RENEWAL THEORY:

SIMPLE AND ELEGANT DERIVATIONS

THEORIE DU RENOUVELLEMENT:

DERIVATIONS SIMPLES ET ELEGANTS

A Thesis Submitted

to the Division of Graduate Studies of the Royal Military College of Canada

by

Brent David Fisher, MBA, B.Sc., rmc
Sub-Lieutenant

In Partial Fulfillment of the Requirements for the Degree of

Master of Science

May 2014

ROYAL MILITARY COLLEGE OF CANADA
COLLÈGE MILITAIRE ROYAL DU CANADA

DIVISION OF GRADUATE STUDIES AND RESEARCH
DIVISION DES ÉTUDES SUPÉRIEURES ET DE LA RECHERCHE

This is to certify that the thesis prepared by / Ceci certifie que la thèse rédigée par

BRENT DAVID FISHER

entitled / intitulée

RENEWAL THEORY: SIMPLE AND ELEGANT DERIVATIONS

complies with the Royal Military College of Canada regulations and that it meets the accepted standards of the Graduate School with respect to quality, and, in the case of a doctoral thesis, originality, / satisfait aux règlements du Collège militaire royal du Canada et qu'elle respecte les normes acceptées par la Faculté des études supérieures quant à la qualité et, dans le cas d'une thèse de doctorat, l'originalité,

for the degree of / pour le diplôme de

MASTER OF SCIENCE

Signed by the final examining committee: /
Signé par les membres du comité examinateur de la soutenance de these

___________________________, Chair / Président

___________________________, External Examiner / Examinateur externe

___________________________, Main Supervisor / Directeur de thèse principal

____

Approved by the Head of Department: /
Approuvé par le Directeur du Département: _______________ Date: ________

To the Librarian: This thesis is not to be regarded as classified. /
Au Bibliothécaire : Cette thèse n'est pas considérée comme à publication restreinte.

____
Main Supervisor / Directeur de thèse principal

ACKNOWLEDGEMENTS

I am truly fortunate to have had an opportunity to complete a second graduate degree by this point in my life, and I owe a great deal of thanks to Dr. Mohan Chaudhry for encouraging me to undertake this course of study. I have appreciated his guidance and mentorship throughout the completion of this thesis, and the commitment that he displays to his graduate students is inspirational.

Dr. Alain Gosselin was also instrumental in my completion of this program. Working on a degree as a part-time part-distance student while employed with the Royal Canadian Navy certainly presented new challenges to overcome, but as Head of the Department of Mathematics and Computer Science he certainly punched above his weight to assist me in resolving all administrative issues.

As a past graduate of this Department, I have had my education in mathematics influenced by the majority of the faculty. Although there are too many names to list, there have been several professors – Dr. Bill Hurley, Dr. Jack Brimburg, Dr. Bob Johnson, Dr. Lucien Haddad, and Dr. Yawei Liang – who have each taught me on more than one occasion, and have thus had an even more profound impact on the experience that has contributed to this thesis. I look forward to employing this knowledge in the academic and professional opportunities that await me.

ABSTRACT

Fisher, Brent David, M.Sc. Royal Military College of Canada, May 2014, Renewal Theory: Simple and Elegant Derivations. Supervised by Dr. M.L. Chaudhry.

This thesis comprises two principal areas of research: new derivations of asymptotic results in renewal theory and the computation of the distribution for the number of renewals with bulk arrivals.

A simple and elegant solution to determine the asymptotic results for the renewal density as well as for the first and second moments of the number of renewals for the discrete-time renewal process is presented. Using generating functions, the difficult-to-determine constant term in the second moment is also addressed. A similar process using Laplace transforms (LTs) is likewise employed to determine analogous results in continuous time. Further, the solution is extended to determine the asymptotic results for the first and second moments of the number of bulk renewals as well.

The distribution of the number of renewals for both single and bulk arrivals in continuous time is calculated using an algorithm employed through MAPLE. These numerical results are acquired by considering rational as well as non-rational LTs and Padé-approximated LTs for the distributions of inter-renewal times. The asymptotic results derived in the first part of this thesis then help to validate the accuracy of these numerical results.

Keywords: Renewal theory, Asymptotic results, Numerical results, Bulk arrivals, Generating functions, Laplace transforms

RÉSUMÉ

Fisher, Brent David, M.Sc. Collège militaire royal du Canada, mai 2014, Théorie de renouvellement: dérivations simples et élégantes. Dirigé par Dr. M.L. Chaudhry.

Cette thèse comprend deux domaines de recherche principaux: de nouvelles dérivations de résultats asymptotiques pour la théorie du renouvellement, et le calcul de la distribution pour le nombre de renouvellements pour les arrivées en groupes.

Une solution simple et élégante pour déterminer les résultats asymptotiques pour la densité de renouvellement ainsi que pour les premiers et deuxièmes moments du nombre de renouvellements pour le processus de renouvellement en temps discret est présentée. En utilisant des fonctions génératrices, le terme constant qui est difficile à déterminer pour le second moment est également résolu. Un processus similaire utilisant les transformations de Laplace (LTs) est également utilisé pour déterminer des résultats similaires en temps continu. De plus, la solution est poursuivie afin de déterminer les résultats asymptotiques pour les premiers et deuxièmes moments du nombre de renouvellements en groupes.

La distribution du nombre de renouvellements pour les arrivées simples et en groupes en temps continu est calculée en utilisant un algorithme fait avec MAPLE. Ces résultats numériques sont acquis en considérant les LTs rationnelles et non-rationnelles ainsi que les LTs de Padé pour les distributions des intervalles de renouvellement. Les résultats asymptotiques dérivés dans la première partie de cette thèse aident ensuite à valider le calcul d'incertitude de ces résultats numériques.

Mots-clés: Théorie de renouvellement, résultats asymptotiques, résultats numériques, arrivées en groupes, fonctions génératrices, transformations de Laplace

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF ACRONYMS

i.i.d.r.v.	Independent Identically Distributed Random Variable
g.f.	Generating Function
LHS	Left Hand Side
LST	Laplace-Stieltjes Transform
LT	Laplace Transform
p.d.f.	Probability Density Function
p.m.f	Probability Mass Function
p.g.f.	Probability Generating Function
RHS	Right Hand Side
r.v.	Random Variable

LIST OF EQUATIONS

(1) $$m(z)=\sum_{k=1}^{\infty} m_k z^k = \frac{f(z)}{1-f(z)}$$

(2) $$m(z)=\frac{C_{-1}}{(1-z)}+O(1)$$

(3) $$M_n^{(2)}=\left(\frac{(n+2)!}{2!n!}\right)C_{-3}+\left(n+1\right)C_{-2}+C_{-1}+o(1)$$

(4) $$M_n^{(2)}=\frac{n^2}{\mu^2}+n\left(\frac{2\sigma^2}{\mu^3}+\frac{1}{\mu^2}-\frac{1}{\mu}\right)+\left(\frac{3\mu_2^2}{2\mu^4}+\frac{3\mu_2-2\mu_3}{3\mu^3}+\frac{1-9\mu_2}{6\mu^2}-\frac{3}{2\mu}+1\right)+o(1)$$

(5) $$\tilde{m}(s)=\int_0^{\infty} e^{-st} m(t)\,dt=\frac{\tilde{f}(s)}{\left(1-\tilde{f}(s)\right)}$$

(6) $$\tilde{m}(s)=\frac{C_{-1}}{s}+O(1)$$

(7) $$M(t)=t\cdot C_{-2}+C_{-1}+o(1)$$

(8) $$M^{(2)}(t)=\frac{t^2\cdot C_{-3}}{2}+t\cdot C_{-2}+C_{-1}+o(1)$$

(9) $$M(t)=t\cdot C_{-2}+C_{-1}+o(1)$$

(10) $$M^{(2)}(t)=\frac{t^2\cdot C_{-3}}{2}+t\cdot C_{-2}+C_{-1}+o(1)$$

(A.1) $$\frac{\tilde{p}_n(s)}{s}=\frac{\tilde{N}^n(s)\left[\tilde{D}(s)-\tilde{N}(s)\right]}{s\tilde{D}^{n+1}(s)}.$$

1 INTRODUCTION

1.1 Problem Description

Renewal theory has been of interest for several decades due to a wide variety of applications in business and engineering, and although many well-known theoretical results have been available for quite some time, limitations in computing power have prevented further advances in the field. Certain theoretical results, including the use of constant terms in the various moments of asymptotic results, are incomplete or have complex derivations that have not been succinctly articulated for new students of the field. The traditional lack of computing power has likewise limited the availability of numerical results in renewal theory. Acquiring the distributions for the number of renewals based on common probability distributions for both renewal time and group size could assist in finding new applications of renewal theory in modern society.

