_j \, v^k$$

$$= \sum_{k=1}^{\infty} f_k v^k + \left(\sum_{k=j}^{\infty} m_{k-j} v^{k-j} \right) \left(\sum_{j=1}^{\infty} f_j v^j \right)$$

or alternately expressed as

$$= f(v) + m(v)f(v)$$

and the g.f. of renewal mass function is

$$m(v) = \frac{f(v)}{1 - f(v)}, (|v| < 1) \tag{1}$$

2.2.3 Number of renewals

With the understanding of time duration between renewals (renewal periods) and the likelihood of a renewal at a particular point in time (renewal mass function), the probability of the number of renewals over a time interval can be discussed. Let $\{N_k, k \geq 1\}$ be the discrete-time single-renewal processes where N_k counts the number of renewals in the time interval $(0, k]$. The average (mean) number of renewals in $(0, k]$ is referred to as the renewal function and defined as $M_k = E[N_k], (k \geq 1)$. There exists a relation between M_k and m_k, such that

$$M_k = \sum_{i=1}^{k} m_i$$

The proof for the above relation is provided in Appendix B.1.1. To count the number of renewals in a window of time, the following three relations between N_k and W_n must be used:

a) $N_k \geq n \leftrightarrow W_n \leq k$

b) $N_k \leq n \leftrightarrow W_{n+1} > k$

both of which are true for $n \geq 0$ and $k \geq 1$. The two relations above between N_k and W_n can be each explained as follows:

First relation: There are at least n renewals during $(0, k]$ if and only if the time until the n-th renewal is at most k.

Second relation: There are at most n renewals during $(0, k]$ if and only if the time until the $(n+1)$-th renewal is at least $k+1$.

The importance of the above two relations is reflected in the fact that the number of renewals can be described in terms of renewal periods (thus confirming the statement in Subsection 2.2.1 that the fundamental building blocks of $\{N_k, k \geq 1\}$ are renewal periods). Let the p.m.f. of N_k be $P_n(k) = P(N_k = n), (n \geq 0)$. Using the first relation above, it can be expressed as

$$P_n(k) = P(N_k = n) = P(N_k \geq n) - P(N_k \geq n+1)$$
$$= P(W_n \leq k) - P(W_{n+1} \leq k)$$

Since N_k is a random variable (r.v.) of stochastic processes, the number of renewals depends on the renewal periods. The stochastic nature of N_k allows it to have a double generating function (d.g.f.) such that it becomes $P(z,v) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} P_n(k) v^k z^n$. This d.g.f. can be found by first taking a g.f. of k followed by a p.g.f. of n, which becomes

$$P(z,v) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} P_n(k) v^k z^n = \frac{1 - f(v)}{(1-v)[1 - zf(v)]}, (|z| < 1, |v| < 1) \qquad (2)$$

The complete derivation for (2) is provided in Appendix B.2.1.

2.3 Discrete-time bulk-renewal processes

The discrete-time single-renewal processes can be extended to the discrete-time bulk-renewal processes in terms of the size of renewals. In the extended processes, renewals occur in groups which results in interesting changes to the existing properties of the single-renewal processes. Some of the previous derivations for the discrete-time single-renewal processes can be either reused or extended to the discrete-time bulk-renewal processes.

2.3.1 Renewal periods

The renewal periods between individual renewals are also the renewal periods between bulk-renewals. Since a renewal period measures the time elapsed between two consecutive renewals (whether those renewals are single or bulk in size), the renewal periods of the discrete-time single-renewal processes are the same as that of the discrete-time bulk-renewal processes.

2.3.2 Renewal mass function

The renewal mass function of a single-renewal can also be the renewal mass function of a bulk-renewal. This is possible since the renewal mass function provides the probability of a renewal at time k, regardless of the size of that renewal.

2.3.3 Number of renewals

Assume that there are bulk-renewals at time $s_1, s_2, \ldots$, with size X_i . The r.v.'s X_i are i.i.d.r.v.'s that are distributed as X through a p.m.f. $b_n = P(X = n), (n \geq 1)$. The p.m.f. b_n has a p.g.f. $P_X(z) = E[z^X] = \sum_{n=1}^{\infty} b_n z^n,$ where $\mu_X = P_X'(1)$ and $P_X''(1) = \frac{d^2}{dz^2} P_X(z)|_{z=1}$. In addition, N_k in the discrete-time single-renewal processes can be re-interpreted as the number of bulk-renewals (not the number of renewals) over the time interval $(0, k]$ in the discrete-time bulk-renewal processes. The number of renewals over the time interval $(0, k]$ is $Y_{N_k} = \sum_{i=1}^{N_k} X_i$ with p.m.f. $B_n(k) = P\left(Y_{N_k} = n\right),\ (n \geq 0)$. Let $M_k^{(i)} = E\left[Y_{N_k}^i\right], (i \geq 1)$ be the i-th moment of Y_{N_k}. In addition, Y_{N_k} has a d.g.f. $B(z, v), (|z| < 1, |v| < 1)$ that can be found by taking the p.g.f. of $B_n(k)$ with respect to n, such that

$$\sum_{n=0}^{\infty} B_n(k) z^n = E\left[z^{Y_{N_k}}\right] = E\left[E\left[z^{\sum_{i=1}^{N_k} X_i}\middle| N_k\right]\right] = \sum_{n=0}^{\infty} E\left[z^{\sum_{i=1}^{N_k} X_i}\middle| N_k = n\right] P_n(k)$$

$$= \sum_{n=0}^{\infty} [P_X(z)]^n\, P_n(k), \quad (|z| < 1, k \geq 1) \tag{3}$$

where $P_n(k)$ is the probability of n bulk-renewals occurring in $(0, k]$. By taking the g.f. of (3) with respect to k, it becomes

$$\sum_{k=1}^{\infty} \sum_{n=0}^{\infty} B_n(k) z^n\, v^k = \sum_{k=1}^{\infty} \left\{ \sum_{n=0}^{\infty} [P_X(z)]^n P_n(k) \right\} v^k$$

$$= \sum_{n=0}^{\infty} [P_X(z)]^n \sum_{k=1}^{\infty} P_n(k) v^k$$

Substituting $\sum_{k=1}^{\infty} P_n(k) v^k = \frac{f^n(v)}{1-v} [1 - f(v)], (|v| < 1)$ (see Appendix B.1.2 for proof) in the above leads to

$$= \sum_{n=0}^{\infty} \frac{1 - f(v)}{1 - v} [P_X(z) f(v)]^n$$

Thus the d.g.f. of Y_{N_k} is found as

$$B(z, v) = \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} B_n(k) z^n\, v^k = \frac{1 - f(v)}{(1 - v)[1 - P_X(z) f(v)]}, \qquad (|z|, |v| < 1) \tag{4}$$

where if $P_X(z) = z$, then $B(z, v)$ reduces to $P(z, v)$ of the discrete-time single-renewal processes. Interestingly, in the discrete-time single-renewal processes, a g.f. then p.g.f. are taken to find (2) whereas in the discrete-time bulk-renewal processes, the steps are reversed such that a p.g.f. then g.f. are taken to find (4). This reverse in procedure is due to the composition of (4), where $B_n(k)$ is partially in terms of $P_n(k)$.

2.3.4 Conclusion

In the discrete-time bulk-renewal processes, the renewal periods, renewal mass function, and the d.g.f. of the number of renewals are derived. Given that both the renewal periods and the renewal mass function strictly focus on the event that a renewal occurs (rather than its size), previous derivations in single-renewal processes can be reused. However, the d.g.f. of the number of renewals is an extension of that in single-renewal processes since it takes the size of each bulk-renewal into consideration. This d.g.f. has several advantages over its p.m.f. counterpart when considered as a tool in deriving the asymptotic results. It is for this reason that (4) is derived.

The derivation of the d.g.f. in Section 2.3 is discussed in the manuscript that has been accepted for publication in the Journal of Mathematics and System Science (Kim and Chaudhry, 2014).

2.4 Asymptotic results in the discrete-time bulk-renewal processes

In this section, the asymptotic theory (see Appendix B.3.1) is applied to the discrete-time bulk-renewal processes. Such application leads to the asymptotic results in the discrete-time bulk-renewal processes that consist of the asymptotic result of renewal mass function and the asymptotic moments. Asymptotic results provide a tangible way of describing the number of renewals in the long run. For instance, the asymptotic result of renewal mass function is the probability of a bulk-renewal at time k as $k \to \infty$. In addition, the asymptotic first and second moments of Y_{N_k} are used to find the mean and standard deviation of the number of renewals in the long run, respectively.

2.4.1 Asymptotic result of renewal mass function

Consider the discrete-time bulk-renewal processes $\{Y_{N_k}, k \geq 1\}$ that are recurrent with $\mu < \infty$. The asymptotic result of renewal mass function can be described as

$$\lim_{k\to\infty} m_k = \frac{1}{\mu}$$

where μ is the mean renewal period. The proof of this result is as follows:

The renewal mass function $m_k, (k \geq 1)$ is a probability for which $0 \leq m_k \leq 1$ holds. Its p.g.f. $m(v)$ is absolutely convergent in $|v| < 1$ since

$$|m(v)| = \left|\sum_{k=1}^{\infty} m_k v^k\right| = \sum_{k=1}^{\infty} |m_k||v|^k \leq \sum_{k=1}^{\infty} |v|^k = \frac{v}{1-v}$$

is true for $|v| < 1$. Since $m(v)$ converges in $|v| < 1$, a procedure similar to the one discussed by Cox (1962) in the continuous-time single-renewal processes can be used to express $m(v)$ as

$$m(v) = \frac{C}{1-v} + O(1)$$

and

$$m_k = C + o(1)$$

where C is a positive constant $(0 < C < \infty)$. In addition, $O(1)$ indicates a function of $(1-v)$ bounded as $v \to 1^-$ and $o(1)$ indicates a function of k that tends to zero as $k \to \infty$. The first of the above expression can be rearranged to

$$(1-v)m(v) - (1-v)O(1) = C$$

By taking a limit as $v \to 1^-$, it becomes

$$\lim_{v\to1^-}\{(1-v)m(v) - (1-v)O(1)\} = C$$

Since $O(1)$ is bounde near $v = 1$, the above expression simplifies to

$$\lim_{v \to 1^-} (1 - v)m(v) = C$$

By substituting (1) in the above expression, and then applying L'Hôpital's rule gives

$$C = \lim_{v \to 1^-} \frac{\frac{d}{dv}[(1 - v)f(v)]}{\frac{d}{dv}[1 - f(v)]}$$

$$= \frac{1}{\mu}$$

Substituting $C = {}^{1}/_{\mu}$ into $m_k = C + o(1)$ and as $k \to \infty$, the asymptotic renewal density becomes

$$\lim_{k \to \infty} m_k = \frac{1}{\mu}$$

There are several other ways of determining $\lim_{k \to \infty} m_k$. For one such method, see Kohlas (1982). The same result can also be found using a theorem in Karlin and Taylor (1975).

2.4.2 Asymptotic first moment in discrete-time

The asymptotic first moment in the discrete-time bulk-renewal processes $\{Y_{N_k}, k \geq 1\}$ is

$$M_k^{(1)} = E[Y_{N_k}] = \left(\frac{\mu_x}{\mu}\right)k + \mu_x\left(\frac{\sigma^2 - \mu^2 + \mu}{2\mu^2}\right) + o(1) \qquad (5)$$

where $o(1) \to 0$ as $k \to \infty$, and these processes are assumed to be recurrent with $\sigma < \infty$ and $\mu_X < \infty$. The proof of (5) is as follows:

Let the g.f. of $M_k^{(1)}$ with respect to k be $M^{(1)}(v), (|v| < 1)$, such that

$$M^{(1)}(v) = \sum_{k=1}^{\infty} M_k^{(1)} v^k = \frac{\partial}{\partial z} B(z, v)|_{z=1}$$

$$= \frac{f(v)}{(1-v)[1-f(v)]}\mu_X, \quad (|v|<1) \tag{6}$$

where $B(z, v)$ is provided in (4). Now, following the procedure similar to the one used by Cox (1962) in the continuous-time single-renewal processes, $M^{(1)}(v)$ and $M_k^{(1)}$ can be alternatively expressed as

$$M^{(1)}(v) = \frac{C_{-2}}{(1-v)^2} + \frac{C_{-1}}{(1-v)} + O(1) \tag{7}$$

and

$$M_k^{(1)} = (k+1)C_{-2} + C_{-1} + o(1) \tag{8}$$

with $O(1)$ indicating a function of v bounded as $v \to 1^-$ and $o(1)$ indicating a function of k that tends to zero as $k \to \infty$. The procedure to prove (5) is to first rearrange (7) and then leverage (6) to solve for the unknown constant terms (C_{-1} and C_{-2}) one at a time. In doing so, both sides of (7) are multiplied by $(1-v)^2$ and as $v \to 1^-$, it becomes

$$C_{-2} = \lim_{v\to 1^-}(1-v)^2 M^{(1)}(v) = \lim_{v\to 1^-}(1-v)^2 \frac{f(v)}{(1-v)\big(1-f(v)\big)} P_X'(1)$$

L'Hôpital's rule can be applied to determine C_{-2} such that

$$C_{-2} = \frac{P_X'(1)}{f'(1)} = \frac{\mu_X}{\mu}$$

C_{-1} can be found using a similar procedure. In doing so, both sides of (7) can be multiplied by $(1-v)$ and as $v \to 1^-$ it becomes

$$C_{-1} = \lim_{v\to 1^-}\left\{(1-v)\frac{f(v)P_X'(1)}{(1-v)\big(1-f(v)\big)} - \frac{P_X'(1)}{\mu(1-v)}\right\}$$

$$= P_X'(1)\lim_{v\to 1^-}\frac{\mu f(v)(1-v) - (1-f(v))}{\mu(1-v)(1-f(v))}$$

Applying L'Hôpital's rule, it leads to

$$C_{-1} = P_X'(1)\left(\frac{f''(1)}{2\mu^2} - 1\right) = \mu_X\left(\frac{\sigma^2 + \mu^2 - \mu}{2\mu^2} - 1\right)$$

Substituting C_{-1} and C_{-2} into (8) gives

$$M_k^{(1)} = \left(\frac{\mu_x}{\mu}\right)k + \mu_x\left(\frac{\sigma^2 - \mu^2 + \mu}{2\mu^2}\right) + o(1)$$

where $o(1) \to 0$ as $k \to \infty$. When $\mu_X = 1$ in (5), it simplifies to the asymptotic first moment in the discrete-time single-renewal processes that corresponds to that of Feller (1968), Hunter (1983), and Chaudhry and Fisher (2012). The above finding leads to the well-known result in renewal theory, $\lim_{k\to\infty} \frac{M_k^{(1)}}{k} = \frac{\mu_X}{\mu}$, which gives the arrival rate for the discrete-time bulk-renewal processes.

2.4.3 Asymptotic second moment in discrete-time

The asymptotic second moment of the discrete-time bulk-renewal processes $\{Y_{N_k}, k \geq 1\}$ is

$$\begin{aligned}
M_k^{(2)}& \\
= E\left[Y_{N_k}^2\right] &= k^2\left(\frac{\mu_x}{\mu}\right)^2 + k\left(\frac{\mu_x^2}{\mu^2} + \frac{\mu_x}{\mu} - \frac{2\mu_x^2}{\mu} + \frac{P_X''(1)}{\mu} + \frac{2\mu_x^2\sigma^2}{\mu^3}\right) \\
&+ \left(2\mu_x^2 - \mu_x - P_X''(1) + \frac{8\mu_x^2}{3\mu^2} - \frac{2\mu_x^2\mu_3}{3\mu^3} + \frac{\mu_x}{\mu} - \frac{4\mu_x^2}{\mu} + \frac{P_X''(1)}{\mu} + \frac{P_X''(1)\sigma^2}{2\mu^2} - \frac{2(\sigma\mu_x)^2}{\mu^2}\right. \\
&\left. + \frac{\mu_x\sigma^2}{2\mu^2} + \frac{4(\sigma\mu_x)^2}{\mu^3} + \frac{3\mu_x^2\sigma^4}{\mu^4}\right) \\
&+ o(1) \qquad (9)
\end{aligned}$$

where $o(1) \to 0$ as $k \to \infty$ and the processes are assumed to be recurrent with $\mu_3 < \infty$ and $P_X''(1) < \infty$. The proof of this result is as follows:

Let the g.f. of $M_k^{(2)}$ with respect to k be $M^{(2)}(v), (|v| < 1)$, such that

$$M^{(2)}(v) = \sum_{k=1}^{\infty} M_k^{(2)} v^k = \frac{\partial^2}{\partial z^2} B(z,v)|_{z=1} + \frac{\partial}{\partial z} B(z,v)|_{z=1}$$

$$= \frac{f(v)}{(1-v)\big(1-f(v)\big)} \left(\frac{2f(v)\mu_X^2 + \mu_X - \mu_X f(v)}{1-f(v)} + P_X''(1) \right), (|v| < 1) \qquad (10)$$

where $B(z,v)$ is provided in (4). Similar to the first moment, a procedure like the one used by Cox (1962) in the continuous-time single-renewal processes can be used to express $M^{(2)}(v)$ and $M_k^{(2)}$ as

$$M^{(2)}(v) = \frac{D_{-3}}{(1-v)^3} + \frac{D_{-2}}{(1-v)^2} + \frac{D_{-1}}{(1-v)} + O(1) \qquad (11)$$

and

$$M_k^{(2)} = \frac{(k+2)!}{2!\,k!} D_{-3} + (k+1)D_{-2} + D_{-1} + o(1) \qquad (12)$$

with $O(1)$ indicating a function of v bounded as $v \to 1^-$ and $o(1)$ indicating a function of k that tends to zero as $k \to \infty$. (9) can be proven by first rearranging (11), substituting (10) in that rearranged expression, and then solving for the unknown constant terms (D_{-1}, D_{-2}, and D_{-3}) one at a time. This procedure is demonstrated as follows: Both sides of (11) are multiplied by $(1-v)^3$ and as $v \to 1^-$ it leads to

$$D_{-3} = \lim_{v \to 1^-} (1-v)^3 M^{(2)}(v) = \frac{P_X'(1)}{\mu} \lim_{v \to 1^-} \frac{2(1-v)f(v)P_X'(1) + (1-v)(1-f(v))}{1-f(v)}$$