1.2 Thesis Objectives

The goals of this thesis are three-fold:

1. To derive simple and elegant asymptotic results in both discrete and continuous time, including the difficult-to-determine constant terms of the second moments;
2. To complete similar derivations for the bulk renewal process in continuous time;
3. To compute numerical results for both single and bulk renewal processes in continuous time for a variety of distributions.

2 LITERATURE REVIEW

The theory of renewal processes and renewal theorems plays a fundamental role in many areas such as failure and replacement of equipment, risk-based asset management models, and queues. Rackwitz (2001) and van Noortwijk (2003) discuss some of these applications throughout their respective works.

In a recent study, van der Weide et al. (2007) give asymptotic results for the first and second moments for the number of renewals. They provide a constant term in the second moment and state that it is not clear from Feller (1949) as to how to obtain the constant term using generating functions (g.f.s). The same arguments apply to the results presented in Feller (1968). Hunter (1983) likewise uses g.f.s to estimate the moments of the asymptotic results in renewal theory. In this regard, the derivations of results presented in both Feller (1949) and Hunter (1983) are such that the complexity of the procedure involved increases while proceeding from the density function to higher order moments. This is expected to a certain degree, but the derivation of the second moment, for example, is such that the constant term is not explicitly stated. Brown (2008) provides explicit formulae for the moments of all orders of the number of renewals in discrete time, but this is done using difficult combinatorial methods.

Cox (1962) and Feller (1968) are among the most prominent of the various authors who discuss the theoretical aspects of renewal theory throughout their works, and their ideas are repeated through other publications such as Heyman and Sobel (1982), Tijms (2003) and Beichelt (2006). With that being said, there remains a deficiency in attempts to apply these results in practice. Historically, the computation of numerical results has been limited by the availability of computing power, but the emergence of powerful modelling software such as MAPLE has facilitated the use of algorithms to compute these results. Abate and Whitt's (2006) paper is a culmination of several earlier works providing reviews on the numerical inversion of both g.f.s and Laplace transforms (LTs).

Chaudhry and Templeton (1983) discuss basic renewal theory in the context of queueing theory. Although the majority of their work concerns bulk queues and queueing systems, all essential elements of renewal theory in continuous time are presented. The notations used throughout this thesis most closely reflect those used throughout their work, although certain elements have been changed in the interest of clarity for readers unfamiliar with the field of study.

Parzen (1962) likewise includes a succinct discussion of the continuous time renewal process, but since his work is primarily focussed on stochastic processes, he includes discussions on other basic topics such as probability distribution types and Markov chains. Kohlas (1982) assumes a similar approach in his work, but he is one of the few notable authors to include descriptions of renewal theory in both discrete and continuous time. Further, the work of Karlin and Taylor (1975) amplifies their basic description of renewal theory with several additional theorems.

In a recent paper by Chaudhry, Yang, and Ong (2013), LTs are employed to calculate the distribution function, mean, and variance of the number of renewals in a continuous-time renewal process. In part an extension of some of the results presented in Chaudhry (1995), their work is limited to distributions that have a rational LT or can be approximated as rational, and it is assumed that all renewals are single arrivals. Despite these limitations, in many ways the authors build upon previous results by Baxter et al. (1982), who use a generalized cubic splining algorithm to evaluate convolution integrals for various probability distribution functions. The work by Baxter et al. (1982) is still significant since it provides a variety of numerical results for distributions such as gamma, truncated normal, and inverse Gaussian, each evaluated at time values of $t \leq 1.25$ with which to compare in the case of single arrivals.

The speaking notes from a presentation to the University of Melbourne by Fackrell (2004) provide information on matrix-exponential distributions, which are another class of distributions of interest in renewal theory. Further, an Honours Mathematics Senior Project by the author of this thesis (Fisher, 2010) describes several distributions of interest, as well as details describing how the Padé approximation can be used to find the distribution of renewals assuming single arrivals.

3 RENEWAL THEORY BASICS

Refer to Appendix A for descriptions of elementary concepts in probability theory, including the generating function, Laplace transform, inversion of Laplace transforms, the Padé method, and asymptotic theory.

Renewal theory is a notion in probability theory concerned with the number of random, or stochastic, *events* that occur prior to a specific time. These events can be indicative of real-life occurrences such as equipment failure or replacement, arrivals or departures in queues, as well as monetary gains or losses, and they are typically called *renewals* within this field of study. For example, consider the inventory of a particular item at a store: The length of time that passes between purchases cannot be known with certainty, so both the number of sales that are made as well as the number of items to be restocked are examples of random processes.

Mathematically, this situation can be modelled by random variables (r.v.s) that are based on probabilistic laws. Stochastic processes model the change of a system and its corresponding values over time, whereas counting processes are a type of stochastic process since they count the total number of stochastic events that have occurred up to a specific time. A renewal process is a type of counting process, and can be modelled in either discrete or continuous time, depending on whether time is modelled as progressing finitely or infinitesimally. The term *recurrent process* is used interchangeably with renewal process throughout the literature (Chaudhry and Templeton, 1983).

3.1 Discrete Renewal Theory

A *renewal process* is a process $\{N_n, n \geq 1\}$ for which the state space belongs to a denumerable set $\{0, 1, 2, \ldots\}$. N_n can count the number of renewals within a time period $(0,n]$, and the intervals between renewals are called the *renewal periods*. Renewals occur at instants of time $s_1', s_2', s_3', \ldots$, and *renewal intervals* $T_i = s_i' - s_{i-1}'$, $i = 1, 2, 3, \ldots$ $s_0' = 0$, are independent identically distributed random variables (i.i.d.r.v.s) distributed as T with common probability mass function (p.m.f.)

$$f_k = P(T = k), \; k \geq 1, \;\; f_0 = 0,$$

g.f.

$$f(z) = \sum_{k=1}^{\infty} f_k z^k, |z| \leq 1,$$

and mean

$$\mu = f'(1) = \sum_{k=1}^{\infty} k f_k < \infty.$$

In discrete time, the following notations are used to describe common statistical measures:

$$a_n = f^{(n)}(1) < \infty, \; n = 1, 2, 3$$

and

$$\sigma^2 = E[T^2] - E^2[T].$$

A renewal process is *periodic* if there is an integer $d > 1$ such that $f_k = 0$ except when $k = d, 2d, 3d, \ldots$ such that renewals can take place in intervals that are multiples of d. As a result, $f_k \neq 0$ at $k = d, 2d, 3d, \ldots,$ and the greatest integer d is the period of the renewal process. A non-periodic renewal process has a period of 1 and is called *aperiodic*.

The *renewal equation* is defined as $m_n = f_n + \sum_{j=1}^{n} m_{n-j} f_j$, $n = 1, 2, 3, \ldots$ where m_n = P(renewal at time n) with $m_1 = f_1$ and $m_0 = 0$ (implying no renewal at time 0). This is an important notion in renewal theory since the left-hand side (LHS) illustrates the probability of a renewal taking place at time n while the right-hand side (RHS) illustrates that this is either a *first renewal* occurring at time n or a renewal occurring at time $j \geq 1$ with probability f_j and a subsequent renewal at time $(n - j)$ with probability m_{n-j}.

The *waiting time* until the k^{th} renewal is given by the partial sum

$$W_k = \sum_{r=1}^{k} T_r, \qquad W_0 = 0.$$

Waiting time is used to derive the distribution of N_n and its moments due to the following expression for $n \geq 0,\ k \geq 0$:

$$N_n \geq k \Leftrightarrow W_k \leq n.$$

Consequently,

$$\begin{aligned} p_{N_n}(k) &= P(N_n = k) = P(N_n \geq k) - P(N_n \geq k+1) \\ &= P(W_k \leq n) - P(W_{k+1} \leq n) \\ &= F_{W_k}(n) - F_{W_{k+1}}(n). \end{aligned}$$

By employing the use of g.f.s on $p_{N_n}(k)$, it follows that

$$\sum_{k=0}^{\infty} P\left(N_n = k\right) z^k = \frac{f^n(z) - f^{n+1}(z)}{1-z} = \frac{f^n(z)\left(1 - f(z)\right)}{1-z},\ n \geq 0,\ |z| < 1,$$

where $f(z)$ is the g.f. of f_k and $f^n(z)$ is g.f. of the n^{th} convolution of $f(z)$. The coefficients of z^n provide the probabilities $p_{N_n}(k)$ that compose the distribution of N_n.

The *mean value* of the discrete-time renewal process N_n is referred to as the *renewal function*, and is defined as $M_n \equiv E[N_n]$. A great portion of renewal theory is concerned with properties of the renewal function, and it is for this reason that its asymptotic results (see Appendix A.4) are of such interest.

3.2 Continuous Renewal Theory

In continuous time, a *renewal process* is a process $\{N(t), t \geq 0\}$ for which the state space belongs to a denumerable set {0, 1, 2, . . .} and for which the inter-renewal times $s_i' - s_{i-1}'$, i = 1,2,3, . . ., $s_0' = 0$, between successive renewal groups are positive i.i.d.r.v.s. $N(t)$ can count the number of renewal groups within a time period $(0,t]$, and the intervals between renewal groups are called the *renewal periods*.

Renewals occur at instants of time s_1', s_2', s_3', . . ., and *renewal intervals* $T_i = s_i' - s_{i-1}'$, i = 1, 2, 3, . . . are i.i.d.r.v.s distributed as T with common probability density function (p.d.f.) $f(t) = \lim_{\Delta t \to 0} P\left(t < T \leq t + \Delta t\right) / \Delta t$, $f(0) = 0$, LT $\tilde{f}(s) = \int_0^\infty f(t) e^{-st} dt, s \geq 0$, and mean $\mu = -\tilde{f}'(0) < \infty$. In continuous time, the following notations are used to describe common statistical measures:

$$\mu_n = E[T^n] < \infty,\ n = 1, 2, 3$$

and

$$\sigma^2 = E[T^2] - E^2[T] < \infty.$$

In addition,

$$\mu_1 = \mu.$$

It is often possible that only the renewal intervals T_2, $T_3, \ldots$ are i.i.d.r.v.s, particularly when the observation of a system commences after the process has been operating previously. The period prior to the first renewal, denoted by T_1, is commonly referred to as a *residual lifetime*. A renewal process with the characteristic that $f_1(t) \neq f(t)$, whereby the period prior to the first renewal is uniquely distributed compared to all other periods, is considered to be *delayed*.

The *renewal equation* is defined as $m(t) = f(t) + \int_0^t m(t-y) f(y) dy$, where $f(t)$ is the common density of T's. Specifically, $m(t)$ = renewal density at time t. For details, see Chaudhry and Templeton (1983).

The *waiting time* until the n^{th} renewal is given by the partial sum

$$W(n) = \sum_{r=1}^{n} T_r, \quad W_0 = 0.$$

Waiting time is used to derive the distribution of $N(t)$ and its moments due to the equivalency of the following expression for $t \geq 0,\ n \geq 0$:

$$N(t) < n \Leftrightarrow W(n) > t$$

Let $F_n(t) = P(\mathrm{W}(n) \leq t),\ F_0(t) = 1.$ It follows that

$$F_n(t) = F^{*n}(t),\ n = 0,1,2,\ldots,$$

which is the n-fold convolution of $F(t)$ with itself as described in Chaudhry and Templeton (1983).

Also note that

$$P_n(t) = F_n(t) - F_{n+1}(t), \; t \geq 0, \; n = 0,$$

and let $\tilde{f}(s)$ and $\tilde{p}_n(s)$ be the LST of $F(t)$ and $P_n(t)$, respectively. By taking the LST on both sides of the equation, it follows that

$$\tilde{p}_n(s) = \tilde{f}^{\,n}(s) - \tilde{f}^{\,n+1}(s) = \tilde{f}^{\,n}(s)\left(1 - \tilde{f}(s)\right).$$

Since $\tilde{p}_n(s) = \int_0^\infty e^{-st} dP_n(t),$ the desired result is obtained:

$$P_n(t) = L^{-1}\left[\frac{\tilde{p}_n(s)}{s}\right] = L^{-1}\left[\frac{\tilde{f}^{\,n}(s)\left(1 - \tilde{f}(s)\right)}{s}\right], \; n \geq 0, \; s > 0,$$

where inversion provides the probabilities $P_n(t)$ that compose the distribution of $N(t)$. Note that a very similar notation is employed in Chaudhry, Yang, and Ong (2013), and for further details on the inversion, see Appendix A.3.

The *mean value* of the continuous-time renewal process $N(t)$ is referred to as the *renewal function*. The renewal density is the first derivative of the renewal function, but these are defined in Section 5.1 and 5.2, respectively. A great portion of renewal theory is concerned with properties of the renewal function, and it is for this reason that its asymptotic results (see Appendix A.4) are of such great interest. Let $M(t) = E[N(t)]$ and $m(t) = M'(t)$.

It is important to note two key distinctions between the notation used in discrete time and continuous time: Whereas in discrete time n is used to describe the time, in continuous time the symbol t is used instead. Likewise, when counting the number of renewals in discrete time, the symbol k is employed, whereas n is used in continuous time.

3.3 Bulk Renewal Theory in Continuous Time

In order to limit the focus of this thesis, as well as to provide other graduate students with adequate research opportunities in renewal theory, all work regarding bulk arrivals has been restricted to continuous time.

Although modelled by pre-determined distributions, renewals are generally assumed to occur as single events. It is simple to conceptualize a single piece of equipment failing or one new customer arriving in a store at a time. These events are examples of a single-arrival renewal process, and are said to have renewal events of a *group size* or *batch size* of one. In both renewal theory and the real world, events with a group size greater than one can occur as well. For example, a bus full of tourists can cause multiple customers to arrive at a location such as a restaurant or an amusement park at the same time. Just as the frequency of renewal events can be modelled using probability distributions, so can the group size of these renewals. Bulk renewal theory is required to model either of these examples, and this has a direct impact on the distribution of the total number of renewals and its corresponding moments.

Assume that groups arrive at times $s_1', s_2', s_3', \ldots,$ with X_i being the size of the i^{th} group and $N(t)$ being the number of groups arriving during the time interval $(0,t]$. As such, the total number of renewals is

$$Y(t) = \sum_{i=1}^{N(t)} X_i.$$

Let the probability generating function (p.g.f.) of $Y(t)$ be

$$P(z,t) = E\left[z^{Y(t)}\right] = E\left[z^{\sum_{i=1}^{N(t)} X_i}\right].$$

Assuming that X_i's are i.i.d.r.v.s distributed as X with p.g.f. $P_X(z)$ and the first two moments of X are finite, $P(z,t)$ can be rewritten as

$$\begin{aligned}P(z,t) &= E\left[E\left[z^{\sum_{i=1}^{N(t)} X_i} \mid N(t)\right]\right] \\ &= \sum_{n=0}^{\infty} E\left[z^{\sum_{i=1}^{N(t)} X_i} \mid N(t)=n\right] P\left(N(t)=n\right) \\ &= \sum_{n=0}^{\infty} \left(P_X(z)\right)^n P_n(t).\end{aligned}$$

The model can be reduced to the single-arrival renewal process if $P_X(z)=z,$ where $Y(t)$ becomes $N(t)$ and $P(z,t)=\sum_{n=0}^{\infty} P_n(t)z^n.$