By applying L'Hôpital's rule, it leads to

$$D_{-3} = \frac{P_X'(1)}{\mu} \left(\frac{2P_X'(1)}{f'(1)} \right) = 2\left(\frac{\mu_x}{\mu}\right)^2$$

Similarly, multiplying both sides of (11) by $(1-v)^2$ and as $v \to 1^-$, it becomes

$$D_{-2} = \lim_{v \to 1^-} \left\{ (1-v)^2 M^{(2)}(v) - \frac{D_{-3}}{(1-v)} \right\}$$

$$= \lim_{v\to1^-}\left\{(1-v)^2M^{(2)}(v)-\frac{2\left(P_X'(1)\right)^2}{\mu^2(1-v)}\right\}$$

$$= \lim_{v\to1^-}\left\{\frac{(1-v)f(v)(P_X'(1)-P_X'(1)f(v)+2f(v)\left(P_X'(1)\right)^2)}{(1-f(v))^2}+\frac{(1-v)^2f(v)P_X''(1)}{(1-v)\left(1-f(v)\right)}\right.$$

$$\left.-\frac{2\left(P_X'(1)\right)^2}{\mu^2(1-v)}\right\}$$

$$= \lim_{v\to1^-}\frac{1}{(1-f(v))^2(1-v)\left(1-f(v)\right)\mu^2(1-v)}\{(1-v)f(v)(P_X'(1)-P_X'(1)f(v)$$

$$+2f(v)\{P_X'(1)\}^2)(1-v)\left(1-f(v)\right)\mu^2(1-v)+(1-v)^2f(v)P_X''(1)(1$$

$$-f(v))^2\mu^2(1-v)-2\{P_X'(1)\}^2(1-f(v))^2(1-v)\left(1-f(v)\right)\}$$

Applying L'Hôpital's rule, it leads to

$$D_{-2}=\frac{\mu^2\mu_x-4\mu_x^2\mu^2+2\mu_x^2a_2+\mu^2P_X''(1)}{\mu^3}$$

Lastly, multiplying both sides of (11) by $(1-v)$ and as $v\to1^-$ it becomes

$$D_{-1}=\lim_{v\to1^-}\left\{(1-v)M^{(2)}(v)-\frac{D_{-3}}{(1-v)^2}-\frac{D_{-2}}{(1-v)}\right\}$$

$$= \lim_{v\to1^-}\left\{\frac{f(v)(P_X'(1)-P_X'(1)f(v)+2f(v)\{P_X'(1)\}^2)(1-v)}{(1-v)(1-f(v))^2}+\frac{P_X''(1)f(v)(1-f(v))^2}{(1-f(v))^3}\right.$$

$$-\frac{2(\{P_X'(1)\}^2-f(v)\{P_X'(1)\}^2)(1-f(v))}{\mu^2(1-v)^2(1-f(v))^2}$$

$$\left.-\frac{\mu^2P_X'(1)-4\{P_X'(1)\}^2\mu^2+2\{P_X'(1)\}^2a_2+\mu^2P_X''(1)}{\mu^3(1-v)}\right\}$$

$$= \lim_{v \to 1^-} \frac{1}{(1-v)(1-f(v))^2(1-f(v))^3\mu^2(1-v)^2(1-f(v))^2\mu^3(1-v)} [f(v)(P_X'(1)$$

$$- P_X'(1)f(v) + 2f(v)\{P_X'(1)\}^2)(1-v)^4(1-f(v))^5\mu^5 + P_X''(1)f(v)(1$$

$$- f(v))^6\mu^2(1-v)^4\mu^3 - 2(\{P_X'(1)\}^2 - f(v)\{P_X'(1)\}^2)(1-v)^2(1$$

$$- f(v))^3\mu^3]$$

Again applying L'Hôpital's rule, above expression becomes

$$D_{-1} = \frac{3\mu_x^2}{2} - \frac{\mu_x}{2} - \frac{P_X''(1)}{2} + \frac{\mu_x^2}{6\mu^2} - \frac{2\mu_x^2(a_3 + 3a_2 + \mu)}{3\mu^3} + \frac{\mu_x}{2\mu} - \frac{\mu_x^2}{\mu} + \frac{P_X''(1)}{2\mu}$$

$$+ \frac{P_X''(1)(a_2 + \mu - \mu^2) + 2\mu_x^2(a_2 + \mu - \mu^2) + \mu_x(a_2 + \mu - \mu^2)}{2\mu^2}$$

$$+ \frac{\mu_x^2(a_2 + \mu - \mu^2)}{\mu^3} + \frac{3\mu_x^2(a_2^2 + 2a_2\mu - 2a_2\mu^2 + \mu^2 - 2\mu^3 + \mu^4)}{2\mu^4}$$

Now substituting $D_{-1,}$ D_{-2} and D_{-3} into (12) with $a_3 \equiv E[(T-2)(T-1)T] = E[T^3] - 3E[T^2] + 2E[T] = \mu_3 - 3\mu_2 + 2\mu$ and $a_2 = E[(T-1)T] = E[T^2] - E[T] = \sigma^2 + \mu^2 - \mu$ leads to

$$M_k^{(2)} = E\left[Y_{N_k}^{(2)}\right] = k^2\left(\frac{\mu_x}{\mu}\right)^2 + k\left(\frac{\mu_x^2}{\mu^2} + \frac{\mu_x}{\mu} - \frac{2\mu_x^2}{\mu} + \frac{P_X''(1)}{\mu} + \frac{2\mu_x^2\sigma^2}{\mu^3}\right)$$

$$+ \left(2\mu_x^2 - \mu_x - P_X''(1) + \frac{8\mu_x^2}{3\mu^2} - \frac{2\mu_x^2\mu_3}{3\mu^3} + \frac{\mu_x}{\mu} - \frac{4\mu_x^2}{\mu} + \frac{P_X''(1)}{\mu} + \frac{P_X''(1)\sigma^2}{2\mu^2}\right.$$

$$\left. - \frac{2(\sigma\mu_x)^2}{\mu^2} + \frac{\mu_x\sigma^2}{2\mu^2} + \frac{4(\sigma\mu_x)^2}{\mu^3} + \frac{3\mu_x^2\sigma^4}{\mu^4}\right) + o(1)$$

where $o(1) \to 0$ as $k \to \infty$. The first two terms of the above expression correspond to Feller (1968) and Hunter (1983) when $P_X(z) = z$. However, in addition to the first two terms, (9) provides extra constant which are unavailable in previous literature. In addition, (9) matches exactly with its equivalent result in the discrete-time single-renewal processes by Chaudhry and Fisher (2012) when $P_X(z) = z$. Higher asymptotic moments can be found in a similar manner.

2.4.4 Conclusion

The asymptotic results in the discrete-time bulk-renewal processes are the asymptotic result of renewal mass function and asymptotic moments of the number of renewals. All asymptotic results (including the extra constant terms of the asymptotic second moment) are found using the g.f. method, hence it addresses the statement made by van der Weide et al. (2007): “It is not clear from Feller (1949) as to how to obtain those terms using g.f.’s.”

The asymptotic results in the discrete-time bulk-renewal processes presented in Section 2.4 are part of the manuscript that has been accepted for publication in the Journal of Mathematics and System Science (Kim and Chaudhry, 2014).

2.5 Asymptotic results in the continuous-time bulk-renewal processes

The asymptotic results in the discrete-time bulk-renewal processes can be used to derive their equivalent results in the continuous-time bulk-renewal processes. The two processes are fundamentally different since one has a discrete-time parameter k, (k is a nonnegative integer) while the other has a continuous-time parameter t, (t is a nonnegative real number). The two time parameters can be related using the relation

$$t = \Delta k \tag{13}$$

where Δ in (13) is a small, positive, and real number. Through this relation, t encompasses all characteristics of a continuous-time parameter. Based on this notion, there exists continuous-time ‘equivalents’ of f_k, m_k, and $M_k^{(i)}$ in the discrete-time bulk-renewal processes. Let $\{Y_{N(t)}, t > 0\}$ be the recurrent continuous-time bulk-renewal processes, then the equivalencies across two different time domains can be summarized as follows

Table 1: Summary of equivalencies between the discrete and the continuous-time bulk-renewal processes

Time domain	Discrete-time	Continuous-time
Time parameter	k	t
Bulk-renewal processes	$\{Y_{N_k}, k \geq 1\}$	$\{Y_{N(t)}, t > 0\}$
D.f. of time parameter	f_k	$f(t)$
D.f. of a renewal at time	m_k	$m(t)$
Asymptotic i-th moment	$M_k^{(i)}$	$M^{(i)}(t)$

All asymptotic results in $\{Y_{N(t)}, t > 0\}$ are available in Fisher and Chaudhry (2014). The purpose of Section 2.5 is to obtain the same results using a different and independent approach by leveraging the asymptotic results in $\{Y_{N_k}, k \geq 1\}$. The renewal density in continuous-time, denoted by $m(t)$, is equivalent to m_k in discrete-time. As stated in the Master's thesis by Fisher (2014), $\lim_{k\to\infty} m_k$ and $\lim_{t\to\infty} m(t)$ lead to the same result, however that is not the case for the asymptotic moments. It is for this reason that the derivation to manipulate $M_k^{(i)}$ into $M^{(i)}(t)$ is provided.

2.5.1 Asymptotic first moment in continuous-time

The asymptotic first moment in continuous-time $M^{(1)}(t)$ can be derived by letting $\mu = \frac{\hat{\mu}}{\Delta}, \sigma^2 = \left(\frac{\hat{\sigma}}{\Delta}\right)^2$ and $k = \frac{t}{\Delta}$ in (5), where $\hat{\mu}, \hat{\sigma}$, and t are the parameters of $\{Y_{N(t)}, t > 0\}$. Then as $\Delta \to 0$, the asymptotic first moment in the continuous-time bulk-renewal processes becomes

$$M^{(1)}(t) = \lim_{\Delta\to 0}\left[\left(\frac{\frac{t}{\Delta}}{\frac{\hat{\mu}}{\Delta}}\right)\mu_x + \mu_x\left(\frac{\left(\frac{\hat{\sigma}}{\Delta}\right)^2 - \left(\frac{\hat{\mu}}{\Delta}\right)^2 + \left(\frac{\hat{\mu}}{\Delta}\right)}{2\left(\frac{\hat{\mu}}{\Delta}\right)^2}\right)\right] + o(1)$$

which can be simplified to

$$M^{(1)}(t) = \left(\frac{\mu_x}{\hat{\mu}}\right)t + \mu_x\left(\frac{\hat{\sigma}^2 - \hat{\mu}^2}{2\hat{\mu}^2}\right) + o(1)$$

where $o(1) \to 0$ as $t \to \infty$ and $t > 0$. This result coincides with that of Fisher and Chaudhry (2014), and Fisher (2014).

2.5.2 Asymptotic second moment in continuous-time

Similar to $M^{(1)}(t)$, $M^{(2)}(t)$ can be derived by letting $\mu = \frac{\hat{\mu}}{\Delta}, \sigma^2 = \left(\frac{\hat{\sigma}}{\Delta}\right)^2, \mu_3 = \frac{\hat{\mu}_3}{\Delta^3}$ and $k = \frac{t}{\Delta}$ in (9). Then as $\Delta \to 0$, the asymptotic second moment in the continuous-time bulk-renewal processes becomes

$$M^{(2)}(t) = \lim_{\Delta\to 0}\left[\frac{\mu_x t}{\left(\frac{\hat{\mu}}{\Delta}\right)\Delta}\right]^2 + \lim_{\Delta\to 0}\frac{t}{\Delta}\left(\frac{\mu_x^2}{\left(\frac{\hat{\mu}}{\Delta}\right)^2} + \frac{\mu_x}{\left(\frac{\hat{\mu}}{\Delta}\right)} - \frac{2\mu_x^2}{\left(\frac{\hat{\mu}}{\Delta}\right)} + \frac{P_X''(1)}{\left(\frac{\hat{\mu}}{\Delta}\right)} + \frac{2\mu_x^2\left(\frac{\hat{\sigma}}{\Delta}\right)^2}{\left(\frac{\hat{\mu}}{\Delta}\right)^3}\right)$$

$$+ \lim_{\Delta\to 0}\left(2\mu_x^2 - \mu_x - P_X''(1) + \frac{8\mu_x^2}{3\left(\frac{\hat{\mu}}{\Delta}\right)^2} - \frac{2\mu_x^2\left(\frac{\hat{\mu}_3}{\Delta^3}\right)}{3\left(\frac{\hat{\mu}}{\Delta}\right)^3} + \frac{\mu_x}{\left(\frac{\hat{\mu}}{\Delta}\right)} - \frac{4\mu_x^2}{\left(\frac{\hat{\mu}}{\Delta}\right)} + \frac{P_X''(1)}{\left(\frac{\hat{\mu}}{\Delta}\right)}\right.$$

$$\left. + \frac{P_X''(1)\left(\frac{\hat{\sigma}}{\Delta}\right)^2}{2\left(\frac{\hat{\mu}}{\Delta}\right)^2} - \frac{2\left[\left(\frac{\hat{\sigma}}{\Delta}\right)\mu_x\right]^2}{\left(\frac{\hat{\mu}}{\Delta}\right)^2} + \frac{\mu_x\left(\frac{\hat{\sigma}}{\Delta}\right)^2}{2\left(\frac{\hat{\mu}}{\Delta}\right)^2} + \frac{4\left[\left(\frac{\hat{\sigma}}{\Delta}\right)\mu_x\right]^2}{\left(\frac{\hat{\mu}}{\Delta}\right)^3} + \frac{3\mu_x^2\left(\frac{\hat{\sigma}}{\Delta}\right)^4}{\left(\frac{\hat{\mu}}{\Delta}\right)^4}\right) + o(1)$$

which can be simplified to

$$M^{(2)}(t) = t^2\left(\frac{\mu_x}{\hat{\mu}}\right)^2 + t\left(\frac{P_X''(1) - 2\mu_x^2 + \mu_x}{\hat{\mu}} + \frac{2\hat{\sigma}^2\mu_x^2}{\hat{\mu}^3}\right)$$

$$+ \left(\frac{\hat{\sigma}^2 P_X''(1)}{2\hat{\mu}^2} + \frac{\hat{\sigma}^2\mu_x}{2\hat{\mu}^2} - \frac{\mu_x}{2} - \frac{2\hat{\mu}_3\mu_x^2}{3\hat{\mu}^3} + \frac{3\hat{\sigma}^4\mu_x^2}{2\hat{\mu}^4} + \frac{\hat{\sigma}^2\mu_x^2}{\hat{\mu}^2} + \frac{3\mu_x^2}{2} - \frac{P_X''(1)}{2}\right)$$

$$+ o(1)$$

where $o(1) \to 0$ as $t \to \infty$ and $t > 0$. This result coincides with that of Fisher and Chaudhry (2014), and Fisher (2014). Higher asymptotic moments in the continuous-time bulk-renewal processes can be found in a similar manner.

2.5.3 Conclusion

Using the asymptotic results in the discrete-time bulk-renewal processes, the equivalent results in the continuous-time bulk-renewal processes are derived. The continuous-time parameter is built by multiplying the discrete-time parameter by a delta (very small, real, and positive number) to make it possess the characteristics of a continuous-time parameter. By doing so, the asymptotic first and second moments in the continuous-time bulk-renewal processes are completely determined.

All derivations in Section 2.5 are part of the manuscript that has been accepted for publication in the Journal of Mathematics and System Science (Kim and Chaudhry, 2014).

3 NUMERICAL EXAMPLES IN RENEWAL THEORY

In this chapter, various numerical examples in discrete-time renewal theory are presented. They are organized in the following manner: The discrete-time single-renewal processes in Section 3.1 and the discrete-time bulk-renewal processes in Section 3.2. All computations were done using MAPLE software that was configured to compute up to the ninth decimal place. Final results were rounded to four decimal places in the tables below.

3.1 Discrete-time single-renewal processes

In computing $P_n(k)$ in $\{N_k, k \geq 1\}$, the p.m.f. of renewal periods (f_k) was considered as a geometric, negative binomial, and Poisson distribution (see Appendix A.3). The numerical computations of $P_n(k)$ at various values of (n, k) were done by first performing a Taylor's series expansion (see Appendix A.3.6) of (2) with respect to z. This resulted in a power series of z, and for the coefficient of each term, second Taylor's series expansion but with respect to v was performed. The final product is a power series of v, where the coefficient of each term are the probabilities $P_n(k)$.

3.1.1 Geometric renewal periods

The p.m.f. of the renewal period is a geometric distribution such that $f_k = pq^{k-1}, (k \geq 1)$ with p.g.f. $f(v) = \frac{pv}{(1-qv)}, |v| < 1$ and $p = 0.3, q = 0.7$. $P_n(k)$ was computed at $k = 1, 5, 10, 15, 20$ and $n = 0, 1, 2, 3, 4, 5, 6$.

Table 2: $\{N_k, k \geq 1\}$ with geometric renewal periods

k	$P_0(k)$	$P_1(k)$	$P_2(k)$	$P_3(k)$	$P_4(k)$	$P_5(k)$	$P_6(k)$	...	$E[N_k]$	$E[N_k^2]$

1	0.7000	0.3000	0.0000	0.0000	0.0000	0.0000	0.0000	…	0.3000	0.3000
5	0.1681	0.3602	0.3087	0.1323	0.0284	0.0024	0.0000	…	1.5000	3.3000
10	0.0283	0.1211	0.2335	0.2668	0.2001	0.1029	0.0368	…	3.0000	11.1000
15	0.0048	0.0305	0.0916	0.1700	0.2186	0.2061	0.1472	…	4.5000	23.4000
20	0.0008	0.0068	0.0279	0.0716	0.1304	0.1789	0.1916	…	6.0000	40.2000

3.1.2 Negative binomial renewal periods

The p.m.f. of renewal periods is a negative binomial distribution such that $f_k = \binom{k+r-2}{k-1} p^r q^{k-1}, (k \geq 1)$ with p.g.f. $f(v) = v\left(\frac{p}{1-qv}\right)^r, |v| < 1$ and $p = 0.75$, $q = 0.25$ and $r = 13$. $P_n(k)$ was computed at $k = 1, 10, 20, 30$ and $n = 0, 1, 2, 3, 4, 5$.

Table 3: $\{N_k, k \geq 1\}$ with negative binomial renewal periods

k	$P_0(k)$	$P_1(k)$	$P_2(k)$	$P_3(k)$	$P_4(k)$	$P_5(k)$	…	$E[N_k]$	$E[N_k^2]$
1	0.9762	0.0238	0.0000	0.0000	0.0000	0.0000	…	0.0238	0.0238
10	0.0295	0.4571	0.4317	0.0772	0.0045	0.0001	…	1.5703	2.9526
20	$7.9845\text{x}10^{-6}$	0.0064	0.1343	0.4058	0.3328	0.1041	…	3.4453	12.7406
30	$7.4214\text{x}10^{-11}$	$6.0263\text{x}10^{-6}$	0.0015	0.0354	0.1929	0.3535	…	5.3203	29.5570

3.1.3 Poisson renewal periods

The p.m.f. of renewal periods is a Poisson distribution such that $f_k = \frac{\alpha^{k-1}}{(k-1)!} e^{-\alpha}, (k \geq 1)$ with p.g.f. $f(v) = ve^{-\alpha(1-v)}, |v| < 1$, where $\alpha = 2$. $P_n(k)$ was computed at $k = 1, 5, 10, 15$ and $n = 0, 1, 2, 3, 4$.

Table 4: $\{N_k, k \geq 1\}$ with Poisson renewal periods

k	$P_0(k)$	$P_1(k)$	$P_2(k)$	$P_3(k)$	$P_4(k)$	…	$E[N_k]$	$E[N_k^2]$
1	0.8647	0.1353	0.0000	0.0000	0.0000	…	0.1353	0.1353
5	0.0527	0.5139	0.3715	0.0590	0.0030	…	1.4459	2.5791
10	$4.6498\text{x}10^{-5}$	0.0213	0.2347	0.4306	0.2463	…	3.1111	10.5432

15	$4.2000x10^{-9}$	$7.6325x10^{-5}$	0.0088	0.1031	0.3050	…	4.7778	24.0617

3.2 Discrete-time bulk-renewal processes with binomial bulk-size

In computing $B_n(k)$ in $\{Y_{N_k}, k \geq 1\}$, a similar procedure from Section 3.1 was used where the Taylor's series expansion was performed twice, once with respect to z and another with respect to v on (4). The same p.m.f. of renewal periods from Section 3.1 were used while incorporating a binomial bulk-size distribution. The p.m.f. of the bulk-size (b_n) follows a binomial distribution such that $b_n = \binom{r}{n-1} p^n q^{r-n+1}$, $(1 \leq n \leq 4)$ with p.g.f. $P_X(z) = z(q + pz)^r$ where $p = 0.45, q = 0.55$ and $r = 3$. The numerical results of asymptotic first and second moments in discrete-time bulk-renewal processes are also presented in this section. $M_k^{(1)}$ and $M_k^{(2)}$ were computed by substituting different values of k in (5) and (9).

3.2.1 Geometric renewal periods and binomial bulk-size

Table 5: $\{Y_{N_k}, k \geq 1\}$ with geometric renewal periods and binomial bulk-size

k	$B_0(k)$	$B_1(k)$	$B_2(k)$	$B_3(k)$	$B_4(k)$	$B_5(k)$	$B_6(k)$	…
1	0.7000	0.0499	0.1225	0.1002	0.0273	0.0000	0.0000	…
5	0.1681	0.0599	0.1556	0.1629	0.1231	0.1085	0.0864	…
10	0.0283	0.0201	0.0559	0.0734	0.0851	0.1019	0.1069	…
15	0.0048	0.0051	0.0150	0.0234	0.0342	0.0483	0.0608	…
20	0.0008	0.0011	0.0036	0.0064	0.0109	0.0174	0.0251	…

k	$M_k^{(1)}$	$M_k^{(2)}$
1	0.7050	1.8795
5	3.5250	19.3380
10	7.0500	63.5273
15	10.5750	132.5678

20	14.1000	226.4595

3.2.2 Negative binomial renewal periods and binomial bulk-size

Table 6: $\{Y_{N_k}, k \geq 1\}$ with negative binomial renewal periods and binomial bulk-size

k	$B_0(k)$	$B_1(k)$	$B_2(k)$	$B_3(k)$	$B_4(k)$	$B_5(k)$	$B_6(k)$	...
1	0.9762	0.0040	0.0097	0.0079	0.0022	0.0000	0.0000	...
2	0.8990	0.0167	0.0410	0.0336	0.0093	0.0002	0.0001	...
3	0.7639	0.0386	0.0948	0.0780	0.0223	0.0013	0.0008	...
4	0.5950	0.0646	0.1591	0.1320	0.0400	0.0050	0.0031	...
5	0.4261	0.0879	0.2170	0.1826	0.0606	0.0136	0.0085	...

k	$M_k^{(1)}$	$M_k^{(2)}$
1	N/A	0.7995
2	0.1652	1.1005
3	0.6059	1.7898
4	1.0465	2.8674
5	1.4871	4.3333

As a remark, not applicable (N/A) applies to the cases where $M_k^{(1)} < 0$.

3.2.3 Poisson renewal periods and binomial bulk-size

Table 7: $\{Y_{N_k}, k \geq 1\}$ with Poisson renewal periods and binomial bulk-size

k	$B_0(k)$	$B_1(k)$	$B_2(k)$	$B_3(k)$	$B_4(k)$	$B_5(k)$	$B_6(k)$	...
1	0.8647	0.0225	0.0553	0.0452	0.0123	0.0000	0.0000	...
5	0.0527	0.0855	0.2201	0.2225	0.1521	0.1192	0.0817	...
10	$4.6498\text{x}10^{-5}$	0.0036	0.0152	0.0410	0.0820	0.1208	0.1433	...
15	$4.1957\text{x}10^{-9}$	$1.2699\text{x}10^{-5}$	0.0003	0.0017	0.0062	0.0164	0.0343	...

k	$M_k^{(1)}$	$M_k^{(2)}$
1	0.2611	1.2415
5	3.3945	15.3219
10	7.3111	60.5349

15	11.2278	136.4284

3.3 Discrete-time bulk-renewal processes with 1-3-6-9 bulk-size

In computing $B_n(k)$ in $\{Y_{N_k}, k \geq 1\}$, the same procedure from Section 3.2 was used. The same p.m.f. of renewal periods from Section 3.1 were used while incorporating a 1-3-6-9 bulk-size distribution. The p.m.f. of the bulk-size (b_n) follows a 1-3-6-9 distribution where $b_1 = 0.1, b_3 = 0.25, b_6 = 0.45, b_9 = 0.2$ with p.g.f. $P_X(z) = 0.1z + 0.25z^3 + 0.45z^6 + 0.2z^9$. The numerical results of asymptotic first and second moments in discrete-time bulk-renewal processes are also presented in this section. $M_k^{(1)}$ and $M_k^{(2)}$ were computed by substituting different values of k in (5) and (9).

3.3.1 Geometric renewal periods and 1-3-6-9 bulk-size

Table 8: $\{Y_{N_k}, k \geq 1\}$ with geometric renewal periods and 1-3-6-9 bulk-size

k	$B_0(k)$	$B_1(k)$	$B_2(k)$	$B_3(k)$	$B_4(k)$	$B_5(k)$	$B_6(k)$	...
1	0.7000	0.0300	0.0000	0.0750	0.0000	0.0000	0.1350	...
5	0.1681	0.0360	0.0031	0.0902	0.0154	0.0010	0.1814	...
10	0.0282	0.0121	0.0023	0.0305	0.0117	0.0020	0.0693	...
15	0.0048	0.0031	0.0009	0.0078	0.0046	0.0013	0.0197	...
20	0.0008	0.0007	0.0003	0.0018	0.0014	0.0005	0.0050	...

k	$M_k^{(1)}$	$M_k^{(2)}$
1	1.6050	10.4250
5	8.0250	103.6455
10	16.0500	336.0923
15	24.0750	697.3403
20	32.1000	1187.3895

3.3.2 Negative binomial renewal periods and 1-3-6-9 bulk-size

Table 9: $\{Y_{N_k}, k \geq 1\}$ with negative binomial renewal periods and 1-3-6-9 bulk-size

k	$B_0(k)$	$B_1(k)$	$B_2(k)$	$B_3(k)$	$B_4(k)$	$B_5(k)$	$B_6(k)$	...
1	0.9762	0.0024	0.0000	0.0059	0.0000	0.0000	0.0107	...
2	0.8990	0.0100	5.6441x10^{-6}	0.0251	2.8220x10^{-5}	0.0000	0.0452	...
3	0.7639	0.0232	4.2197x10^{-2}	0.0580	0.0002	1.0057x10^{-7}	0.1046	...
4	0.5950	0.0388	0.0002	0.0971	0.0008	1.0787x10^{-6}	0.1758	...
5	0.4261	0.0528	0.0005	0.1321	0.0022	5.9502x10^{-6}	0.2406	...

k	$M_k^{(1)}$	$M_k^{(2)}$
1	N/A	3.8765
2	0.3762	5.8639
3	1.3793	9.8639
4	2.3824	15.8764
5	3.3856	23.9014

As a remark, not applicable (N/A) applies to the cases where $M_k^{(1)} < 0$.

3.3.3 Poisson renewal periods and 1-3-6-9 bulk-size

Table 10: $\{Y_{N_k}, k \geq 1\}$ with Poisson renewal periods and 1-3-6-9 bulk-size

k	$B_0(k)$	$B_1(k)$	$B_2(k)$	$B_3(k)$	$B_4(k)$	$B_5(k)$	$B_6(k)$	...
1	0.8647	0.0134	0.0000	0.0338	0.0000	0.0000	0.0609	...
5	0.0527	0.0514	0.0037	0.1285	0.0186	0.0004	0.2545	...
10	0.0005	0.0021	0.0024	0.0058	0.0118	0.0032	0.0245	...
15	4.1957x10^{-9}	7.6325x10^{-6}	8.7512x10^{-5}	0.0001	0.0005	0.0008	0.0009	...

k	$M_k^{(1)}$	$M_k^{(2)}$
1	0.5944	6.6880
5	7.7278	82.7040
10	16.6444	320.8364
15	25.5611	717.9827

3.4 Conclusion

Most practical applications of renewal theory are done in terms of numerical computations. After discussing the analytical aspect of renewal theory in Chapter 2, the numerical examples in discrete-time renewal theory are presented in Chapter 3.

Section 3.1 covered the discrete-time single-renewal processes by presenting the probabilities of the number of renewals over a time interval. In considering geometric, negative binomial, and Poisson renewal periods, there exists an intuitive pattern that is reflected in the probabilities of table 2, 3, and 4: If the time interval is long then more renewals are likely to occur, whereas if the time interval is short then less renewals are likely to occur.

Section 3.2 and 3.3 covered the discrete-time bulk-renewal processes by considering the same p.m.f.'s of renewal periods as Section 3.1 with additional consideration of binomial and 1-3-6-9 bulk-sizes, respectively. The same pattern from Section 3.1 can be observed in the probabilities of each table in Section 3.2 and 3.3. The asymptotic first and second moments, which were computed using the final results of Subsection 2.4.2 and 2.4.3, are also presented.

All numerical results presented in Section 3.2 and 3.3 are part of the manuscript that has been accepted for publication in the Journal of Mathematics and System Science (Kim and Chaudhry, 2014).

4 QUEUEING THEORY

Readers may refer to Appendix A.1 for a summary on probability theory, stochastic processes, and Markov processes, which are all important topics that lead to queueing theory. The definitions and properties of a continuous r.v. and its moments, Laplace transform (L.T.) and Laplace-Stieltjes transform (L-S.T.) are provided in Appendix A.2. In addition, the basic concepts of queueing systems, as well as all supplementary proof, derivation, and theorems that are used in discussing queueing theory are provided in Appendix C.

4.1 Literature review

As discussed in Chaudhry and Templeton (1983), queueing theory has its origin in the early 20th century and begins with the works of A.A. Markov and A.K. Erlang on stochastic systems. Markov chains and processes remain among the principle analytical tools in the theory of queues, while the telephone systems studied by Erlang constitute one of the principle areas of application of queueing models.

Since the early developments by Markov and Erlang, various queueing models have been studied in the theory of queues. A standard system to describe and classify queueing models known as Kendall's notation (see Appendix C.1) was developed by Kendall (1953), and many peer-reviewed scientific journals began publishing various applications of queueing theory. Applications went beyond telephones to include automotive traffic, computers, military operations of various kinds, medical appointment scheduling, machine repairs, inventory studies, and many more. In 1986, a research journal entirely dedicated to queueing theory named 'Queueing Systems' had emerged, signifying the efforts to share collective advancements in queueing theory among practitioners, engineers, mathematicians and queueing theorists.

Out of many different classifications of queueing models, single-server queues with a server that offers service which is divided into several service stages (multi-staged queues) are particularly useful in modeling areas where service is provided to customers in a phased progression. In application, multi-staged queues are widely used when analyzing manufacturing lines, annual medical check-ups, and scheduled inspections of any sort. Due to their practical importance, multi-staged queues with server that has a fixed number of service stages have been extensively analyzed by several researchers in the past:

Wishart (1956) and Wu Fang (1960) solve the system $GI/E_k/1$ by interpreting the service mechanism as a single server with identically distributed service times and a scale-modified chi-squared distribution of mean b and $2k$ degrees of freedom. In solving $GI/E_k/1$, Wu Fang (1960) uses the embedded Markov chain technique by considering various scenarios of interactions between the r.v.'s of the number of customers in queue, remaining service-stages, and completed service-stages, all between two successive customer arrivals.

As done by Wu Fang (1960), the r.v.'s that represent different aspects of multi-staged queues can be related through the use of a Markov chain. Bux (1979) builds on this concept and introduces a new technique which involves numerical analysis of the embedded Markov chains of the $GI/E_{r,s}/1$ queue with mixed Erlang service times.

Neuts (1981) determines the steady state probability distribution of the model $GI/PH_k/1$ through the use of bivariate Markov chains. His solution procedure is based on the matrix-geometric method, which Ramaswami and Lucantoni (1985) follow to develop an infinite sum expression of the waiting time distribution of a multi-staged queue that involves the rate matrix R. When this matrix R has an order r, it becomes the rate matrix of the $GI/E_r/1$ system. In response to the work by Ramaswami and Lucantoni (1985), Adan and Zhao (1996) state that when considering large values of r, the determination of R may cause memory resource problems and require excessive computation times, the latter of which is especially true for higher traffic loads.

Chaudhry and Templeton (1983) relate $GI^r/M/1$ with $GI/E_r/1$ by regarding the group of customers as being present in the system until all of that group's members have completed their services. Such interpretation allows various d.f.'s in one model to also be true in the other. As an example, the d.f. of the number of customers in $GI^r/M/1$ is the same as the d.f. of the number of uncompleted service stages in $GI/E_r/1$. In addition, they also state that the results when considering instances just before a customer arrival for the system $GI/E_r/1$ can be derived from those for the system $E_r/G/1$.

Chaudhry and Templeton (1983) also discuss basic renewal theory in the context of queueing theory. By interpreting renewals as customer arrivals (similarly, bulk-renewals as bulk-arrivals), several properties and theorems in renewal theory can be applied to solve problems in queueing theory. Examples of such application include distributional Little's law and length-biased sampling phenomenon.

Adan and Zhao (1996) solve $GI/E_r/1$ through the use of Vandermonde matrix where they express the solution as a geometric sum whose terms are the roots of the underlying characteristic equation. Grassmann (2010) gives an alternative solution procedure to $E_n/E_m/1$ by deriving and solving a set of equations that stem from the roots of the underlying characteristic equation.

In the case of finite buffer queues, Ohsone (1981) derives the distributions of the number of customers in $GI/E_k/1/N$ at random and post-departure epochs. In addition, Nobel (1989) uses the embedded Markov Chain technique to determine the solutions of total and partial rejections in $GI^X/E_k/1/N$.

Despite several published works on multi-staged queues with server that has fixed number of service stages ($GI/E_k/1$ or $GI/E_r/1$), no significant work has been done on multi-staged queues with a server that has random number of service stages $(GI/E_X/1)$ (to the best of the author's knowledge). In the review of literature, Yao et al. (1984) state that "**there is no simple way to analyze the queue $\boldsymbol{GI/E_X/1}$**".

4.2 The queueing model $GI/E_k/1$

Consider a queueing model with one server that runs a service that is divided into k fixed number of exponential service stages (E_k). The customer arrival pattern is generic (GI) and the system capacity is infinite $(N \rightarrow \infty)$. When all these conditions are put together in Kendall's notation, it becomes the $GI/E_k/1$ queueing model. Although this model has been extensively studied by several researchers in the past using various techniques (see Section 4.1), a mathematical description of $GI/E_k/1$ is deemed necessary prior to discussing its extended version $GI/E_X/1$.