By taking the LT of $P(z,t),$ the following compound distribution is obtained:

$$\begin{aligned}\tilde{P}(z,s) &= \int_0^{\infty} P(z,t)e^{-st}dt \\ &= \int_0^{\infty}\left[\sum_{n=0}^{\infty}\left(P_X(z)\right)^n P_n(t)\right]e^{-st}dt \\ &= \sum_{n=0}^{\infty}\left(P_X(z)\right)^n \int_0^{\infty} P_n(t)e^{-st}dt \\ &= \sum_{n=0}^{\infty}\left(P_X(z)\right)^n \tilde{P}_n(s),\end{aligned}$$

where $\tilde{P}_n(s)=\int_0^{\infty} P_n(t)e^{-st}dt, n=0,1,2,\ldots$ is the LT of $P_n(t).$ Assuming that the inter-renewal times of the process $\{N(t),\ t \geq 0\}$ are i.i.d.r.v.s with distribution function $F_T(t)$ and Laplace-Stieltjes transform (LST) $\tilde{f}(s)=\int_0^{\infty} e^{-st}dF_T(t),$ it can be shown that (see, for example, Cox (1962))

$$\tilde{P}_n(s)=\frac{\tilde{f}^{\,n}(s)\left(1-\tilde{f}(s)\right)}{s},$$

where $\tilde{P}_n(s)$ and $\tilde{f}(s)$ are the LSTs of $P_n(t)$ and $f(t),$ respectively. It then follows that

$$\begin{aligned}\tilde{P}(z,s) &= \frac{1-\tilde{f}(s)}{s}\sum_{n=0}^{\infty}\left(\tilde{f}(s)P_X(z)\right)^n \\ &= \frac{1-\tilde{f}(s)}{s-s\tilde{f}(s)P_X(z)}.\end{aligned}$$

Although this p.g.f. could be inverted to determine the distribution of the number of renewals for bulk arrivals, the complexity of the equations makes this process quite cumbersome. Instead, by first calculating the $P_n(t)$ from the single arrival case and expanding $P(z,t)$ as a Taylor's series, it is then possible to collect the coefficients of z^n and quickly determine numerical results for various values of t.

4 ASYMPTOTIC RESULTS IN THE DISCRETE-TIME RENEWAL PROCESS

A simple and elegant solution to determining the asymptotic results for the renewal density as well as for the first and second moments of the number of renewals for the discrete time renewal process is presented. Using g.f.s, the difficult-to-determine constant term, as stated in Van der Weide et al. (2007), the second moment is also addressed.

The Tauberian theorem as indicated in Cox (1962) for the continuous time renewal case is employed to expand the g.f. of the probability of a renewal at time *n*, of the renewal function and of the first and second moments. This quickly leads to the final asymptotic results for the probability of a renewal at time *n* as well as for both the first and second moments. Some easy steps could have been avoided, but are included here for the sake of clarity.

4.1 Renewal Density

The use of the term renewal density for m_n is employed from continuous time renewal theory. Since the renewal time is at least one unit and assuming that $\lim_{n\to\infty} m_n$ exists, it can be shown that $\lim_{n\to\infty} m_n = 1/\mu,\ \mu < \infty$:

Proof: The renewal equation defined as $m_n = f_n + \sum_{j=1}^{n} m_{n-j} f_j,\ n = 1, 2, 3, \ldots$ has g.f.

$$m(z) = \sum_{k=1}^{\infty} m_k z^k = \frac{f(z)}{1 - f(z)}. \qquad (1)$$

Since m_k are probabilities, $0 \le m_k \le 1$, and $m(z)$ is at least convergent in $|z| < 1$. Now, assuming $f(1) = 1$ (the recurrent case), $\mu < \infty$ (used in finding C_{-1}), and using the Tauberian theorem (since $\lim_{z \to 1^-}(1-z)m(z) = 1/\mu$ as shown below), the intuitively obvious relation

$$m(z) = \frac{C_{-1}}{(1-z)} + O(1), \tag{2}$$

where C_{-1} is a constant derived below and $O(1)$ indicates a function of $(1-z)$ bounded as $z \to 1^-$, is obtained. This follows from Eq. (1), since $\lim_{z \to 1^-}(1-z)m(z) = 1/\mu$ is derived from multiplying Eq. (2) by $(1-z)$ and taking the limit as $z \to 1^-$. The result is based on the assumption that $O(1)$ near $z = 1$ leads to $o(1)$ as $n \to \infty$. This assumption is similar to the one used in continuous-time renewal theory results such as in Cox (1962). It should also be noted that in all asymptotic expressions for the renewal density, first moment, and second moment, it is implied that $n \to \infty$.

There are several ways of determining $\lim_{k \to \infty} m_k$. For one such method, see Kohlas (1982). The existence of the limit can also be obtained using a theorem in Karlin and Taylor (1975). From Eq. (1) and (2), it follows that

$$C_{-1} = \lim_{z \to 1^-}(1-z)m(z) = 1/\mu.$$

4.2 First Moment

It follows from both Feller (1949) and Hunter (1983) that given the number of renewals N_n in $(0,n]$, the g.f. for the mean number of renewals $M_n = E[N_n]$ is

$$M(z)=\sum_{n=1}^{\infty}M_n z^n=\frac{f(z)}{(1-z)(1-f(z))}=\frac{m(z)}{(1-z)},\ |z|<1.$$

Assuming that the renewal event is aperiodic recurrent with μ < ∞ and σ < ∞, it can be shown that

$$M_n=\frac{n}{\mu}+\left(\frac{\sigma^2-\mu^2+\mu}{2\mu^2}\right)+o(1).$$

Proof:

Since $\lim_{z\to 1^-}(1-z)^2 M(z)=\frac{1}{\mu}$, $M(z)$ can be written as

$$M(z)=\frac{C_{-2}}{(1-z)^2}+\frac{C_{-1}}{(1-z)}+O(1).$$

Proceeding as in the case of the renewal density, it follows that

$$M_n=(n+1)C_{-2}+C_{-1}+o(1).$$

Now

$$C_{-2}=\lim_{z\to 1^-}(1-z)^2 M(z)=1/\mu,$$

and

$$C_{-1}=\lim_{z\to 1^-}(1-z)\left[M(z)-\frac{C_{-2}}{(1-z)^2}\right]=\lim_{z\to 1^-}\left[\frac{1-z-(1/\mu)(1-f(z))}{(1-z)(1-f(z))}\right]-1$$
$$=\frac{(1/\mu)\,f''(1)}{2f'(1)}-1=\frac{a_2}{2\mu^2}-1.$$

Substituting these constant values into the preceding equation, it follows that

$$M_n=\frac{n}{\mu}+\left(\frac{a_2}{2\mu^2}-1+\frac{1}{\mu}\right)+o(1).$$

By substituting $a_2=\sigma^2+\mu^2-\mu,$ the desired result is obtained:

$$M_n = \frac{n}{\mu} + \left(\frac{\sigma^2 - \mu^2 + \mu}{2\mu^2} \right) + o(1).$$

This matches with the results in Feller (1968) and Hunter (1983). It can also be shown to match the result in van der Weide et al. (2007). In addition, it easily leads to the well-known result $\lim_{n\to\infty} M_n / n = 1/\mu$.

4.3 Second Moment

An identical approach is taken for solving for the asymptotic results for the second moment. Given that the number of renewals in (0,n] is N_n, it follows from Feller (1968) and Hunter (1983) that

$$M^{(2)}(z) \equiv \sum_{n=1}^{\infty} E[N_n^2] z^n = \frac{f(z) + f^2(z)}{(1-z)(1-f(z))^2} = \frac{M(z)(1+f(z))}{1-f(z)}.$$

Using the same arguments as before, it can be shown that

$$\lim_{n\to\infty} M_n^{(2)} \equiv \lim_{n\to\infty} E[N_n^2] \to \frac{n^2}{\mu^2} + n\left(\frac{2\sigma^2}{\mu^3} + \frac{1}{\mu^2} - \frac{1}{\mu} \right) + \left(\frac{3\mu_2^2}{2\mu^4} + \frac{3\mu_2 - 2\mu_3}{3\mu^3} + \frac{1-9\mu_2}{6\mu^2} - \frac{3}{2\mu} + 1 \right),$$

or, as $n \to \infty$,

$$M_n^{(2)} = \frac{n^2}{\mu^2} + n\left(\frac{2\sigma^2}{\mu^3} + \frac{1}{\mu^2} - \frac{1}{\mu} \right) + \left(\frac{3\mu_2^2}{2\mu^4} + \frac{3\mu_2 - 2\mu_3}{3\mu^3} + \frac{1-9\mu_2}{6\mu^2} - \frac{3}{2\mu} + 1 \right) + o(1).$$

Proof:

Since $\lim_{z\to 1^-} (1-z)^3 M^{(2)}(z) = 2/\mu^2$, proceeding as in the case of the first moment and assuming that $\mu, \mu_2,$ *and* $\mu_3 < \infty$, $M^{(2)}(z)$ can be written as

$$M^{(2)}(z) = \frac{C_{-3}}{(1-z)^3} + \frac{C_{-2}}{(1-z)^2} + \frac{C_{-1}}{(1-z)} + O(1),$$

and thus

$$M_n^{(2)} = \left(\frac{(n+2)!}{2!n!} \right) C_{-3} + \left(n+1 \right) C_{-2} + C_{-1} + o(1). \tag{3}$$

Now,

$$\begin{aligned} C_{-3} &= \lim_{z \to 1^-} \left(1-z\right)^3 M^{(2)}(z) = \lim_{z \to 1^-} \frac{\left(1-z\right)^2 \left(f^2(z) + f(z) \right)}{\left(1-f(z)\right)^2} \\ &\vdots \\ &= 2 \Big/ \left(f'(1) \right)^2 = 2 \big/ \mu^2, \end{aligned}$$

$$\begin{aligned} C_{-2} &= \lim_{z \to 1^-} \left(1-z\right)^2 \left[M^{(2)}(z) - \frac{C_{-3}}{\left(1-z\right)^3} \right] = \lim_{z \to 1^-} \left[\frac{\left(1-z\right)\left(f^2(z) + f(z) \right)}{\left(1-f(z)\right)^2} - \frac{2}{\mu^2 \left(1-z\right)} \right] \\ &= \lim_{z \to 1^-} \left[\frac{\mu^2 \left(1-z\right)^2 \left(f^2(z) + f(z) \right) - 2\left(1-f(z)\right)^2}{\mu^2 \left(1-z\right)\left(1-f(z)\right)^2} \right] \\ &\vdots \\ &= \frac{-3\mu^2 + 2a_2}{\mu^3}, \end{aligned}$$

and

$$\begin{aligned} C_{-1} &= \lim_{z \to 1^-} \left(1-z\right) \left[M^{(2)}(z) - \frac{C_{-3}}{\left(1-z\right)^3} - \frac{C_{-2}}{\left(1-z\right)^2} \right] = \lim_{z \to 1^-} \left[\frac{f^2(z) + f(z)}{\left(1-f(z)\right)^2} - \frac{2}{\mu^2 \left(1-z\right)^2} - \frac{2a_2 + 3\mu^2}{\mu^3 \left(1-z\right)} \right] \\ &= \lim_{z \to 1^-} \left[\frac{\mu^3 \left(1-z\right)^2 \left(f^2(z) + f(z) \right) - 2\mu\left(1-f(z)\right)^2 - \left(2a_2 + 3\mu^2\right)\left(1-z\right)\left(1-f(z)\right)^2}{\mu^3 \left(1-z\right)^2 \left(1-f(z)\right)^2} \right] \\ &\vdots \\ &= \frac{9a_2^2 - 9\mu^2 a_2 - 4\mu a_3}{6\mu^4} + 1. \end{aligned}$$

After substituting the values of C_{-1}, C_{-2}, and C_{-3} into Eq. (3) it follows that

$$M_n^{(2)} = \frac{2}{\mu^2}\frac{(n+2)!}{2!n!} + \frac{\left(2a_2 - 3\mu^2\right)(n+1)}{\mu^3} + \frac{9a_2^2 - 9\mu^2 a_2 - 4\mu a_3}{6\mu^4} + 1 + o(1)$$

$$= \frac{n^2}{\mu^2} + n\left(\frac{3}{\mu^2} + \frac{2a_2}{\mu^3} - \frac{3}{\mu}\right) + \left(\frac{2}{\mu^2} + \frac{2a_2}{\mu^3} - \frac{3}{\mu} + \frac{9a_2^2}{6\mu^4} - \frac{9a_2}{6\mu^2} - \frac{4a_3}{6\mu^3} + 1\right) + o(1)$$

$$= \frac{n^2}{\mu^2} + n\left(\frac{2\sigma^2}{\mu^3} + \frac{1}{\mu^2} - \frac{1}{\mu}\right) + \left(\begin{array}{l} \dfrac{3\left(\mu_2 - 2\mu_2\mu + \mu^2\right)}{2\mu^4} - \dfrac{2\mu_2 - 2\mu - 3\mu^2}{\mu^3} \\ - \dfrac{3(\mu_2 - \mu)}{2\mu^2} - \dfrac{2(\mu_3 - 3\mu_2 + 2\mu)}{3\mu^3} + 1 \end{array}\right) + o(1),$$

where $a_3 \equiv E[(T-2)(T-1)T] = E[T^3] - 3E[T^2] + 2E[T] = \mu_3 - 3\mu_2 + 2\mu_1$,

$a_2 \equiv E[(T-1)T] = E[T^2] - E[T] = \mu_2 - \mu_1$, and $\mu = \mu_1$.

This implies that as $n \to \infty$,

$$M_n^{(2)} = \frac{n^2}{\mu^2} + n\left(\frac{2\sigma^2}{\mu^3} + \frac{1}{\mu^2} - \frac{1}{\mu}\right) + \left(\frac{3\mu_2^2}{2\mu^4} + \frac{3\mu_2 - 2\mu_3}{3\mu^3} + \frac{1 - 9\mu_2}{6\mu^2} - \frac{3}{2\mu} + 1\right) + o(1). \qquad (4)$$

The first two terms on the RHS of Eq. (4) are identical to the first two terms presented in Hunter (1983), but now there is an expression including the constant terms. For an expression entirely in terms of ordinary moments, refer to van der Weide et al. (2007). As in the case of the first moment, $\lim_{n\to\infty} M_n^{(2)} \big/ n^2 = 1\big/\mu^2$, as expected.

4.3 Conclusion

The techniques illustrated in this chapter provide a shorter and simpler alternative to determining several asymptotic results in discrete-time renewal theory. By first expressing $m(z)$, $M(z)$, and $M^{(2)}(z)$ in a particular form, the desired asymptotic results are easily derived. Further, if the first renewal period has a different distribution than the other renewal periods, then the renewal density as well as the first and second moments can be dealt with along similar lines.

A paper based on this result has been published in the journal *Statistics and Probability Letters* (Chaudhry and Fisher, 2012). It has already influenced new research by other graduate students studying renewal theory (Chaudhry and Kim, 2013).

5 ASYMPTOTIC RESULTS IN THE CONTINUOUS-TIME RENEWAL PROCESS

As in discrete time, an elegant solution to determining the asymptotic results for the renewal density as well as for the first and second moments of the number of renewals in continuous time has been derived. The purpose of this chapter is not only to give the asymptotic results for the second moment with a constant term using LTs, but also to give an elegant derivation of the asymptotic results for the renewal density as well as for both the first and second moments using LTs in continuous time.

The method requires inversion on the LT of the probability of a renewal at time t of the renewal function and of the second moment (see Appendix A.3). This quickly leads to the final asymptotic results for the probability of a renewal at time t as well as for both the first and second moments. Some easy steps could have been avoided, but are included for the sake of clarity.

5.1 Renewal Density

The limit of the renewal density can be expressed as $\lim_{t\to\infty} m(t) = 1/\mu,\ 1 \le \mu < \infty$:

Proof:

The renewal equation, defined as $m(t) = f(t) + \int_0^t m(t-y) f(y) dy$, has LT

$$\tilde{m}(s) = \int_0^\infty e^{-st} m(t)\, dt = \frac{\tilde{f}(s)}{\left(1 - \tilde{f}(s)\right)}. \quad (5)$$

Assuming $f(1) = 1$ (the persistent recurrent case), $\mu < \infty$ (used in finding C_{-1}), and using the Tauberian theorem (since $\lim_{s \to 0^-} s \cdot \tilde{m}(s) = 1/\mu$ as shown below), from Eq. (5) an intuitively obvious relation is derived:

$$\tilde{m}(s) = \frac{C_{-1}}{s} + O(1), \tag{6}$$

where $O(1)$ indicates a function of s bounded as $s \to 0$. This is based on the assumption that $O(1)$ near $s = 0$ leads to $o(1)$ as $t \to \infty$. For details, see Cox (1962). As in the case of discrete time, it should also be noted that in all asymptotic expressions for the renewal density, first moment, and second moment, it is implied that $t \to \infty$.

From the inversion of Eq. (6), it follows that

$$m(t) = C_{-1} + o(1),$$

where $o(1)$ indicates a function of t tending to zero as $t \to \infty$.