4.2.1 Model description

The queueing model $GI/E_k/1$ has inter-arrival times (time periods measured between each pair of consecutive customer arrivals) $T_i, (i \geq 1)$ that are i.i.d.r.v.'s such that $T_i \sim T$. It has a cumulative distribution function (c.d.f.) $A(t) = P(T \leq t), (t > 0)$, L-

S.T. $\bar{a}(\omega) = \int_0^\infty e^{-\omega t} dA(t)$, and mean $1/_{\lambda}$. In Figure 2, the $(n+1)$-th customer arrives T time units after the n-th customer's arrival. The server in $GI/E_k/1$ consists of k exponential service stages with k being a positive constant.

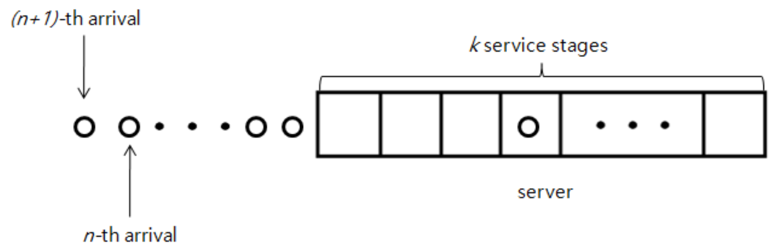

Figure 2: Visual illustration of $GI/E_k/1$.

The dynamics (everything that happens inside the model) of $GI/E_k/1$ can be described in terms of the number of uncompleted service stages in the system. For instance, the customer inside the server in Figure 2 has $k-3$ uncompleted service stages remaining until his/her departure. Another example of this concept would be an arrival of a customer resulting in an increase of the number of uncompleted service stages in the system by k. The waiting-time-in-queue of the n-th customer would be equivalent to the number of uncompleted service stages in the system immediately prior to the n-th customer's arrival. In addition, the number of customers in the queue, as well as in the system can be found in terms of the number of uncompleted service stages in the system.

The observation of the number of uncompleted service stages in the system can be made at three different time epochs. Let N_n^-, N_n, and N_n^+ be the r.v.'s that count the number of uncompleted service stages in the system at the following specific time instances:

N_n^-: Just before an arrival of the n-th customer (the n-th pre-arrival epoch)

N_n: Any time instance between the moment just before the arrival of the n-th customer to the moment just before the arrival of the $(n+1)$-th customer (the n-th random epoch)

N_n^+: Just after a departure of the n-th customer (the n-th post-departure epoch)

The r.v.'s N_n^-, N_n, and N_n^+ become steady-state r.v.'s (see Appendix B.3.1 for explanation) N^-, N, and N^+ as $n \to \infty$. Each of these steady-state r.v.'s counts the number of uncompleted service stages at following generic time instances:

N^-: Just before an arrival of a customer (a pre-arrival epoch)

N: Any time instance between the moment just before the arrival of a customer to the moment just before the arrival of the next customer (a random epoch)

N^+: Just after a departure of a customer (a post-departure epoch)

The steady-state r.v.'s have respective p.m.f.'s $p_j^- = \lim_{n\to\infty} P(N_n^- = j)$, $p_j = \lim_{n\to\infty} P(N_n = j)$, and $p_j^+ = \lim_{n\to\infty} P(N_n^+ = j)$ for $j \geq 0$. The service mechanism of $GI/E_k/1$ is such that the server serves each customer independently of previous customers and of the queue-length. Furthermore, since the duration of each service stage follows the exponential distribution (see Appendix A.2.1), service that consists of k service stages (or service time of the server) follows the Erlang-k distribution (see Appendix A.2.3). Let $B(t)$ be the c.d.f. of the Erlang-k distribution where

$$dB(t) = \frac{\mu(\mu t)^{k-1}}{(k-1)!} e^{-\mu t} dt, \quad (0 < t < \infty)$$

holds and has a L-S.T. $\bar{\beta}(\omega) = \int_0^\infty e^{-\omega t} dB(t) = \left(\frac{\mu}{\mu+\omega}\right)^k$. The mean of $B(t)$ is $b = \int_0^\infty t\, dB(t) = \frac{k}{\mu} < \infty$. The service times of the server are also independent of the inter-arrival times.

Let D_n be the number of completed service stages between the instances just before the n-th and the $(n+1)$-th customer arrivals. It has the steady-state p.m.f.

$$k_j = \int_0^{\infty} \lim_{n\to\infty} P(D_n = j \,|T = t) dA(t)$$

Since the d.f. of D_n follows a Poisson distribution (see Appendix A.3.4), k_j can be expressed as

$$= \int_0^{\infty} \frac{e^{-\mu t}(\mu t)^j}{j!} dA(t), \quad j \geq 0$$

and has a p.g.f.

$$K(z) = \sum_{j=0}^{\infty} k_j z^j = \int_0^{\infty} e^{-\mu(1-z)t} dA(t)$$

$$= \bar{a}\big(\mu(1-z)\big), \ \ |z| < 1$$

The traffic intensity (see Appendix C.1) of $GI/E_k/1$ is $\rho = \frac{\lambda k}{\mu} < 1$. The relations between N_n^-, N_{n+1}^-, and D_n can be expressed as

$$N_{n+1}^- = (N_n^- + k - D_n)^+ = \begin{cases} N_n^- + k - D_n, & N_n^- + k - D_n > 0 \\ 0, & N_n^- + k - D_n \leq 0 \end{cases} \tag{14}$$

where $(a)^+ = \max(a, 0)$ with a being an integer.

4.3 The queueing model $GI/E_X/1$

Consider an extension of $GI/E_k/1$ where the server consists of X exponential service stages with X being a random number between 1 and r, $(1 < r < \infty)$. In Kendall's notation, this extension is represented by the $GI/E_X/1$ queueing system. In describing the model $GI/E_X/1$, some r.v.'s from $GI/E_k/1$ can be kept the same given that their definitions do not change.

4.3.1 Model description

The definition of inter-arrival time (T), the number of completed service stages (D_n), and the number of uncompleted service stages in the system (N_n^-, N_n, or N_n^+) in $GI/E_k/1$ remain unchanged in $GI/E_X/1$. The service mechanism also remains unchanged since in $GI/E_X/1$, the server serves each customer independently of previous customers and of the queue-length.

The key difference between $GI/E_k/1$ and $GI/E_X/1$ is in the service pattern since the number of service stages the n-th customer has to go through is extended from k to X_n. The r.v. X_n has a p.m.f. $P(X_n = j) = s_j,\ (1 \le j \le r)$ and a p.g.f. $S(z) = E[z^{X_n}] = \sum_{h=1}^{r} s_h z^h$, $(|z| < 1)$ where r is the maximum number of service stages that the n-th customer has to complete. The mean of X_n is $\bar{s} = S'(1) = S^{(1)}(1)$ where $S^{(i)}(1)$ for $i \ge 1$ is the i-th derivative of $S(z)$ evaluated at $z = 1$. Since X_n for $n \ge 1$ are i.i.d.r.v.'s such that $X_n \sim X$, every customer in $GI/E_X/1$ must go through X service stages as indicated in Figure 3.

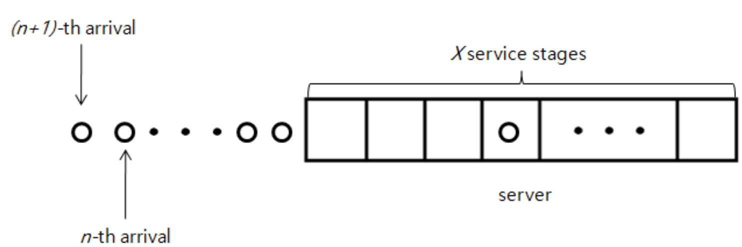

Figure 3: Visual illustration of $GI/E_X/1$.

Since the duration of each service stage follows the exponential distribution, the duration of service that consists of X exponential service stages follows the modified Erlang distribution (see Appendix A.2.4). In extending the service pattern of $GI/E_k/1$ to $GI/E_X/1$, $B(t)$ from Subsection 4.2.1 can be extended such that it becomes the c.d.f. of the modified Erlang distribution where

$$dB(t) = \sum_{j=1}^{r} s_j \frac{\mu(\mu t)^{j-1}}{(j-1)!} e^{-\mu t} dt, \quad (0 < t < \infty)$$

holds and has a L-S.T. $\bar{b}(\omega) = \int_0^\infty e^{-\omega t} dB(t) = \sum_{j=1}^{r} s_j \left(\frac{\mu}{\mu+\omega}\right)^j$. The mean of $B(t)$ is $b = \int_0^\infty t\, dB(t) = \frac{\bar{s}}{\mu} < \infty$. The service times of the server are independent of the inter-arrival times. The traffic intensity of $GI/E_X/1$ is $\rho = \frac{\lambda \bar{s}}{\mu} < 1$ and the relations between N_n^-, N_{n+1}^-, X_n and D_n can be expressed as

$$N_{n+1}^- = (N_n^- + X_n - D_n)^+ = \begin{cases} N_n^- + X_n - D_n, & N_n^- + X_n - D_n > 0 \\ 0, & N_n^- + X_n - D_n \leq 0 \end{cases} \tag{15}$$

which is an extension of (14). As done in $GI/E_k/1$ (Subsection 4.2.1), the dynamics of $GI/E_X/1$ can also be described in terms of the number of uncompleted service stages in the system. For this reason, the solution to the model is three-fold. The three parts of the solution are defined in the table below as

Table 11: Three-fold solution to $GI/E_X/1$

Pre-arrival solution	$p_j^-, (j \geq 0)$
Random solution	$p_k, (k \geq 0)$
Post-departure solution	$p_k^+, (k \geq 0)$

4.3.2 Steady-state p.g.f. and its inversion

The pre-arrival solution to $GI/E_X/1$ can be determined using a g.f. where the steady-state p.g.f. of N_n^- is constructed and then inverted to find p_j^- for $j \geq 0$. This is done as follows:

To construct the steady-state p.g.f. of N_n^-, each r.v. in (15) needs to be expressed in its steady-state form as $n \to \infty$. This results in several notational changes in (15) where $N_n^-, N_{n+1}^- \to N^-$, $X_n \to X$ and $D_n \to D$. The steady-state p.g.f. of N_n^- is thus defined as

$$P^-(z) = E[z^{N^-}] = E\left[z^{(N^- + X - D)^+}\right]$$

and using (15), the above expression can be expanded as

$$= E[z^{N^- + X - D} | N^- + X - D > 0]\, P(N^- + X - D > 0)$$
$$+ E[z^{N^- + X - D} | N^- + X - D \leq 0] P(N^- + X - D \leq 0)$$

which leads to

$$= E[z^{N^- + X - D} | N^- + X - D > 0]\, P(N^- + X - D > 0) + P(N^- + X - D \leq 0)$$

In the first term of the above expression, $E[z^{N^- + X - D} | N^- + X - D > 0] = E[z^X] E[z^{N^- - D} | N^- + X - D > 0]$ is true given that $N^- + X - D > 0$. Hence $P^-(z)$ becomes

$$= E[z^X] E[z^{N^- - D} | N^- + X - D > 0]\, P(N^- + X - D > 0) + P(N^- + X - D \leq 0)$$

Let the p.g.f. of $N^- - D$ be

$$E[z^{N^- - D}] = E[z^{N^- - D} \mid N^- + X - D > 0]P(N^- + X - D > 0)$$

$$+E[z^{N^- - D} \mid N^- + X - D \le 0]P(N^- + X - D \le 0)$$

By isolating $E[z^{N^- - D} \mid N^- + X - D > 0]P(N^- + X - D > 0)$ in the above expression and then substituting that into the previous expression, it becomes

$$P^-(z) = P^-(z)E[z^X]E[z^{-D}]$$

$$- \sum_{m=0}^{\infty} E[z^{-m} \mid N^- + X - D = -m]\, P(N^- + X - D = -m)$$

$$+ \sum_{m=0}^{\infty} P(N^- + X - D = -m)$$

Let $q_m = P(N^- + X - D = -m), (m \ge 0)$, then isolating $P^-(z)$ in above gives

$$P^-(z) = \frac{\sum_{m=0}^{\infty} q_m (1 - z^{-m})}{1 - S(z)K(z^{-1})}, (|z| \le 1) \tag{16}$$

which is analytic (can be differentiated and evaluated) on $|z| \le 1$. In general, the inversion of a p.g.f. through Taylor's series expansion (see Appendix A.3.6) requires no unknown probabilities (that is the constant coefficient of each term) in that p.g.f.. Taylor's series expansion is not a suitable tool to invert (16) since its numerator contains undetermined probabilities (q_m). This problem can be mitigated by employing a technique that is similar to the one given in Chaudhry and Templeton (1983) where they express a rational p.g.f. with unknown terms as another form of p.g.f. that is readily invertible through Taylor's series expansion. The illustration of their technique in finding the alternate form of (16) is as follows:

Let the characteristic equation (see Appendix C.1) of $GI/E_X/1$ be

$$0 = 1 - S(z^{-1})K(z)$$

where $S(z^{-1}) = \sum_{h=1}^{r} s_h\, z^{-h}$. This equation has r roots on the inside of a unit circle $|z| = 1$ (see Appendix C.2.1 for proof), which can be easily found using MAPLE. Let these inside roots be $z_1, z_2, \dots, z_r$. Since $1 = S(z^{-1})K(z)$ is a reciprocal polynomial of $1 = S(z)K(z^{-1})$, it can be said that the denominator of (16) has r roots on the outside of a unit circle $|z| = 1$. Let those outside roots be $z_1^{-1}, z_2^{-1}, \dots, z_r^{-1}$. Suppose that there is a new complex function

$$B(z) = P^{-}(z)\prod_{h=1}^{r}(1 - z_h z)$$

which is analytic on $|z| \leq 1$ given that it consists of $P^{-}(z)$. Then by substituting (16) into $B(z)$, it becomes evident that

$$B(z) = \frac{\prod_{h=1}^{r}(1 - z_h z)\sum_{m=0}^{\infty} q_m (1 - z^{-m})}{1 - S(z)K(z^{-1})}$$

is a complex function that is analytic on $|z| > 1$ since the roots $z_1^{-1}, z_2^{-1}, \dots, z_r^{-1}$ of its denominator are also the roots of its numerator. With $B(z)$ being analytic on the inside, outside, and contour of a unit circle $|z| = 1$, by Liouville's theorem (see Appendix C.4.2), it becomes a positive constant B, such that

$$B = P^{-}(z)\prod_{h=1}^{r}(1 - z_h z)$$

Using a property of the p.g.f., $P^{-}(1) = 1$, B can be determined as

$$B = \prod_{h=1}^{r}(1 - z_h)$$

Finally, the alternate form of (16) with no unknowns is found as

$$P^-(z) = \prod_{h=1}^{r}\left(\frac{1 - z_h}{1 - z_h z}\right), \quad |z| \leq 1 \tag{17}$$

where p_j^- for $j \geq 0$ are the set of constant coefficients of each term in the Taylor's series expansion of (17). The technique illustrated in determining (17) is known as the roots method. Once this pre-arrival solution $\left(p_j^-\right)$ is determined, the model $GI/E_X/1$ can be considered as solved. This is true since p_j^- is a key p.m.f. which all other d.f.'s that describe different dynamics of $GI/E_X/1$ are built upon. Two of these d.f.'s are discussed in the next subsection.

4.3.3 Relations between solutions at different time epochs

The pre-arrival solution from Subsection 4.3.2 can be used to determine random and post-departure solutions to $GI/E_X/1$. This can be done through the standard level crossing analysis, which is a technique in queueing theory that is widely used to build relations among d.f.'s of a r.v. at different time epochs. Similar applications in different models are discussed by Yao et al. (1984) and Cordeau and Chaudhry (2009). The standard level crossing analysis in the context of $GI/E_X/1$ is explained as follows:

In Subsection 4.2.1, N is defined as the steady-state r.v. that counts the number of uncompleted service stages in the system at a random epoch. Suppose that throughout some time interval $(0, t]$, the value of *N* varies due to changes in the number of completed service stages (decrease of N) and the number of customer arrivals (increase of N) during $(0, t]$. An example of N varying throughout $(0, t]$ is depicted in Figure 4.

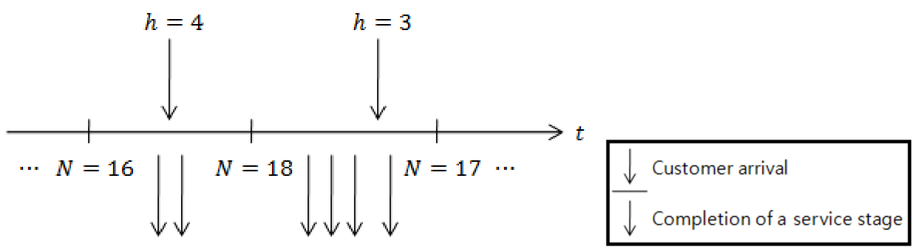

Figure 4: An example illustrating the variation of N throughout some time interval $(0, t]$.

Whenever N decreases, it will decrease by 1 since a customer in the server completes one service stage at a time. Suppose that N is initially at k then decreases to $k - 1$. This is depicted in Figure 5.

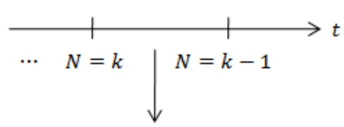

Figure 5: Decrease of N from k to $k - 1$ due to completion of a service stage.

Let $D_k(t)$, $(k \geq 1)$ be the mean number of downward transitions from $N = k$ to $N = k - 1$ throughout $(0, t]$. On the contrary, whenever N increases due to a customer arrival, it will increase by the number of service stages that the arriving customer has to complete, or $h, (1 \le h \le r)$. To illustrate this, suppose that N is initially at $j, (0 \le j < k)$ and becomes $j + h$ after a customer arrival (see Figure 6).