C_{-1} is then derived as follows:

$$C_{-1} = \lim_{s \to 0^-} s \cdot \tilde{m}(s) = 1/\mu.$$

5.2 First Moment

Given the number of renewals, $N(t)$, in $(0,t)$, from Eq. (5) the LT for the mean number of renewals $M(t) = E[N(t)]$ is

$$\tilde{M}(s) = \int_0^\infty e^{-st} M(t)\, dt = \frac{\tilde{f}(s)}{s\left(1 - \tilde{f}(s)\right)},$$

where it is assumed that the first two moments of $f(t)$ are finite. Assuming that $\mu < \infty$ and $\sigma < \infty$, it can be shown that

$$M(t) = \frac{t}{\mu} + \left(\frac{\sigma^2 - \mu^2}{2\mu^2} \right) + o(1),$$

where $o(1) \to 0$ as $t \to \infty$.

Proof:

Since $\lim_{s\to 0} s^3 \tilde{M}(s) = 0$, $\tilde{M}(s)$ has a pole of order 2 at s = 0, and $\tilde{M}(s)$ can be written as

$$\tilde{M}(s) = \frac{C_{-2}}{s^2} + \frac{C_{-1}}{s} + O(1).$$

By inverting this function as in the case of the renewal density, it follows that

$$M(t) = t \cdot C_{-2} + C_{-1} + o(1). \tag{7}$$

Now

$$C_{-2} = \lim_{s\to 0} s^2 \tilde{M}(s) = 1/\mu,$$

and

$$\begin{aligned} C_{-1} &= \lim_{s\to 0} s \left[\tilde{M}(s) - \frac{C_{-2}}{s^2} \right] \\ &= \lim_{s\to 0} \left[\frac{\tilde{f}(s)\mu s - 1 + \tilde{f}(s)}{\left(1 - \tilde{f}(s)\right)\mu s} \right] \\ &\quad \vdots \\ &= (\sigma^2 - \mu^2 - \mu)/2\mu^2. \end{aligned}$$

By substituting the values of C_{-1} and C_{-2} into Eq. (7) it follows that

$$M(t) = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + o(1).$$

As expected, this easily leads to the well-known result $\lim_{t\to\infty} M(t)/t = 1/\mu$.

5.3 Second Moment

An identical approach is taken for deriving the asymptotic results for the second moment. Given that the number of renewals in (0,*t*) is *N*(*t*), it follows that

$$\tilde{M}^{(2)}(s) \equiv \int_0^\infty e^{-st} M^{(2)}(t)\,dt = \int_0^\infty e^{-st} E[N^{(2)}(t)]\,dt = \frac{\tilde{f}(s)\left(\tilde{f}(s)+1\right)}{s\left(1-\tilde{f}(s)\right)^2}.$$

It can then be shown that

$$M^{(2)}(t) = \frac{t^2}{\mu^2} + \left(\frac{2\sigma^2}{\mu^3} - \frac{1}{\mu}\right)t + \frac{3\sigma^4}{2\mu^4} - \frac{2\mu_3}{3\mu^3} + \frac{3\sigma^2}{2\mu^2} + 1 + o(1).$$

Proof:

Since $\lim_{s\to 0} s^4 \tilde{M}^{(2)}(s) = 0$, $\tilde{M}^{(2)}(s)$ has a pole of order 3 at s =0.

Proceeding as in the case of the first moment,

$$\tilde{M}^{(2)}(s) = \frac{C_{-3}}{s^3} + \frac{C_{-2}}{s^2} + \frac{C_{-1}}{s} + O(1),$$

where

$$C_{-3} = \lim_{s\to 0} s^3 \tilde{M}^{(2)}(s) = \lim_{s\to 0} \frac{s^2\tilde{f}(s)\left(\tilde{f}(s)+1\right)}{\left(1-\tilde{f}(s)\right)^2} = 2/\mu^2,$$

$$\begin{aligned}
C_{-2} &= \lim_{s\to 0} s^2\left[\tilde{M}^{(2)}(s) - \frac{C_{-3}}{s^3}\right] = \lim_{s\to 0}\left[\frac{s\,\tilde{f}(s)\left(\tilde{f}(s)+1\right)}{\left(1-\tilde{f}(s)\right)^2} - \frac{2}{s\mu^2}\right] \\
&= \lim_{s\to 0}\left[\frac{\mu^2 s^2\tilde{f}(s)\left(\tilde{f}(s)+1\right) - 2\left(1-\tilde{f}(s)\right)^2}{\mu^2 s\left(1-\tilde{f}(s)\right)^2}\right] \\
&\ \ \vdots \\
&= \frac{2(\sigma^2+\mu^2) - 3\mu^2}{\mu^3},
\end{aligned}$$

and

$$
\begin{aligned}
C_{-1} &= \lim_{s\to 0} s\left[\tilde{M}^{(2)}(s) - \frac{C_{-3}}{s^3} - \frac{C_{-2}}{s^2}\right] \\
&= \lim_{s\to 0}\left[\frac{\tilde{f}(s)\left(\tilde{f}(s)+1\right)}{\left(1-\tilde{f}(s)\right)^2} - \frac{2}{\mu^2 s^2} - \frac{3\mu^2 - 2a_2}{\mu^3 s^2}\right] \\
&\;\vdots \\
&= \frac{6\mu^4 - 4\mu\cdot\mu_3 + 9\sigma^4 + 9\mu^2\sigma^2}{6\mu^4}.
\end{aligned}
$$

As in the case of the first moment and the renewal density, from inversion of the LT it follows that

$$
M^{(2)}(t) = \frac{t^2\cdot C_{-3}}{2} + t\cdot C_{-2} + C_{-1} + o(1), \tag{8}
$$

and after substituting the values of $C_{-1}, C_{-2},$ and C_{-3} into Eq. (8), the desired result is obtained:

$$
M^{(2)}(t) = \frac{t^2}{\mu^2} + \left(\frac{2\sigma^2}{\mu^3} - \frac{1}{\mu}\right)t + \frac{3\sigma^4}{2\mu^4} - \frac{2\mu_3}{3\mu^3} + \frac{3\sigma^2}{2\mu^2} + 1 + o(1).
$$

As in the case of the first moment, $\lim_{t\to\infty} M^{(2)}(t)\big/t^2 = 1\big/\mu^2$, as expected.

5.4 Conclusion

The techniques illustrated in this chapter provide a shorter and simpler alternative to determining several asymptotic results for single arrivals in continuous-time renewal theory. By first expressing $\tilde{m}(s)$, $\tilde{M}(s)$, and $\tilde{M}^{(2)}(s)$ in a particular form, the desired asymptotic results are easily derived. Further, if the first renewal period has a different distribution than the other renewal periods, then the renewal density as well as the first and second moments can be dealt with along similar lines.

6 ASYMPTOTIC RESULTS FOR THE BULK RENEWAL PROCESS IN CONTINUOUS TIME

The purpose of this chapter is to give an elegant derivation of the asymptotic results for the first and second moments of the bulk-arrival renewal process using LTs. Inversion on the LT of the probability of a renewal at time *t* of the renewal function and of the second moment quickly leads to the final asymptotic results for the probability of a renewal at time *t* as well as for both the first and second moments. Some easy steps could have been avoided, but are included here for the sake of clarity.

6.1 Renewal Density

As defined in Chaudhry and Templeton (1983), the renewal density, *m(t)*, is the derivative of the renewal function, *M(t).* This does not change in the case of bulk arrivals.

6.2 First Moment

The LT for the mean number of renewals $M(t) = E[Y(t)]$ is

$$\begin{aligned}\tilde{M}(s) &= \frac{d}{dz}\left(\frac{1-\tilde{f}(s)}{s - s\tilde{f}(s)P_X(z)}\right)\Big|_{z=1} \\ &= \frac{\tilde{f}(s)}{s\left(1-\tilde{f}(s)\right)} P_X^{'}(1).\end{aligned}$$

Remark: By substituting $P_X(z) = z$ with $P_X^{'}(1) = 1$, it follows that

$$\tilde{M}(s) = \int_0^\infty e^{-st} E[N(t)]\,dt = \frac{\tilde{f}(s)}{s\left(1-\tilde{f}(s)\right)},$$

which is the LT for the mean number of renewals for the single-arrival renewal process.