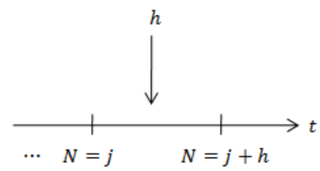

Figure 6: Increase of N from j to $j + h$ due to a customer arrival.

Let $U_j(t), (j \ge 0)$ be the mean number of upward transitions from $N = j$ to $N = j + h$ where h varies according to its p.m.f. $s_h, (1 \le h \le r)$. Since $j < k$, whenever N increases from j, it will either become k or greater than k. Let $\overline{U}_k(t)$ be the mean number of upward transitions of N from j to and over k. By intuition, $\overline{U}_k(t)$ would be in terms of $U_j(t)$ and $s_h, (1 \le h \le r)$ since the number of times N increases from j to k throughout$(0, t]$ is same as the number of times N increases from j to $j + h$ when $h = k - j$ throughout $(0, t]$. Similarly, the number of times N increases from j to any value greater than k throughout $(0, t]$ is same as the number of times N increases from j to $j + h$ when $h > k - j$ throughout $(0, t]$. Based on this notion, $\overline{U}_k(t)$ can be defined as

$$\overline{U}_k(t) = \sum_{j=0}^{k-1} U_j(t) \sum_{h=k-j}^{r} s_h$$

Given that a stable queueing system has an upward transition rate which is the same as its downward transition rate (Foster and Perera, 1965), the expression

$$\lim_{t\to\infty}\frac{D_k(t)}{t}=\lim_{t\to\infty}\frac{\overline{U}_k(t)}{t} \tag{18}$$

must hold. The definition of $\overline{U}_k(t)$ can be substituted into (18) such that

$$\lim_{t\to\infty}\frac{D_k(t)}{t}=\lim_{t\to\infty}\frac{\sum_{j=0}^{k-1}U_j(t)\sum_{h=k-j}^{r}s_h}{t} \tag{19}$$

Multiplying and dividing the left-hand side of (19) by $\sum_{k=1}^{\infty}D_k(t)$ and doing the same on the right-hand side by $\sum_{j=0}^{\infty}U_j(t)$ gives

$$\lim_{t\to\infty}\left(\frac{\sum_{k=1}^{\infty}D_k(t)}{t}\right)\left(\frac{D_k(t)}{\sum_{k=1}^{\infty}D_k(t)}\right)=\lim_{t\to\infty}\sum_{j=0}^{k-1}\left(\frac{\sum_{j=0}^{\infty}U_j(t)}{t}\right)\left(\frac{U_j(t)}{\sum_{j=0}^{\infty}U_j(t)}\right)\sum_{h=k-j}^{r}s_h$$

where $\lim_{t\to\infty}\frac{\sum_{k=1}^{\infty}D_k(t)}{t}$ is the service rate (μ) and $\lim\limits_{t\to\infty}\frac{\sum_{j=0}^{\infty}U_j(t)}{t}$ is the arrival rate (λ) . In addition, as defined by Foster and Perera (1965), $p_j^- = \lim\limits_{t\to\infty}\frac{U_j(t)}{\sum_{j=0}^{\infty}U_j(t)}$ and $p_k = \lim\limits_{t\to\infty}\frac{D_k(t)}{\sum_{k=1}^{\infty}D_k(t)}$ can be substituted into above expression, which gives

$$\mu p_k=\sum_{j=0}^{k-1}\lambda p_j^-\sum_{h=k-j}^{r}s_h$$

or

$$p_k=\frac{\lambda}{\mu}\sum_{j=0}^{k-1}p_j^-\left(1-\sum_{h=1}^{k-j-1}s_h\right), \tag{20}$$

where $k \geq 1$ and $p_0 = 1 - \sum_{k=1}^{\infty} p_k = 1 - \left(\frac{\lambda \bar{s}}{\mu}\right) = 1 - \rho$. By renewal theory, there exists an alternate way of determining p_k using p_j^- (see Appendix C.3.1).

With p_k known from (20), the standard level crossing analysis enables the finding of p_k^+ in terms of p_k. This is done as follows: Given the definition of $\bar{U}_k(t)$, both of its sides are summed over k, such that

$$\sum_{k=1}^{\infty} \bar{U}_k(t) = \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} U_j(t) \sum_{h=k-j}^{r} s_h$$

$$= \bar{s} \sum_{j=0}^{\infty} U_j(t)$$

Similarly, sum of (18) on both of its sides over k gives

$$\sum_{k=1}^{\infty} \lim_{t\to\infty} \frac{D_k(t)}{t} = \sum_{k=1}^{\infty} \lim_{t\to\infty} \frac{\bar{U}_k(t)}{t}$$

By rearranging $\sum_{k=1}^{\infty} \bar{U}_k(t) = \bar{s} \sum_{j=0}^{\infty} U_j(t)$ and $\sum_{k=1}^{\infty} \lim_{t\to\infty} \frac{D_k(t)}{t} = \sum_{k=1}^{\infty} \lim_{t\to\infty} \frac{\bar{U}_k(t)}{t}$, it leads to

$$\lim_{t\to\infty} \frac{\sum_{k=1}^{\infty} \bar{U}_k(t)}{\sum_{k=1}^{\infty} D_k(t)} = \lim_{t\to\infty} \frac{\bar{s} \sum_{j=0}^{\infty} U_j(t)}{\sum_{k=1}^{\infty} D_k(t)} = 1$$

Multiplying both sides of the above by ${D_k(t)}/{t}$ results in

$$\lim_{t\to\infty} \left[\frac{\sum_{j=0}^{\infty} U_j(t)}{t}\right] \left[\frac{\bar{s} D_k(t)}{\sum_{k=1}^{\infty} D_k(t)}\right] = \lim_{t\to\infty} \frac{D_k(t)}{t} \tag{21}$$

The right-hand side of (21) is μp_k as can be seen in deriving (20). The left-hand side of (21) indicates that if $\lim_{t\to\infty}\frac{\sum_{j=0}^{\infty}U_j(t)}{t}$ is the arrival rate (λ) then $\lim_{t\to\infty}\frac{D_k(t)}{\sum_{k=1}^{\infty}D_k(t)}$ for $k \geq 1$ is the probability of there being $k-1$ uncompleted service stages in the system just after the departure of a customer (p_{k-1}^{+}). Based on this, (21) can be rewritten as

$$\lambda\bar{s}p_{k-1}^{+} = \mu p_k$$

or

$$p_{k-1}^{+} = \frac{1}{\rho}p_k, \quad (k \geq 1) \tag{22}$$

Hence the solution to $GI/E_X/1$ (defined in Table 11) is completely found.

4.3.4 Conclusion

In queueing theory, queues with multi-staged services are useful tools of application when considering a setting where service is provided in a phased manner. In the past, such queues with server that has fixed number of exponential service stages ($GI/E_k/1$) have been analyzed by various researchers. In extending $GI/E_k/1$ to $GI/E_X/1$, some r.v.'s and d.f.'s are kept the same both in terms of notation and definition for consistency sake. What changed is the number of service stages that every customer has to go through, which is extended from a constant to a r.v.. Relations between different r.v.'s of $GI/E_X/1$ are derived, which are used to build the steady-state p.g.f. of the model. This p.g.f. is then inverted to determine the pre-arrival solution, which is used to find its counterpart solutions at random and post-departure epochs. Importantly, the pre-arrival solution can be further leveraged to explore other d.f.'s within $GI/E_X/1$ which is discussed in the next section.

All derivations and relations in Section 4.3 are part of the manuscript that has been accepted for publication in the American Journal of Operations Research (Chaudhry and Kim, 2015).

4.4 Additional findings in $GI/E_X/1$

The pre-arrival solution from Section 4.3 can be leveraged to determine other important steady-state d.f.'s that describe different dynamics of $GI/E_X/1$. In this section, the distribution of the waiting-time-in-queue of an incoming customer and the number of customers in queue and the system are derived in terms of the pre-arrival solution. This derivation is considered rigorous and lengthy, hence an analytical example of a special case of $GI/E_X/1$ is provided for the purpose of demonstration. Lastly, the performance measures of $GI/E_X/1$ are introduced, which were used in Chapter 5 to compute various numerical results.

4.4.1 Waiting-time-in-queue

Let w_q^- be the r.v. of the amount of time an incoming customer has to spend in queue (waiting-time-in-queue) until entering service. The c.d.f. of w_q^- is $W_q^-(t) = P(w_q^- \leq t),\ (t \geq 0)$. Intuitively, if an incoming customer enters service immediately upon arrival, then there must be no uncompleted service stages in the system prior to his or her arrival. In other words,

$$W_q^-(0) = p_0^-$$

must be true. Alternatively, if an incoming customer has to spend at least some amount of time in queue (say $t > 0$) before entering service then there must be $i, (i > 0)$

uncompleted service stages in the system prior to his or her arrival (see Figure 7 for visual illustration).

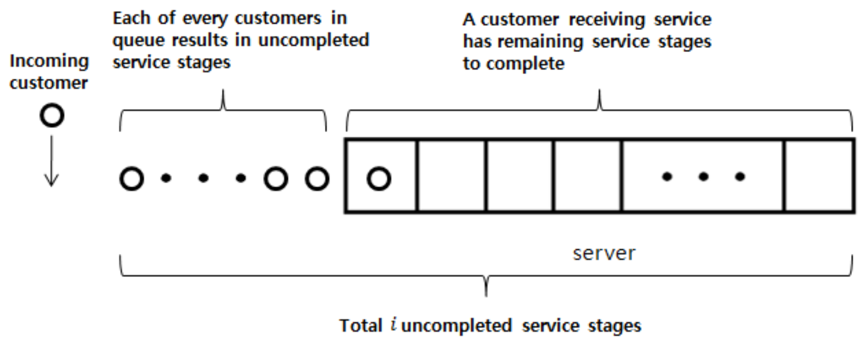

Figure 7: Visual illustration of the composition of waiting-time-in-queue of an incoming customer.

Based on this notion, when $t > 0$, the waiting-time-in-queue of an incoming customer is the durations of $i, (i \geq 1)$ uncompleted exponential service stages. The probability distribution of that duration follows the Erlang-i distribution with a c.d.f.

$$W_q^-(t) = \int_0^t \frac{(\mu x)^{i-1}}{(i-1)!} e^{-\mu x} \mu dx$$

which holds for $t > 0$. To complete the expression of $W_q^-(t)$, the c.d.f. above can be multiplied with p_i^- and then summed over i in order to span all possible values of i and associated probabilities. Thereby $W_q^-(t)$ becomes

$$W_q^-(t) = \sum_{i=1}^{\infty} p_i^- \int_0^t \frac{(\mu x)^{i-1}}{(i-1)!} e^{-\mu x} \mu dx, \quad (t > 0)$$

Finally, by combining $W_q^-(t)$ when $t = 0$ and $t > 0$, the c.d.f. of waiting-time-in-queue of an incoming customer is completely expressed as

$$W_q^-(t) = p_0^- + \mu \sum_{i=1}^{\infty} p_i^- \int_0^t \frac{(\mu x)^{i-1}}{(i-1)!} e^{-\mu x} dx \quad (t \geq 0) \tag{23}$$

4.4.2 Number of customers in queue and the system

It is evident that (23) is in terms of the pre-arrival solution. The steady-state p.m.f. of the number of customers in queue can be found in terms of (23), hence by composition, it is also in terms of the pre-arrival solution to $GI/E_X/1$. This confirms the statement in Subsection 4.3.2 that p_j^- is a key d.f. which all other d.f.'s in $GI/E_X/1$ are built upon.

Distributional Little's law is a technique in queueing theory that is used to build connection between the steady p.m.f. of the number of customers in queue at a random epoch and (22). In the following, distributional Little's law is explained in the context of $GI/E_X/1$.

Intuitively, if an incoming customer has to wait in queue for a long time, it indicates that queue is large. On the contrary, if the waiting-time-in-queue of a customer is short, it signifies that queue is small. It is evident that the two r.v.'s (waiting-time-in-queue and queue-length) are directly proportional and nicely meet the criteria for the application of distributional Little's Law. This can be done by introducing renewal theory in queueing theory: Processes that counts the number of renewals over duration of time can be manipulated into processes that count the number of customers in queue, where that number is proportional to the duration of waiting-time-in-queue. This is done as follows:

Let there be some continuous-time single-renewal processes $\{M_a(t), t > 0\}$ where $M_a(t)$ is a r.v. that counts the number of renewals during the time interval $(0, t]$. Let the p.m.f. of $M_a(t)$ be $Q_n(t) = Q(n,t) = P(M_a(t) = n)$ with the p.g.f. $Q(z,t) = \sum_{n=0}^{\infty} P(M_a(t) = n)\, z^n$, (|z| < 1) and a L.T.

$$\bar{Q}(z,\omega) = \int_0^{\infty} e^{-\omega t} Q(z,t)dt = \frac{1}{\omega} - \lambda \frac{\left(1-\bar{a}(\omega)\right)(1-z)}{\omega^2\left(1-z\bar{a}(\omega)\right)} \tag{24}$$

where $\bar{a}(\omega)$ is a L-S.T. of the c.d.f. of renewal periods $A(t)$ (for a proof of (24) readers may see Cox,1962 or Bertsimas and Nakazato,1995). Interestingly, expression (24) is what would be equivalent to (2) in the renewal theory portion of this thesis. The p.m.f. $Q_n(t)$ can be manipulated into the steady-state p.m.f. of the number of customers in queue at a random epoch (Q_n) by conditioning then un-conditioning $M_a(t)$ on w_q^- such that

$$Q_n(t) = P(M_a(t) = n)$$

becomes

$$Q_n = \begin{cases} 1 - \rho + P\left(M_a\left(w_q^-\right) = 0\right), & (n = 0) \\ \displaystyle\int_0^{\infty} P\left(M_a\left(w_q^-\right) = n \middle| w_q^- = t\right) dW_q^-(t), & (n \geq 1) \end{cases} \tag{25}$$

Although (25) is an explicit expression of the steady-state p.m.f. of the number of customers in queue at random epoch, its direct computation is inconvenient and potentially time consuming since it requires the simultaneous finding of the inverse L.T. as well as z-transform (see Appendix A.3.5) of (24). To mitigate this difficulty, suppose that Q_n has a p.g.f. $G_{N_q}(z)$ such that

$$G_{N_q}(z) = \sum_{n=0}^{\infty} Q_n \, z^n, \quad (|z| < 1)$$

$$= 1 - \rho + \int_0^{\infty} \sum_{n=0}^{\infty} P(M_a(t) = n) \, z^n \, dW_q^-(t)$$

or

$$G_{N_q}(z) = 1 - \rho + \int_0^{\infty} Q(z,t) \, dW_q^-(t) \tag{26}$$

where $Q(z,t)$ in (26) is an inverse-L.T. of (24). Additionally, in (26), the term $1 - \rho$ is added to account for the case where the queue and server of $GI/E_X/1$ are empty and idle, respectively (there exists an alternate case where the queue is empty but the server is busy). In the case when $\bar{Q}(z,\omega)$ cannot be inverted directly to $Q(z,t)$, Padé approximation (see Appendix 2.6) can be used to accurately estimate $\bar{Q}(z,\omega)$ into a rational form that is readily invertible. Once $Q(z,t)$ is found, it can be substituted into (26). By doing so, its Taylor's series expansion becomes:

$$G_{N_q}(z) = Q_0 + Q_1 z + Q_2 z^2 + Q_3 z^3 + \ldots, (|z| < 1) \tag{27}$$

where $\{Q_n\}$ for $n \geq 0$ is a sequence of the coefficients in (27). Also, the steady-state p.m.f. of the number of customers in the system at a random epoch, say R_n, $(n \geq 0)$, can be found directly from Q_n such that

$$R_n = \begin{cases} 1 - \rho, & (n = 0) \\ \rho - 1 + Q_0, & (n = 1) \\ Q_{n-1}, & (n \geq 2) \end{cases} \tag{28}$$

where $R_0 = p_0 = 1 - \rho$ holds (probability of no customer in the system at a random epoch equating to probability of no uncompleted service stages in the system at random epoch).

Let R_{n-1}^{+} for $n \geq 1$ be the p.m.f. of the number of customers in the system at a post-departure epoch. Since the relation (22) is also true between R_n and R_{n-1}^{+} (see Yao et.al, 1984), another relation can be established as

$$R_{n-1}^{+} = \frac{1}{\rho} R_n, \quad (n \geq 1)$$

where $R_n^{+} = R_n^{-}$ for $n \geq 0$.

4.4.3 Analytical example of $GI/E_X/1$

In this subsection, a demonstration is provided on how one would analyze $GI/E_X/1$ using various findings in Subsection 4.3.2 (steady-state p.g.f. and its inversion), 4.3.3 (relations between solutions at different time epochs), 4.4.1 (waiting-time-in-queue), and 4.4.2 (number of customers in queue and the system).

Consider the model $M/E_X/1$ with p.m.f. of X as $s_h, (1 \leq h \leq r)$. Customer arrivals follow exponential distribution, hence the probability density function (p.d.f.) of the inter-arrival time is

$$a(t) = \lambda e^{-\lambda t}, (t > 0)$$

with a L.T. $\bar{a}(\omega) = \frac{\lambda}{\lambda + \omega}$. The p.g.f. of $k_j, (j \geq 0)$ is

$$K(z) = \sum_{j=0}^{\infty} k_j z^j = \bar{a}\big(\mu(1 - z)\big), \quad |z| < 1$$

and letting $\omega = \mu(1-z)$ in the above leads to

$$K(z) = \frac{\lambda}{\lambda + \mu(1-z)}$$

The equation $0 = 1 - S(z^{-1})K(z)$ of $M/E_X/1$ can be solved to obtain the roots to build (17). In doing so, the equation

$$0 = 1 - \left(\frac{s_1}{z} + \frac{s_2}{z^2} + \cdots + \frac{s_r}{z^r}\right)\left(\frac{\lambda}{\lambda + \mu(1-z)}\right)$$

can be solved using MAPLE or MATHEMATICA to determine the roots $z_1, z_2, z_3, \dots, z_r$. These roots are then substituted into (17) which form the steady-state p.g.f. of $M/E_X/1$:

$$P^-(z) = \prod_{h=1}^{r}\left(\frac{1-z_h}{1-z_h z}\right), \quad |z| \le 1$$

where the coefficients of its Taylor's series expansion are the pre-arrival solution $p_j^-, (j \ge 0)$. With $p_j^-, (j \ge 0), p_k, p_k^+, (k \ge 0)$, and $W_q^-(t), (t \ge 0)$ can be determined using (20), (22), and (23), respectively. In determining Q_n, $\bar{a}(\omega) = \frac{\lambda}{\lambda+\omega}$ can be substituted into (24), which results in

$$\bar{Q}(z,\omega) = \int_0^\infty e^{-\omega t} Q(z,t)dt = \frac{1}{\lambda + \omega - \lambda z}$$

The above expression has an inverse L.T.

$$Q(z,t) = e^{-\lambda(1-z)t}, \quad (|z| \le 1, t \ge 0)$$

The Taylor's series expansion of $Q(z,t)$ is

$$e^{-\lambda(1-z)t} = e^{-\lambda t}\left\{1 + (\lambda t)z + \frac{1}{2}(\lambda t)^2 z^2 + \frac{1}{6}(\lambda t)^3 z^3 + \frac{1}{24}(\lambda t)^4 z^4 + \cdots\right\}$$

Hence, by substituting the above series into (26), it becomes (27) such that

$$G_{N_q}(z) = 1 - \rho + \int_0^\infty Q(z,t)\, dW_q^-(t)$$

$$= \left[1 - \rho + \int_0^\infty e^{-\lambda t} dW_q^-(t)\right] + \left[\lambda \int_0^\infty t e^{-\lambda t} dW_q^-(t)\right] z$$
$$+ \left[\frac{\lambda^2}{2}\int_0^\infty t^2 e^{-\lambda t} dW_q^-(t)\right] z^2 + \left[\frac{\lambda^3}{6}\int_0^\infty t^3 e^{-\lambda t} dW_q^-(t)\right] z^3 + \cdots$$

The coefficients of $G_{N_q}(z)$ form the steady-state p.m.f. of the number of customers in queue at a random epoch

$$Q_n = \begin{cases} 1 - \rho + \int_0^\infty e^{-\lambda t} dW_q^-(t)\,, (n = 0) \\ \dfrac{\lambda^n}{n!}\int_0^\infty t^n e^{-\lambda t} dW_q^-(t), \quad (n \geq 1) \end{cases}$$

Using (28), the steady-state p.m.f. of the number of customers in the system at a random epoch can be found as

$$R_n = \begin{cases} 1 - \rho, & (n = 0) \\ \rho - 1 + Q_0, & (n = 1) \\ \dfrac{\lambda^{n-1}}{(n-1)!}\int_0^\infty t^{n-1} e^{-\lambda t} dW_q^-(t), & (n \geq 2) \end{cases}$$

The steady-state p.m.f. of the number of customers in the system at pre-arrival and post-departure epochs can be determined using (22):

$$R_{n-1}^{+} = R_{n-1}^{-} = \begin{cases} \dfrac{Q_0 - 1}{\rho} + 1, & (n = 1) \\ \dfrac{\lambda^{n-1}}{\rho(n-1)!} \displaystyle\int_0^{\infty} t^{n-1} e^{-\lambda t} dW_q^{-}(t), & (n \geq 2) \end{cases}$$

4.4.4 Performance measures

In queueing theory, performance measures are important since they provide the best way of interpreting the different dynamics of the system. For example, to observe the congestion level of the server, the average number of customers in queue would be a far better indicator than the probability of each likely number of customers in queue. In addition, there exist interesting relations among performance measures such as Little's law. The relations are often used as cross-checking tools to confirm that their respective d.f.'s are correct.

In exploring the performance measures of $GI/E_X/1$, let M_{phase}, L_q, L_s, and $E_{W_q^-}$ be the mean number of uncompleted service stages in the system at a random epoch, mean number of customers in queue at a random epoch, mean number of customers in the system at a random epoch and mean waiting-time-in-queue of an incoming customer, respectively. First, the mean number of uncompleted service stages in the system at a random epoch is

$$M_{phase} = \sum_{i=1}^{\infty} i\, p_i \tag{29}$$

and similarly, $M_{phase}^{-} = \sum_{i=1}^{\infty} i\, p_i^{-}$ and $M_{phase}^{+} = \sum_{i=1}^{\infty} i\, p_i^{+}$. The mean number of customers in queue at a random epoch is

$$L_q = \sum_{n=1}^{\infty} n\, Q_n \tag{30}$$

and the mean number of customers in the system at a random epoch is

$$L_s = \sum_{n=1}^{\infty} n\, R_n \tag{31}$$

The mean number of customers in the system at pre-arrival and post-departure epochs are identical such that $L_s^+ = L_s^- = \sum_{n=1}^{\infty} n\, R_n^+$. The relation between (30) and (31) can be established by expressing L_q in terms of R_n, such that

$$L_q = \sum_{n=2}^{\infty} (n-1)\, R_n$$

$$= \sum_{n=2}^{\infty} n\, R_n - \sum_{n=2}^{\infty} R_n$$

$$= L_s - 1 + R_0$$

and using the relation $R_0 = 1 - \rho$ gives

$$L_q = L_s - \rho \tag{32}$$

which is one of the well-known properties of single-server queues in queueing literature. The mean waiting-time-in-queue of an incoming customer can be defined as

$$E_{W_q^-} = \int_0^{\infty} td\; W_q^-(t) \tag{33}$$

and the mean waiting-time-in-system of an incoming customer is

$$E_{W^-} = E_{W_q^-} + \frac{\overline{s}}{\mu} \tag{34}$$

In addition, Little's law in the context of $GI/E_X/1$ can be defined as

$$L_q = \lambda E_{W_q^-} \tag{35}$$

and

$$L_s = \lambda E_{W^-} \tag{36}$$

where both (35) and (36) can be used as crosschecking tools when doing numerical computations since (L_q, L_s) and $\left(E_{W_q^-}, E_{W^-}\right)$ can be separately found.

4.4.5 Conclusion

Using the pre-arrival solution, other d.f.'s within $GI/E_X/1$ are found. These distributions describe different aspects of $GI/E_X/1$, including the c.d.f. of the waiting-time-in-queue of an incoming customer, and the p.m.f. of the number of customers in queue and in the system. In determining the p.m.f. of the number of customers in queue at a random epoch, renewal theory is introduced in queueing theory where a p.m.f. of a r.v. in the continuous-time single-renewal processes is manipulated into a p.m.f. of the number of customers in queue. This technique is known as distributional Little's law. By combining Subsection 4.3.2 and 4.3.3 with Subsection 4.4.1 and 4.4.2, a complete analysis to $GI/E_X/1$ is presented, where various distributions are all expressed fundamentally in terms of the pre-arrival solution. A demonstration of how one would solve $GI/E_X/1$ is provided using $M/E_X/1$. Lastly, performance measures of $GI/E_X/1$ are found using various d.f.'s.

All additional findings in $GI/E_X/1$ are part of the manuscript that has been accepted for publication in the American Journal of Operations Research (Chaudhry and Kim, 2015).

5 NUMERICAL EXAMPLES IN QUEUEING THEORY

In this chapter, various numerical examples of $GI/E_X/1$ are presented. They are organized in the following manner: $M/E_X/1$ in Section 5.1, $E_m/E_X/1$ in Section 5.2, and $D/E_X/1$ in Section 5.3. A numerical comparison against existing results is provided in Section 5.4. All computations were performed on MAPLE, calibrated to compute up to the ninth decimal place. All results were rounded to four decimal places in the following tables.

5.1 Computing $M/E_X/1$

Inter-arrival pattern is exponential (M) with a p.d.f. $f(t) = \lambda e^{-\lambda t}, t > 0$. Parameters taken were $b_1 = 0.4, b_2 = 0.2, b_3 = 0.2, b_4 = 0.2, \mu = 12$ and $\rho = 0.5$ (with $\lambda \cong 2.7272$).

Table 12: Computing $M/E_X/1$ with $b_1 = 0.4, b_2 = 0.2, b_3 = 0.2, b_4 = 0.2, \mu = 12$ and $\rho = 0.5$ (with $\lambda \cong 2.7272$)

j	p_j^-	p_j	p_j^+	Q_j	R_j	R_j^+	t	$W_q^-(t)$
0	0.5000	0.5000	0.2272	0.7632	0.5000	0.5264	0.0000	0.5000
1	0.1136	0.1136	0.1880	0.1282	0.2632	0.2564	0.0114	0.5154
2	0.0940	0.0940	0.1646	0.0596	0.1282	0.1192	0.1544	0.6807
3	0.0823	0.0823	0.1291	0.0271	0.0596	0.0542	0.4566	0.8815
4	0.0645	0.0645	0.0792	0.0122	0.0271	0.0244	0.6940	0.9470
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	2.6666	0.9999

$M_{phase}^- = 1.9091$	$L_s = 0.9339$	$E_{W^-} = 0.3424$
$M_{phase} = 1.9091$	$L_q = 0.4339$	$E_{W_q^-} = 0.1591$
$M_{phase}^+ = 2.8182$	$L_s^+ = 0.8678$	$\lambda E_{W^-} = 0.9339$
		$\lambda E_{W_q^-} = 0.4339$

5.2 Computing $E_m/E_X/1$

Inter-arrival pattern is Erlang-m (E_m) with a p.d.f. $f(t) = \frac{\lambda^m t^{m-1} e^{-\lambda t}}{(m-1)!}, t > 0$.

Parameters taken were $b_1 = 0.25, b_2 = 0.25, b_3 = 0.4, b_4 = 0.1,\ \mu = 4, m = 3$, and $\rho = 0.75$ (with $\lambda \cong 1.2766$).

Table 13: Computing $E_m/E_X/1$ with $b_1 = 0.25, b_2 = 0.25, b_3 = 0.4, b_4 = 0.1$, $\mu = 4, m = 3$, and $\rho = 0.75$ (with $\lambda \cong 1.2766$)

j	p_j^-	p_j	p_j^+	Q_j	R_j	R_j^+	t	$W_q^-(t)$
0	0.9234	0.2500	0.3929	0.9589	0.2500	0.9452	0.0000	0.9233
1	0.0390	0.2947	0.3113	0.0395	0.7089	0.0527	0.0114	0.9251
2	0.0233	0.2335	0.2188	0.0015	0.0395	0.0020	0.1544	0.9446
3	0.0107	0.1641	0.0596	$5.1763\text{x}10^{-5}$	0.0015	$6.9018\text{x}10^{-5}$	0.4566	0.9720
4	0.0028	0.0447	0.0112	$1.6313\text{x}10^{-6}$	$5.1763\text{x}10^{-5}$	$2.1751\text{x}10^{-6}$	0.6940	0.9842
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	2.9999	0.9999

$M_{phase}^- = 0.1339$	$L_s = 0.7927$	$E_{W^-} = 0.6210$
$M_{phase} = 1.5047$	$L_q = 0.0427$	$E_{W_q^-} = 0.0334$
$M_{phase}^+ = 1.0006$	$L_s^+ = 0.0567$	$\lambda E_{W^-} = 0.7927$
		$\lambda E_{W_q^-} = 0.0427$

5.3 Computing $D/E_X/1$

Inter-arrival pattern is deterministic (D) with the inter-arrival time fixed at 1. Parameters taken were $b_1 = 0.65, b_2 = 0.10, b_3 = 0.20, b_4 = 0.05,\ \mu = 7$ and $\rho = 0.4$ (with $\lambda \cong 1.6970$).

Table 14: Computing $D/E_X/1$ with $b_1 = 0.65, b_2 = 0.10, b_3 = 0.20, b_4 = 0.05$, $\mu = 7, T = 1$, and $\rho = 0.4$ (with $\lambda \cong 1.6970$)

j	p_j^-	p_j	p_j^+	Q_j	R_j	R_j^+	t	$W_q^-(t)$
0	0.9882	0.6000	0.5989	0.9961	0.6000	0.9903	0.0000	0.9882
1	0.0086	0.2396	0.2148	0.0038	0.3961	0.0095	0.0114	0.9889

2	0.0026	0.0859	0.1531	$1.4\text{x}10^{-5}$	0.0038	$3.5000\text{x}10^{-5}$	0.1544	0.9947
3	0.0006	0.0613	0.0321	$3.6915\text{x}10^{-8}$	$1.4\text{x}10^{-5}$	$9.2288\text{x}10^{-8}$	0.4566	0.9990
4	$6.1979\text{x}10^{-5}$	0.0129	0.0008	$8.1220\text{x}10^{-11}$	$3.6915\text{x}10^{-8}$	$2.0305\text{x}10^{-10}$	0.6940	0.9992
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.8888	0.9999

$M^{-}_{phase} = 0.0157$	$L_s = 0.4038$	$E_{W^-} = 0.2380$
$M_{phase} = 0.6487$	$L_q = 0.0038$	$E_{W_q^-} = 0.0022$
$M^{+}_{phase} = 0.6218$	$L_s^{+} = 0.0096$	$\lambda E_{W^-} = 0.4038$
		$\lambda E_{W_q^-} = 0.0038$

5.4 Computing $E_n/E_m/1$

The numerical results of a simpler model $E_n/E_m/1$ by Grassmann (2010) were compared with the results obtained by using the method introduced in this paper. Parameters taken were $\mu = 1$ and the remaining (ρ, n and m) were computed at various parameters.

Table 15: Comparison against Grassmann's numerical results in $E_n/E_m/1$

ρ	n	m	Grassmann	L_q	L_s
0.5	2	3	0.1585	0.1585	0.6585
0.5	2	6	0.1228	0.1228	0.6228
0.5	4	3	0.0800	0.0800	0.5800
0.5	4	6	0.0505	0.0505	0.5505
0.9	2	3	3.2570	3.2570	4.1570
0.9	2	6	2.5930	2.5930	3.4930
0.9	4	3	2.1948	2.1948	3.0948
0.9	4	6	1.5417	1.5417	2.4417

All results matched with those of Grassmann (2010) up to 9 decimal places.

5.5 Conclusion

Numerical computations in queueing theory serve as an important proof of concept for theoretical derivations. Specifically, numerical examples confirm that the found solution to the model is correct. Thus, it is common in literature to publish analytical results followed by numerical examples. In this thesis, Chapter 4 contains the theoretical aspect of new and extended results in $GI/E_X/1$. Chapter 5 serves as a proof of concept of Chapter 4 by computing numerical results at different parameter values.

Section 5.1, 5.2, and 5.3 cover numerical examples of $GI/E_X/1$ with exponential, Erlang-m, and deterministic inter-arrival time patterns, respectively. Various probabilities and performance measures are presented along with confirmation of results using Little's law (for both in queue and in the system).

Section 5.4 covers the comparison of results with that of Grassmann (2010) in solving the model $E_n/E_m/1$. Although $E_n/E_m/1$ is considered a special case of $GI/E_k/1$, a numerical comparison against it is made anyway since there are no numerical examples of $GI/E_X/1$ available in literature.

All numerical examples in Chapter 5 are part of the manuscript that has been accepted for publication in the American Journal of Operations Research (Chaudhry and Kim, 2015).

6. CONCLUSION

6.1 Thesis contribution

6.1.1 Contribution to renewal theory

- Extension of the d.g.f. of the number of renewals from the discrete-time single-renewal processes to the discrete-time bulk-renewal processes.
- New derivation for the asymptotic results in the discrete-time bulk-renewal processes, including the extra constant term in the asymptotic secon d moment.
- New derivation for the connection between asymptotic results of the disc rete and continuous-time bulk-renewal processes.
- New numerical examples of the discrete-time bulk-renewal processes.

6.1.2 Contribution to queueing theory

- Extension of the steady-state p.g.f. of the number of uncompleted service stages at a pre-arrival epoch from $GI/E_k/1$ to $GI/E_X/1$.
- New derivation for the solution to $GI/E_X/1$ at different epochs and relations among them.
- New derivation for the c.d.f. of the waiting-time-in-queue and the steady-state p.m.f. of the number of customers in $GI/E_X/1$.
- New numerical examples of $GI/E_X/1$.

6.2 Summary

In this thesis, new and extended results in renewal and queueing theories are presented. In Chapter 2, the discrete-time single-renewal processes are discussed where the concept of renewal periods, renewal mass function, and the number of renewal functions are introduced. These concepts are reintroduced in the discrete-time bulk-renewal processes, and confirmation is made for the fact that renewal periods are the fundamental building blocks of both processes. The asymptotic results in the discrete-time bulk-renewal processes are found and their connections to the asymptotic results in the continuous-time bulk-renewal processes are also derived.

In Chapter 3, numerical examples in discrete-time renewal theory are presented. Different d.f.'s of renewal periods and bulk-sizes are considered in computing the p.m.f. of the number of renewals. The asymptotic first and second moments of the number of renewals are also presented.