By assuming that the renewal event is recurrent with $\mu < \infty$, $\sigma < \infty$, and $P_X^{'}(1) < \infty$, it can be shown that

$$M(t) = \frac{P_X^{'}(1)}{\mu} t + P_X^{'}(1)\left(\frac{\sigma^2 - \mu^2}{2\mu^2}\right) + o(1),$$

where $o(1) \rightarrow 0$ as $t \rightarrow \infty$.

Proof:

Following Cox's (1962) work on the single-arrival renewal process, $\tilde{M}(s)$ can be re-written as

$$\tilde{M}(s) = \frac{C_{-2}}{s^2} + \frac{C_{-1}}{s} + O(1).$$

Inverting this LT as in the case of the renewal density, it follows that

$$M(t) = t \cdot C_{-2} + C_{-1} + o(1). \tag{9}$$

Now

$$C_{-2} = \lim_{s \to 0} s^2 \tilde{M}(s) = P_X^{'}(1) / \mu,$$

and

$$\begin{aligned} C_{-1} &= \lim_{s \to 0} s\left[\tilde{M}(s) - \frac{C_{-2}}{s^2}\right] \\ &= \lim_{s \to 0}\left[\frac{P_X^{'}\left(2\tilde{f}(s) + \tilde{f}''(s)\right)}{\left(\tilde{f}'(s)\right)^2}\right] \\ &\quad\vdots \\ &= P_X^{'}(\mu_2 - 2\mu^2) / 2\mu^2 . \end{aligned}$$

By substituting the values of C_{-1} and C_{-2} as well as $\mu_2 = \sigma^2 + \mu^2$ into Eq. (9), it follows that

$$M(t) = \frac{P_X^{'}(1)}{\mu} t + P_X^{'}(1)\left(\frac{\sigma^2 - \mu^2}{2\mu^2}\right) + o(1).$$

As expected, this easily leads to the well-known result $\lim_{t\to\infty} M(t)/t = P_X{'}(1)/\mu$, which gives the arrival rate for the bulk-arrival renewal process. Also, for the single-arrival renewal process, $M(t)$ matches with the known results in Cox (1962).

6.3 Second Moment

The same approach is taken when solving for the asymptotic results for the second moment. Using the fact that $\tilde{P}"(z,s)$ is the LT of

$$E\left[N(t)\left(N(t)-1\right)\right]\Big|_{z=1} = E\left[N^2(t)\right]\Big|_{z=1} - E\left[N(t)\right]\Big|_{z=1},$$

it follows that

$$\tilde{P}"(z,s) = \frac{\tilde{f}(s)}{s\left(1-\tilde{f}(s)\right)^2}\left[\left(1-\tilde{f}(s)\right)P_X^{"}(1) + 2\tilde{f}(s)\left(P_X^{'}(1)\right)^2\right],$$

where $P_X^{"}(1) = E[X^2] - E[X]$.

Consequently,

$$\begin{aligned}
\tilde{M}^{(2)}(s) &= \int_0^\infty E\left[Y^2(t)\right]e^{-st}\,dt \\
&= \frac{\tilde{f}(s)}{s\left(1-\tilde{f}(s)\right)^2}\left[\left(1-\tilde{f}(s)\right)P_X^{"}(1) + 2\tilde{f}(s)\left(P_X^{'}(1)\right)^2\right] + \frac{P_X^{'}(1)\tilde{f}(s)}{s\left(1-\tilde{f}(s)\right)} \\
&= \frac{\tilde{f}(s)}{s\left(1-\tilde{f}(s)\right)}\left[P_X^{"}(1) + \frac{2\tilde{f}(s)\left(P_X^{'}(1)\right)^2}{\left(1-\tilde{f}(s)\right)} + P_X^{'}(1)\right].
\end{aligned}$$

By substituting $P_X(z) = z$ and the corresponding values $P_X^{'}(1) = 1$ and $P_X^{"}(1) = 0$ for the single arrival case, it follows that

$$\tilde{M}^{(2)}(s) = \int_0^\infty e^{-st} E[N^2(t)]\,dt = \frac{\tilde{f}(s)\left(\tilde{f}(s)+1\right)}{s\left(1-\tilde{f}(s)\right)^2}.$$

In the general case, it is possible to illustrate that

$$M^{(2)}(t) = \frac{\left(P_X^{'}(1)\right)^2}{\mu^2} t^2 + \left(\frac{P_X^{''}(1) - 2\left(P_X^{'}(1)\right)^2 + P_X^{'}(1)}{\mu} + \frac{2\sigma^2\left(P_X^{'}(1)\right)^2}{\mu^3} \right) t + \frac{\sigma^2 P_X^{''}(1)}{2\mu^2} + \frac{\sigma^2 P_X^{'}(1)}{2\mu^2} - \frac{P_X^{'}(1)}{2}$$
$$- \frac{2\mu_3\left(P_X^{'}(1)\right)^2}{3\mu^3} + \frac{3\sigma^4\left(P_X^{'}(1)\right)^2}{2\mu^4} + \frac{\sigma^2\left(P_X^{'}(1)\right)^2}{\mu^2} + \frac{3\left(P_X^{'}(1)\right)^2}{2} - \frac{P_X^{''}(1)}{2} + o(1).$$

Proof:

Proceeding as in the case of the first moment, the LT $\tilde{M}^{(2)}(s)$ can be re-written as

$$\tilde{M}^{(2)}(s) = \frac{C_{-3}}{s^3} + \frac{C_{-2}}{s^2} + \frac{C_{-1}}{s} + O(1),$$

where

$$C_{-3} = \lim_{s\to 0} s^3 \tilde{M}^{(2)}(s) = \lim_{s\to 0} \frac{4P_X^{'}(1)}{2\left(\tilde{f}\,'(s)\right)^2} = 2P_X^{'}(1)/\mu^2,$$

$$C_{-2} = \lim_{s\to 0} s^2 \left[\tilde{M}^{(2)}(s) - \frac{C_{-3}}{s^3} \right]$$
$$\vdots$$
$$= \lim_{s\to 0} \left[\frac{6\left(\tilde{f}\,'(s)\right)^3 P_X^{''}(1) - 24\left(\tilde{f}\,'(s)\right)^3 \left(P_X^{'}(1)\right)^2 + 6\left(\tilde{f}\,'(s)\right)^3 P_X^{'}(1) + 12\tilde{f}\,''(s)\tilde{f}\,'(s) P_X^{'}(1)}{6\left(\tilde{f}\,'(s)\right)^4} \right]$$
$$= \frac{\mu^2 P_X^{''}(1) - 4\mu^2\left(P_X^{'}(1)\right)^2 + \mu^2 P_X^{'}(1) + 2\mu_2\mu\left(P_X^{'}(1)\right)^2}{\mu^3},$$

and

$$C_{-1}=\lim_{s\to 0} s\left[\tilde{M}^{(2)}(s)-\frac{C_{-3}}{s^3}-\frac{C_{-2}}{s^2}\right]$$

$$\vdots$$

$$=\lim_{s\to 0}\left[\frac{\left[\begin{array}{l}12\tilde{f}''(s)P_X''(1)\left(\tilde{f}'(s)\right)^3-48\tilde{f}''(s)\left(\tilde{f}'(s)\right)^3\left(P_X'(1)\right)^2+36\tilde{f}'(s)\left(\tilde{f}''(s)\right)^2\left(P_X'(1)\right)^2+\\ 12P_X'(1)\tilde{f}''(s)\left(\tilde{f}'(s)\right)^3-16\tilde{f}'''(s)\left(\tilde{f}'(s)\right)^2\left(P_Y'(1)\right)^2-24P_X''(1)\left(\tilde{f}'(s)\right)^5+\\ 48\left(\tilde{f}'(s)\right)^5\left(P_X'(1)\right)^2-24P_X'(1)\left(\tilde{f}'(s)\right)^5\end{array}\right]}{24\left(\tilde{f}'(s)\right)^5}\right]$$

$$=\frac{\left[\begin{array}{l}3\mu_2\mu^2P_X''(1)-6\mu^4P_X''(1)-4\mu_3\mu^4\left(P_X'(1)\right)^2+9\mu_2\left(P_X'(1)\right)^2\\ -12\mu_2\mu^2\left(P_X'(1)\right)^2+3\mu_2\mu^2P_X'(1)+12\mu^4\left(P_X'(1)\right)^2-6\mu^4P_X'(1)\end{array}\right]}{6\mu^4}.$$