In Chapter 4, queueing theory and different types of queues are introduced. Out of different classes of queues, multi-staged queues with server that has fixed number of service stages $(GI/E_k/1)$ is discussed in detail. Although this model has been solved by several researchers, its discussion is deemed essential prior to introducing the class of multi-staged queues with server that has random number of service stages $(GI/E_X/1)$. A complete analysis of $GI/E_X/1$ is provided using a wide range of techniques including the imbedded Markov chain technique, g.f. method, roots method, standard level crossing analysis, renewal theory, and distributional Little's Law. Thus, Chapter 4 of this thesis addresses the statement by Yao et al. (1984): "**There is no simple way to analyze the queue $\boldsymbol{GI/E_X/1}$.**"

In Chapter 5, the numerical examples of $GI/E_X/1$ are presented. Different d.f.'s of inter-arrival times and the number of service stages within the server are considered in computing the solution, p.m.f. of the number in queue and the system, and various performance measures.

The new and extended results in renewal and queueing theories discussed in the four chapters of this thesis were drafted into two different manuscripts which were accepted for publication. The two papers are pending at the accepted status as the author of this thesis intends on including some of the future extensions discussed in the next section.

6.3 Future extensions

The main contribution of this thesis in renewal and queueing theories can be further extended in several ways. Some possible areas for further research are noted as follows:

By considering the renewal periods of the discrete-time bulk-renewal processes and the inter-arrival times of $GI/E_X/1$ to follow heavy-tailed distributions such as Pareto, inverse-Gaussian, and Weibull, numerical results become more challenging to determine. Such distributions possess attractive properties that are deemed suitable in areas such as insurance, broad-band communications networks, and packet routing optimization.

The discrete-time bulk-renewal processes discussed in this thesis can be generalized by assuming that the event of a first renewal has a different d.f. than the rest of the renewals. This is defined by Chaudhry and Templeton (1983) as the modified (also known as delayed or general) renewal processes. Such modification results in a more general renewal equation, thus it leads to asymptotic results of the generalized discrete-time bulk-renewal processes.

In $GI/E_X/1$, instead of using distributional Little's law, there may be another way of finding the steady-state p.m.f. of the number of customers in queue in terms of the p.m.f. of the number of uncompleted service stages at a pre-arrival epoch. Such an alternate way has already been found in $GI/E_k/1$, thus it could possibly be extended to $GI/E_X/1$.

The model $GI/E_X/1$ can be extended to $GI^X/E_Y/1$ such that customers may arrive in groups, hence a batch arrival results in 'XY' additional uncompleted service stages in the system. In addition, $GI/E_X/1$ can also be extended to $GI/E_X/1/N$ such that it has a finite capacity. Doing so results in an interesting outcome where a customer can be rejected due to system's finite capacity (known as blocking probabilities). Both $GI^X/E_Y/1$ and $GI/E_X/1/N$ remain unsolved in literature.

The model $GI/E_X/1$ has a discrete-time counterpart, $GI/NB_X/1$ where NB is a modified negative binomial distribution. The solution and analysis of $GI/NB_X/1$ are unavailable in literature.

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APPENDIX A

This appendix provides all preliminary knowledge that is required to understand this thesis. This appendix begins with a brief summary of probability theory, stochastic processes, and Markov processes in a progressive manner. The summary is then supplemented by the following:

Under continuous probability theory, the definition of a probability density function (p.d.f.) and several different examples of p.d.f.'s are presented. The Laplace transform (L.T), Laplace-Stieltjes Transform (L-S.T.), and Padé approximation are also discussed.

Under discrete probability theory, the definition of a probability mass function (p.m.f.) and several different examples of p.m.f.'s are presented. The generating function (g.f.), probability generating function (p.g.f.), and Taylor's series expansion are also discussed.

A.1 Brief summary on probability theory, stochastic processes, and Markov processes

Probability theory can be explained with the example of a coin toss. When a coin is tossed, it could lead to two possible outcomes (heads or tails), and each outcome has a probability of 0.5. A random variable (r.v.) represents a group of outcomes (in this case, heads or tails) and the distribution function (d.f.) allocates probability to each outcome (in this case, 0.5 chance of getting heads and the same for getting tails).

In general, the outcomes of a r.v. can be nonnegative real numbers or nonnegative integers. In the case of the former, the d.f. of a r.v. becomes a probability density function (p.d.f.) and in the case of the latter, the d.f. of a r.v. becomes a probability mass function (p.m.f.). The cumulative distribution function (c.d.f.) is a sum of either p.d.f.'s or p.m.f.'s from the smallest valued outcome up to a particular outcome of interest. In a holistic sense, random variables can be added, subtracted, multiplied, divided, or collected to describe a system.

A random variable could also be time sensitive such that its probability of an outcome changes over time. Building on the previous example of a coin toss, the probability of getting tails on the first coin toss (0.5) would be different from the probability of getting five tails in a row $(0.5^5 = 0.03125)$. As explained, it is evident that the probability of an outcome in the future depends on the probabilities of all previous outcomes. In view of this, a collection of time dependent random variables form the stochastic processes, which Parzen (1962) describes as the "dynamic part of probability theory." The concept of stochastic processes is familiar and extensively applied across various fields including statistical physics (Brownian motion, fluctuations and thermal noise), communication and control (automatic tracking of moving objects, reproduction of sound and images), and inventory control (minimizing time-of-delivery lag and deciding when to place an order for replenishment of stock).

There exists a special class of stochastic processes called the Markovian stochastic processes (Markov processes). Markov processes inherit the basic property of the stochastic processes but has an additional consideration known as the Markov property: The probability of an outcome in the future only depends on the probability of the present outcome and not that of the past. As an example, Markov property states that given a car engine that has a mileage of 120,000 kilometers, the probability of this engine lasting for another 50,000 kilometers is the same as the probability of the same engine lasting for 50,000 kilometers from the time it was first built. When comparing the two probabilities, the previous mileage on the engine (past) is simply forgotten when considering additional mileage from the present to the future. The two well known d.f.'s that follow Markov property are exponential and geometric distributions (see Appendix A.2.1 and A.3.1, respectively). Markov property is also referred to as "the forgetfulness property" due to its tendency to ignore the past. Interestingly, Markov processes are a powerful tool when deducing predictions from a limited amount of information. It enables a great degree of simplification of problems as readers will observe in the discussion on queueing theory. A Markov processes are further divided into four sub-categories:

Table 16: Classification of Markov processes

	Discrete-state	Continuous-state
Discrete-time	Discrete parameter Markov chains	Discrete parameter Markov processes
Continuous-time	Continuous parameter Markov chains	Continuous parameter Markov processes

In addition, Markov chains are Markov processes whose state space is discrete.

A.2 Continuous probability theory

Assume that there is a continuous r.v., say T, such that it has a p.d.f. $f(t) = P(T = t)$ where $0 \leq t < \infty$. The n-th moment of T is defined as $E[T^n] = \int_0^\infty t^n f(t)dt$. Some examples of p.d.f.'s in continuous probability theory are provided below.

A.2.1 Exponential distribution

When T is an exponential r.v., its p.d.f. becomes

$$f(t) = \lambda e^{-\lambda t}, t \geq 0$$

where $\lambda > 0$. The exponential distribution is a fundamental distribution of continuous probability theory that is characterized by the forgetfulness property.

A.2.3 Erlang-k distribution

When T is an Erlangian r.v., its p.d.f. becomes

$$f(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}, t \geq 0$$

where $\lambda, k > 0$. The shape parameter k takes positive integers and if k is not a positive integer, then $f(t)$ becomes a Gamma distribution. An Erlangian r.v. with shape parameter k is the sum of k exponential r.v.'s, hence when $k = 1$, the Erlang-k distribution simplifies to the exponential distribution.

A.2.4 Modified Erlang distribution

When T is a modified Erlangian r.v., its p.d.f. becomes

$$f(t) = \sum_{h=0}^{\infty} b_h \frac{\lambda^h t^{h-1} e^{-\lambda t}}{(h-1)!}, t \geq 0$$

where $\lambda > 0$ and b_h are the probabilities indicating the likelihood of each value of $h, (h \geq 0)$. In addition, when $b_k = 1$, modified Erlang distribution simplifies to the Erlang-k distribution.

A.2.5 Laplace transform and Laplace-Stieltjes transform

As indicated in Chaudhry and Templeton (1983), applying Laplace transform (L.T.) in continuous probability theory transforms a p.d.f. into a L.T. In defining L.T., assume that there is a continuous r.v. T with p.d.f. $f(t), (t \geq 0)$. Its L.T. is defined as

$$\bar{f}(\omega) = E[e^{-\omega T}] = \int_0^\infty e^{-\omega t} f(t) dt$$

where $\bar{f}(0) = 1$ and $\bar{f}(\omega)$ is an analytic function in the half-place $\mathrm{Re}(\omega) > \omega_0, (\omega_0 \leq 0)$ since $0 \leq \bar{f}(\omega) \leq 1$ for $\omega \geq 0$. L.T. is a useful tool in both renewal and queueing theories due to its ability to express useful information in a fairly simple form.

When $\bar{f}(\omega)$ is inverted to $f(t)$, the procedure is known as the inverse L.T. and defined as

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\omega t}\bar{f}(\omega)d\omega$$

where the contour is any vertical line $\omega = a$ so that $\bar{f}(\omega)$ has no singularities on, or to the right of it (Abate and Whitt, 1995).

The Laplace-Stieltjes transform (L-S.T) is considered to be more general than the L.T. as it encompasses a wider class of r.v.'s than the simple L.T. The definition of the L-S.T. is as follows: Let T be a non-negative r.v. with a d.f. $F(t) = f(t \leq T)$, then the L-S.T. of $F(t)$ is defined as

$$\bar{f}_T(\omega) = \int_0^{\infty} e^{-\omega t} dF(t)$$

with $\mathrm{Re}(\omega) \geq 0$. The integral on the right-hand side of the definition of L-S.T. is known as the Stieltjes integral. In addition, the L-S.T. of $F(t)$ becomes the L.T. of $f(t)$ if $f(t) = {dF(t)}/{dt}$ exists.

A.2.6 Padé approximation

A p.d.f. of a continuous r.v. may not have an explicit L.T. due to the nature of the r.v.. When this is the case, the L.T. of a p.d.f. can be approximated using the Padé approximation. Assume that a continuous r.v. T has a p.d.f. $f(t)$ that does not have an explicit L.T.. The Padé approximation of the L.T. of $f(t)$ is

$$\bar{f}(\omega) \cong \bar{f}^*(\omega) = \frac{N(\omega)}{D(\omega)} = \frac{\sum_{l=0}^{A} n_l \omega^l}{\sum_{l=0}^{B} d_l\, \omega^l}$$

where $N(\omega)$ and $D(\omega)$ are polynomials of degrees A and B, respectively with unknown constant coefficients n_l and d_l such that the first $A + B$ moments of $\bar{f}(\omega)$ are equal to some rational function, say $\bar{f}^*(\omega)$. For additional details, readers may refer to Harris and Marchal (1998) or Baker Jr and Graves-Morris (1996).

A.3 Discrete probability theory

Assume that there is a discrete r.v., say M, such that it has a p.m.f. $f_m = P(M = m)$ where $0 \leq m < \infty$. The n-th moment of M is defined as $E[M^n] = \sum_{m=0}^{\infty} m^n f_m$. Some examples of p.m.f.'s in discrete probability theory are provided below.

A.3.1 Geometric distribution

When M is a geometric r.v., its p.m.f. becomes

$$f_m = p(1-p)^{m-1}, m \geq 1$$

where $p > 0$. The geometric distribution is a fundamental distribution of discrete probability theory that is characterized by the forgetfulness property. This distribution provides the probability of the number of trials (m) until an event occurs with a probability $p(1-p)^{m-1}$ for $m \geq 1$.

A.3.2 Binomial distribution

When M is a binomial r.v., its p.m.f. becomes

$$f_m = \binom{n}{m} p^m (1-p)^{n-m}, m \geq 0$$

where $p > 0$ and n is a positive integer. The binomial distribution is characterized by providing the probability of m events that occur in n trials where each event follows a geometric distribution. As an example, when $n = 1$ and $m = 1$, it indicates that an event occurs after the first trial with probability p. When $n = 1$ and $m = 0$, it indicates that no event occurs after the first trial with probability $1 - p$.

A.3.3 Negative binomial distribution

When M is a negative binomial r.v., its p.m.f. becomes

$$f_m = \binom{m+n-2}{m-1} p^n (1-p)^{m-1}, m \geq 1$$

where $p > 0$ and n is a positive integer. Negative binomial distribution is characterized by providing the probability of the number trials (m) until the n-th event occurs. When $n = 1$, the negative binomial distribution becomes a geometric distribution.

A.3.4 Poisson distribution

When M is a Poisson r.v., its p.m.f. becomes

$$f_m = \frac{e^{-\lambda}\lambda^{m-1}}{(m-1)!}, m \geq 1$$

where $\lambda > 0$. The Poisson distribution is characterized by providing the probability of m events that occur over some duration of time where events occur on average λ per time unit. Each of these events follows the exponential distribution in A.2.1.

A.3.5 Generating function and probability generating function

Let $\{u_n\}$ be a sequence of real numbers. If $U(z) = \sum_{n=0}^{\infty} u_n z^n$ converges in some interval $|z| < z_0, (0 \leq z_0 \leq \infty)$, then $U(z)$ is called the generating function (g.f.) of the sequence $\{u_n\}$ (see Hunter (1983) for details). Here, z_0 is a unique number called the radius of convergence such that

- $\sum_{n=0}^{\infty} u_n z^n$ converges (absolutely) for $|z| < z_0$
- $\sum_{n=0}^{\infty} u_n z^n$ diverges for $|z| > z_0$
- $\sum_{n=0}^{\infty} u_n z^n$ converges uniformly for $|z| \leq \theta$, where $\theta < z_0$

Similarly, let $\{u_{n,m}\}$ be a double sequence of real numbers, then $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_{n,m} z^n x^m$ is known as the double generating function (d.g.f.) if $|z| < z_0, (0 \leq z_0 \leq \infty)$ and $|x| <$

$x_0, (0 \le x_0 \le \infty)$. Since a g.f. transforms a sequence into a power series (procedure also known as the z-transform) an inverse g.f. returns a power series back into a sequence.

In introducing g.f. in discrete probability theory, let there be a discrete r.v. V. $U(z)$ becomes a probability generating function (p.g.f.) of V if and only if u_n matches the following characteristics of the p.m.f. of V:

- $u_n = P(V = n), (n \ge 0)$
- $0 \le u_n \le 1, (n \ge 0)$
- $\sum_{n=0}^{\infty} u_n = 1$

When above three conditions are met, $U(z)$ becomes the p.g.f. of V, such that

$$U(z) = E[z^V], (|z| < 1)$$

In this regard, a p.g.f. is always a g.f. but a g.f. is not always a p.g.f. In addition, a p.g.f is a power series that has advantages over its p.m.f. counterpart when obtaining moments of a r.v.. For instance, the moments of a discrete r.v. are easy to derive from a p.g.f. as illustrated by the following property:

$$U^{(r)}(1) = \lim_{z \to 1^-} \frac{d^r U(z)}{dz^r} = \frac{d^r}{dz^r} E[z^V]|_{z=1}, (r \ge 1)$$

where $U^{(r)}(1)$ is the r-th derivative of the p.g.f. of V evaluated at $z = 1$. This can be used to find various parameters such as the mean, variance and moments of V.

A.3.6 Taylor's series expansion

In simple cases, inversion of a g.f. can be done analytically, however, in complex cases, they need to be inverted numerically (readers may refer to Kim et al. (2011) for additional details on inversion of g.f.'s). Out of several ways to invert a g.f., the method of inversion used throughout this thesis is Taylor's series expansion at $z = 0$ (also known as the Maclaurin series) which can be easily done with today's mathematical software such as MAPLE or MATLAB. In the context of this thesis, a p.g.f. of V can be expressed in a form of a Maclaurin series such that

$$U(z) = E[z^V] = \frac{U^{(n)}(0)}{n!} z^n, (n \geq 0)$$

where the probabilities $\{u_n\}$ can be extracted directly from the coefficients of each term in the Maclaurin series such that

$$u_n = \frac{U^{(n)}(0)}{n!}, (n \geq 0)$$

APPENDIX B

In this appendix, all supplementary proofs, derivation, and theorem that are used in discussing renewal theory are provided.

B.1 Supplementary proofs

B.1.1 Proof of the relation between renewal function and renewal mass function

In proving the relation between M_k and m_i, let $Z_k, k \geq 1$ be the r.v.'s which take the values 1 if a renewal occurs at time k and 0 otherwise. The number of renewals counted over the time period $(0,k]$ is $N_k = \sum_{i=1}^{k} Z_i$. The renewal function is then written as

$$M_k = E[N_k] = E\left[\sum_{i=1}^{k} Z_i\right] = \sum_{i=1}^{k} E[Z_i]$$

$$= \sum_{i=1}^{k} [0 \cdot (1 - m_i) + 1 \cdot m_i]$$

Therefore it becomes

$$M_k = \sum_{i=1}^{k} m_i$$

where $k \geq 1$.

B.1.2 Proof of the relation between the probability of bulk renewals and p.g.f. of renewal periods

As a first step of the proof, the g.f. of N_k with respect to k is found in terms of the p.g.f. of renewal periods. This is done as follows:

$$\sum_{k=1}^{\infty} P_n(k)v^k = \sum_{k=1}^{\infty} [P(W_n \le k) - P(W_{n+1} \le k)]v^k$$

$$= \sum_{k=1}^{\infty} P(W_n \le k)v^k - \sum_{k=1}^{\infty} P(W_{n+1} \le k)v^k$$

$$= [P(W_n \le 0)v^0 + P(W_n \le 1)v^1 + \dots] - [P(W_{n+1} \le 0)v^0 + P(W_{n+1} \le 1)v^1 + \dots]$$

$$= \{P(W_n = 0) + [P(W_n = 0) + P(W_n = 1)]v$$
$$+ [P(W_n = 0) + P(W_n = 1) + P(W_n = 2)]v^2 + \dots\}$$
$$- \{P(W_{n+1} = 0) - [P(W_{n+1} = 0) + P(W_{n+1} = 1)]v$$
$$- [P(W_{n+1} = 0) + P(W_{n+1} = 1) + P(W_{n+1} = 2)]v^2 + \dots\}$$

$$= [P(W_n = 0) + P(W_n = 0)v + \dots] + [P(W_n = 1)v + P(W_n = 1)v^2 + \dots]$$
$$+ [P(W_n = 2)v^2 + P(W_n = 2)v^3 + \dots]$$
$$- [P(W_{n+1} = 0) + P(W_{n+1} = 0)v + \dots]$$
$$- [P(W_{n+1} = 1)v + P(W_{n+1} = 1)v^2 + \dots]$$
$$- [P(W_{n+1} = 2)v^2 + P(W_{n+1} = 2)v^3 + \dots]$$

$$= P(W_n = 0)(1 + v + v^2 + \dots) + P(W_n = 1)(v + v^2 + v^3 + \dots)$$
$$+ P(W_n = 2)(v^2 + v^3 + v^4 + \dots) - P(W_{n+1} = 0)(1 + v + v^2 + \dots)$$
$$- P(W_{n+1} = 1)(v + v^2 + v^3 + \dots) - P(W_{n+1} = 2)(v^2 + v^3 + v^4 + \dots)$$

$$= \left[\frac{P(W_n = 0)}{1 - v} + \frac{vP(W_n = 1)}{1 - v} + \frac{v^2P(W_n = 2)}{1 - v} + \dots\right]$$
$$- \left[\frac{P(W_{n+1} = 0)}{1 - v} + \frac{vP(W_{n+1} = 1)}{1 - v} + \frac{v^2P(W_{n+1} = 2)}{1 - v} + \dots\right]$$

$$= \frac{1}{1 - v}\{[P(W_n = 0) + vP(W_n = 1) + v^2P(W_n = 2) + \dots]$$
$$- [P(W_{n+1} = 0) + vP(W_{n+1} = 1) + v^2P(W_{n+1} = 2) + \dots]\}$$

$$= \frac{1}{1-v}\left[\sum_{k=0}^{\infty} P(W_n = k)v^k + \sum_{k=0}^{\infty} P(W_{n+1} = k)v^k\right]$$

since $P(W_n = 0) = 0\ \forall n \geq 0$, and using $\sum_{k=0}^{\infty} P(W_n = k)v^k = f^n(v)$, the above simplifies to

$$\sum_{k=1}^{\infty} P_n(k)v^k = \frac{f^n(v)}{1-v}[1 - f(v)], (|v| < 1)$$

This proof is also provided in Feller (1968).

B.2 Supplementary derivation

B.2.1 Derivation for d.g.f. in the discrete-time single-renewal processes

Using the relation that is proved in Appendix B.1.2, the p.g.f. of $\sum_{k=1}^{\infty} P_n(k)v^k$ with respect to n is

$$P(z,v) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} P_n(k)v^k z^n = \frac{1}{1-v}\sum_{n=0}^{\infty}[f^n(v) - f^{n+1}(v)]z^n, (|z| < 1)$$

$$= \frac{1}{1-v}\left[\sum_{n=0}^{\infty} f^n(v)z^n - \sum_{n=0}^{\infty} f^{n+1}(v)z^n\right]$$

$$= \frac{1}{1-v}\left\{\sum_{n=0}^{\infty}[zf(v)]^n - f(v)\sum_{n=0}^{\infty}[zf(v)]^n\right\}$$

$$= \frac{1-f(v)}{1-v}\sum_{n=0}^{\infty}\{zf(v)\}^n$$

Since$|z| < 1$, the d.g.f. is derived as

$$P(z,v) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} P_n(k)v^k z^n = \frac{1-f(v)}{(1-v)[1-zf(v)]}, (|z| < 1, |v| < 1)$$

B.3 Supplementary theorem

B.3.1 Asymptotic theory

In simple terms, asymptotic theory is the study of the behaviour of a time-dependent function as $t \to \infty$. In renewal theory, asymptotic theory can be applied to the renewal mass function and the moments of the number of renewals. When their time parameter assumes a very large value, the functions converge to what is known as the asymptotic results. In queueing theory, asymptotic theory can be applied to time-dependent r.v.'s such that they become steady state r.v.'s as their common time parameter assumes a very large value.

APPENDIX C

In this appendix, the basic concepts and all supplementary proof, derivation, and theorems that are used in discussing queueing theory are provided.

C.1 Basics of queueing theory and Kendall's notation

Queueing theory analyzes the properties that surround a queueing model. In this thesis, a 'queueing system' and a 'queueing model' are synonymous terms that refer to the mathematical construct that composes of the server, the customer in the server (at any stage of service), and the customers in queue (if any). Queueing models can be described as mathematical models that describe the process of customers arriving for service, waiting for service (if service is not immediately available), and receiving of service, followed by leaving once service is complete. In the context of this thesis, a 'system' refers to the space that includes the queue of customers, and the server with a customer under service. A queue refers to only the space which includes the queue of customers. As well, a customer is a generic term that refers to any element (person, product, packet, etc) that participates in a queueing system. 'Queueing model' is a broader term that is described using Kendall's notation. In Chaudhry and Templeton (1983), Kendall's notation is defined as

$$A_n^{X_n}(t)/B_n^{a,b}/c/M$$

where

$A_n(t)$: Inter-arrival time distribution with arrival rate depending

on t, n (if t, n in $A_n(t)$ are missing, it means arrival rate is constant).

X_n: Arrival group size distribution with group size probability depending on n (if n in X_n is missing, it means the group size probability is independent of n).

B_n: Service time distribution with service rate depending on n (if n in B_n is missing, it means service rate is constant).

a: Quorum for service group.

b: Capacity for service group.

c: Number of servers.

M: Storage capacity (if last descriptor is missing, it is assumed to be infinite).

In addition, although not indicated in the Kendall's notation, the arrival rate (λ) of a queueing system indicates the rate at which customers arrive, whereas the service rate (μ) of the system indicates the rate at which customers are departing the system. The traffic intensity (ρ) is a parameter that is uniquely defined for each queueing model. In $GI/E_k/1$, the traffic intensity is defined as $\rho = {}^{\lambda k}/_{\mu}$. Given $0 < \rho < 1$, low traffic intensity is indicated when ρ is closer to 0, whereas high traffic intensity is indicated when ρ is closer to 1. In single-server queueing models, the magnitude of ρ indicates the degree of server utilization. The characteristic equation of a queuing model is a unique equation that is specific to that model.

C.2 Supplementary proof

C.2.1 Proof that the characteristic equation of $GI/E_X/1$ has r roots inside the unit circle

The proof that the characteristic equation of $GI/E_X/1$

$$0 = 1 - S(z^{-1})K(z)$$

has *r* roots inside the unit circle $|z| = 1$ is as follows: Rearrange and multiply z^r on both of its sides such that

$$z^r = z^r\left(\sum_{h=1}^{r} s_h z^{-h}\right)K(z)$$

or

$$0 = z^r - \left(\sum_{h=1}^{r} s_h z^{r-h}\right)K(z)$$

Now let $f(z) = z^r$ and $g(z) = \sum_{h=1}^{r} s_h z^{r-h}\,K(z)$. Consider the absolute values of $f(z)$ and $g(z)$ on the circle $|z| = 1 - \Delta$, where Δ is a positive and sufficiently small number. This gives

$$|f(z)| = |z^r| = (1-\Delta)^r = 1 - \Delta r + o(\Delta)$$

and

$$|g(z)| = \left|\sum_{h=1}^{r} s_h z^{r-h}\,K(z)\right| \leq \sum_{h=1}^{r} s_h |z|^{r-h}\,K(|z|)$$

which leads to

$$= 1 - \Delta(r - \bar{s}) - \frac{\mu}{\lambda}\Delta + o(\Delta)$$

or

$$= 1 - \Delta r - \frac{\mu}{\lambda}(1-\rho)\Delta + o(\Delta)$$

where $\rho = \frac{\lambda\bar{s}}{\mu}$. Since $\rho < 1$ and Δ is sufficiently small, we have $|f(z)| > |g(z)|$ on $|z| = 1 - \Delta$. It is evident that $f(z)$ and $g(z)$ satisfy Rouché's theorem (see Appendix C.4.1) thus the equation $0 = 1 - S(z^{-1})K(z)$ has r roots on the inside of unit circle $|z| = 1$.

C.3 Supplementary derivation

C.3.1 Derivation for alternate relation between pre-arrival and random solutions to $GI/E_X/1$

Instead of using the standard level crossing analysis, there exists another way to determine p_k in terms of p_j^- using the g.f. method. This alternate way is explained in the paragraphs that follow.

In $GI/E_X/1$, let the inter-arrival times be $T_n = \sigma_n - \sigma_{n-1}, (n \geq 1)$ with mean $E[T]$, where σ_n for $n \geq 1$ are the time epochs just before each customer arrival. Let the p.d.f. of T_n be $a(t), (t > 0)$, where $a(t) \equiv dA(t)/dt$. Let there be a random time epoch, say R, between σ_n and σ_{n+1}, which is illustrated in Figure 8 below.

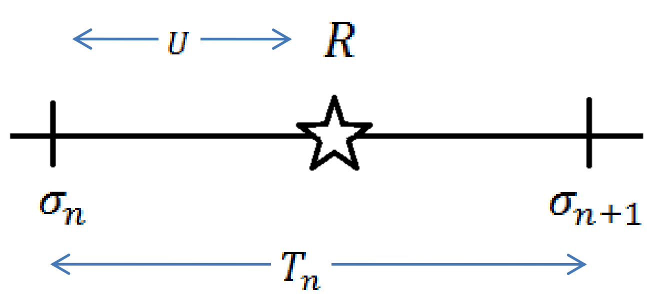

Figure 8: Visual illustration of the n-th pre-arrival epoch to the $(n+1)$-th pre-arrival epoch and the n-th pre-arrival epoch to the $(n+1)$-th random epochs.

In renewal theory, the length-biased sampling phenomenon (see Chaudhry and Templeton, 1983) indicates that the p.d.f. of U, say $a_R(t)$, can be found in terms of $a(t)$ such that

$$a_R(t) = \frac{t}{E[T]} a(t), \ \ (t > 0)$$

and

$$A_R(t) = \frac{1}{E[T]} \int_0^t [1 - A(w)] dw, \ \ (w, t > 0)$$

where $A_R(t)$ and $A(w)$ are the c.d.f.'s of U and T_n, respectively. Let D^* be the number of completed service stages during the time interval U and define $k_j^* = P(D^* = j)$ so that the p.g.f. of $\{k_j^*\}$ becomes

$$K^*(z) = \sum_{j=0}^{\infty} k_j^* z^j = \int_0^{\infty} e^{-\mu(1-z)u} dA_R(u)$$

$$= \bar{a}_R\big(\mu(1-z)\big), \ \ (|z| \leq 1)$$

Using the definition $a_R(t) = \frac{1-A(t)}{E[T]}$ (see modified renewal process in Chaudhry and Templeton, 1983), the above leads to the following relation

$$K^*(z) = \rho \left[\frac{1 - K(z)}{1 - z}\right]$$

where $K(z)$ is from Subsection 4.2.1. The steady-state r.v. N from Subsection 4.3.3 represents the number of uncompleted service stages in the system at a random epoch. Similar to what is done in (15), the relation

$$N = (N^- + X - D^*)^+$$

can be established where $(a)^+ = \max(a, 0)$. Since X and D^* are independent from N^-, the g.f. of above relation becomes

$$P(z) = E[z^N] = E\left[z^{(N^- + X - D^*)^+}\right]$$

$$= E[z^X]E\left[z^{N^- - D^*} \mid N^- + X - D^* > 0\right]P(N^- + X - D^* > 0)$$

$$+ P(N^- + X - D^* \leq 0)$$

Now considering the p.g.f. of $N^- - D^*$, it leads to

$$E\left[z^{N^- - D^*}\right] = E\left[z^{N^- - D^*} \mid N^- + X - D^* > 0\right]P(N^- + X - D^* > 0)$$

$$+E\left[z^{N^- - D^*} \mid N^- + X - D^* \leq 0\right]P(N^- + X - D^* \leq 0)$$

which can be substituted into the previous expression to get

$$P(z) = P^-(z)E[z^X]E\left[z^{-D^*}\right]$$

$$-\sum_{i=0}^{\infty} E[z^{-m} \mid N^- + X - D^* = -i]\, P(N^- + X - D^* = -i)$$

$$+\sum_{i=0}^{\infty} P(N^- + X - D^* = -i)$$

By letting $q_i^* = P(N^- + X - D^* = -i)$ and using the relation between $K^*(z)$ and $K(z)$, it becomes

$$P(z) = P^-(z)S(z)\rho\left[\frac{1 - K(z^{-1})}{1 - z^{-1}}\right] + \sum_{i=0}^{\infty} q_i^*\left(1 - z^{-i}\right)$$

From (16), $K(z^{-1})$ can be isolated and then substituted into the above expression, which leads to

$$P(z) = \rho\frac{P^{-}(z)S(z)}{(1-z^{-1})} - \rho\frac{P^{-}(z)S(z)}{(1-z^{-1})}\left\{\frac{1}{S(z)}\left(1-\frac{\sum_{m=0}^{\infty} q_m(1-z^{-m})}{P^{-}(z)}\right)\right\} + \sum_{i=0}^{\infty} q_i^*(1-z^{-i})$$

$$= \rho\frac{P^{-}(z)(S(z)-1)}{(1-z^{-1})} + \left\{\rho\frac{\sum_{m=0}^{\infty} q_m(1-z^{-m})}{(1-z^{-1})} + \sum_{i=0}^{\infty} q_i^*(1-z^{-i})\right\}$$

Since $P(z)$ is a steady-state p.g.f., it must be a power series with nonnegative powers (see A.3.5). However, the right-hand side of the above expression must not have any terms with negative power. Consequently, the terms inside the bracket $\{\ldots\}$ must cancel out, leaving at the most a nonzero constant, say C. Thus the above expression simplifies to

$$P(z) = C + \rho\frac{P^{-}(z)(S(z)-1)}{(1-z^{-1})}$$

Since $P(1^{-}) = 1$, it leads to final result

$$P(z) = 1 + \rho\left[\frac{P^{-}(z)(S(z)-1)}{1-z^{-1}} - 1\right]$$

which is the alternate way to determine p_k in terms of p_j^{-} using the g.f. method. As a remark, this expression matches with that of a simpler model $GI/E_r/1$ (see Chaudhry and Templeton, 1983) by letting $s_r = 1$, which implies $S(z) = z^r$.

C.4 Supplementary theorems

C.4.1 Rouché's theorem

If $f(z)$ and $g(z)$ are functions of z, which are analytic inside and on a closed countour C, and if $|f(z)| < |g(z)|$ on C, then $g(z)$ and $g(z) + f(z)$ have the same number of roots inside C (Titchmarsh, 1939).

C.4.2 Liouville's theorem

If f is entire and $f(z)$ is bounded for all values of z in the complex plane, then f is a constant (Churchill, 1960).

CURRICULUM VITAE

JAMES JAEHAK KIM

Born 16 March 1989 in Seoul, South Korea

Contact Information:
34 Tylee Pvt., Ottawa, ON, K1V 7M3
s25412@rmc.ca

POST SECONDARY EDUCATION

May 2012 – present **Royal Military College of Canada**

- Completing Master of Science in Mathematics as a part-time student
- Thesis entitled "New and Extended Results in Renewal and Queueing Theories"
- Academic average of 95 percent

September 2008 – May 2012 **Royal Military College of Canada**

- Received an Honours Bachelor of Science in Mathematics with first class distinction
- Finished top of the program with a final-year academic average of 90 percent
- Departmental medal

ACADEMIC RESEARCH EXPERIENCE

June 2013 – Present **Attended international conferences and gave presentations on various findings in renewal and queueing theories**

- The Institute for Operations Research and the Management Sciences (INFORMS) Applied Probability Society (APS) 17th Conference held at San Jose, Costa Rica
- 56th Canadian Operations Research Society (CORS) Annual Conference held at Ottawa, Ontario
- The First European Conference on Queueing Theory (ECQT 2014) held at Ghent, Belgium

November 2013 – Present **Co-authored several accepted and published manuscripts in literature**

- Manuscript titled "Asymptotic Results for the First and Second Moments of Discrete-Time Bulk-Renewal Process" submitted to the Journal of Mathematics and System Science on 28 Nov 13, accepted on 20 Jan 14
- Manuscript titled "Complete Solution to an Extended Version of $GI/E_k/1$: $GI/E_X/1$" submitted to the American Journal of Operations Research on 5 Oct 15, accepted on 12 Oct 15
- Manuscript titled "Analytically Elegant and Computationally Efficient Results In Terms of Roots for the $GI^X/M/c$ Queueing System" submitted to Queueing Systems on 27 Oct 14, accepted on 25 Dec 15, published on 23 Jan 16

- Manuscript titled "Analytically Elegant and Computationally Efficient Solution to $GI^X/Geo/1$ Including Heavy-Tailed Inter-Arrival Times Using Roots" submitted to Operations Research Letters on 2 Nov 15, accepted with minor revision on 8 Feb 16

WORK EXPERIENCE

May 2012 – Present **Member of the Canadian Armed Forces Regular Force as a Communications and Electronics Engineering (Air) Officer**

- Assistant Information Management Officer as the on the job training (OJT) officer at the Canadian Forces Joint Signal Regiment (CFJSR)
- Lead Cryptographic Life Cycle Material Manager at the Canadian Forces Crypto Support Unit (CFCSU)
- Acting Ops O at CFCSU