As in the case of the first moment, from the inversion of the LT it follows that

$$M^{(2)}(t)=\frac{t^2\cdot C_{-3}}{2}+t\cdot C_{-2}+C_{-1}+o(1), \tag{10}$$

and after substituting the values of $C_{-1}, C_{-2},$ and C_{-3} as well as $\mu_2=\sigma^2+\mu^2$ into Eq. (10) the desired result is obtained:

$$M^{(2)}(t)=\frac{\left(P_X'(1)\right)^2}{\mu^2}t^2+\left(\frac{P_X''(1)-2\left(P_X'(1)\right)^2+P_X'(1)}{\mu}+\frac{2\sigma^2\left(P_X'(1)\right)^2}{\mu^3}\right)t+\frac{\sigma^2P_X''(1)}{2\mu^2}+\frac{\sigma^2P_X'(1)}{2\mu^2}-\frac{P_X'(1)}{2}$$
$$-\frac{2\mu_3\left(P_X'(1)\right)^2}{3\mu^3}+\frac{3\sigma^4\left(P_X'(1)\right)^2}{2\mu^4}+\frac{\sigma^2\left(P_X'(1)\right)^2}{\mu^2}+\frac{3\left(P_X'(1)\right)^2}{2}-\frac{P_X''(1)}{2}+o(1).$$

As in the case of the first moment, $\lim_{t\to\infty}M^{(2)}(t)\big/t^2=\left(P_X'(1)\right)^2\big/\mu^2$ as expected. $M^{(2)}(t)$ can likewise be modified to match the results in Cox (1962) for the variance of the number of renewals in single-arrival renewal theory. In the latter case, if considering single arrivals, $M^{(2)}(t)-\left[M(t)\right]^2$ leads to $\frac{\sigma^2 t}{\mu^3}+\frac{5\sigma^4}{4\mu^4}+\frac{2\sigma^2}{\mu^2}-\frac{2\mu_3}{3\mu^3}+\frac{3}{4}+o(1)$, a result given in Heyman and Sobel (1982).

6.4 Conclusion

The techniques illustrated in this chapter provide a shorter and simpler alternative to determining several asymptotic results for the bulk renewal process. By first expressing $\tilde{M}(s)$ and $\tilde{M}^{(2)}(s)$ in a particular form, the desired asymptotic results are easily derived. Further, if the first renewal period has a different distribution than the other renewal periods, then the first and second moments can be dealt with along similar lines. These results for bulk arrivals form part of a paper entitled “Computing the distribution for the number of renewals with bulk arrivals” that has been accepted by *INFORMS Journal on Computing* (Fisher and Chaudhry, 2014).

7 NUMERICAL RESULTS

A common interest of renewal theory is to determine the distribution or moments of the number of renewals during a specific time interval. There have been few recent contributions regarding new techniques for determining numerical results for the distribution of the number of renewals as well as the mean and variance, with the aforementioned paper by Chaudhry, Yang, and Ong (2013) being an exception. It is much more useful to be able to compute the distribution of the number of renewals than to derive information from the mean and variance.

There are several methods for the numerical inversion of generating functions and Laplace transforms. Appendix A.3 illustrates an inversion method when dealing with rational functions or functions that can be approximated using the Padé method. Six different distributions for renewal time are considered in the following sub-chapters. The exponential, mixed generalized Erlang, matrix exponential, gamma, truncated normal, and inverse Gaussian distributions are each used to produce numerical results in continuous time for both single and bulk arrival renewal theory. In the case of the latter, models where the group sizes follow both Poisson or 1-3-5 distributions are considered. Appendix B describes important characteristics of each of these distribution classes.

The Tables presented throughout the following sub-chapters illustrate the first five values of $P_n(t)$, the probability that *n* renewals will have occurred by time *t*, as well as the mean and variance, each for various values of *t*. The values of *t* were selected in part based on previous computations that have been presented in the literature (Chaudhry, Yang & Ong, 2013). Although the results of this work were used to ensure accuracy of the MAPLE algorithm for the cases of single arrivals, an additional comparison between numerical and analytical results for the mean and variance, as well as a verification that $\sum_n P_n(t)=1$ were used in validating the accuracy of the distribution in the case of both single and bulk arrivals.

To account for the bulk arrivals, the following modified equations for the mean and variance of the number of renewals, henceforth referred to as μ_{ANA} and σ^2_{ANA}, respectively, were used:

$$E\left[Y(t)\right]=\sum_{n=0}^{\infty} nP_n(t)=L^{-1}\left[\frac{\tilde{f}(s)\cdot P_X^{'}(1)}{s\left(1-\tilde{f}(s)\right)}\right]$$

and

$$V\left[Y(t)\right]=\sum_{n=0}^{\infty} n^2P_n(t)-E^2\left[Y(t)\right]=L^{-1}\left[\frac{\tilde{f}(s)}{s\left(1-\tilde{f}(s)\right)}\left(P_X^{''}(1)+\frac{2\tilde{f}(s)\cdot\left(P_X^{'}(1)\right)^2}{\left(1-\tilde{f}(s)\right)}+P_X^{'}(1)\right)\right]-\left(L^{-1}\left[\frac{\tilde{f}(s)\cdot P_X^{'}(1)}{s\left(1-\tilde{f}(s)\right)}\right]\right)^2.$$

Note that the analytic results for the mean and variance of the distributions can be determined using other methods such as those described in Cox (1962).

A sample copy of the code and printout from MAPLE for this algorithm is found in Appendix C.

7.1 Single Arrivals

1. Mixed Generalized Erlang Distribution

$f(t)$ follows the Mixed Generalized Erlang distribution with p.d.f. $f(t)=\sum_{j=1}^{k} c_j \frac{\lambda^j t^{j-1} e^{-\lambda t}}{(j-1)!}$, and LT $\tilde{f}(s)=\sum_{j=1}^{k} c_j\left[\frac{\lambda}{\lambda+s}\right]^j$, where $\sum_{j=1}^{k} c_j=1.$ From considering the case where the parameters are $c_1=c_5=0.25,$ $c_{10}=0.5,$ and $\lambda=1,$ the numerical values for $P_n(t)$, the mean, and the variance for various values of t are determined. These results are presented in Table 1.

Table 1: $P_n(t)$ for the Mixed Generalized Erlang distribution with $c_1 = c_5 = 0.25$, $c_{10} = 0.5$, and $\lambda = 1$

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.5	0.9000	0.0939	0.0058	0.0002	0.0000	. . .	0.1063	0.1063	0.1082	0.1082
1.0	0.8433	0.1400	0.0154	0.0012	0.0001	. . .	0.1748	0.1748	0.1835	0.1835
2.0	0.7720	0.1896	0.0333	0.0046	0.0005	. . .	0.2723	0.2723	0.2996	0.2996
3.0	0.7130	0.2262	0.0503	0.0090	0.0013	. . .	0.3601	0.3601	0.4052	0.4052
5.0	0.5969	0.2894	0.0876	0.0210	0.0043	. . .	0.5493	0.5493	0.6196	0.6196
10.0	0.2355	0.4254	0.2238	0.0822	0.0247	. . .	1.2632	1.2632	1.1001	1.1001

2. Matrix Exponential Distribution

f(*t*) follows a Matrix Exponential distribution (non-phase-type) with p.d.f. $f(t) = \left(1 + \frac{1}{4\pi^2}\right)\left(1 - \cos\left(2\pi t\right)\right)e^{-t}$, such that the LT is $\tilde{f}(s) = \frac{4\pi^2 + 1}{(s+1)\left((s+1)^2 + 4\pi^2\right)}$. The numerical values for $P_n(t)$, the mean, and the variance for various values of t are presented in Table 2.

Table 2: $P_n(t)$ for the non-phase-type Matrix Exponential distribution $f(t) = (1+1/4\pi^2)(1-\cos(2\pi t))e^{-t}$

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.5	0.6373	0.3464	0.0162	0.0002	0.0000	. . .	0.3793	0.3793	0.2688	0.2688
1.0	0.3679	0.3915	0.2154	0.0242	0.0009	. . .	0.8988	0.8988	0.6785	0.6785
2.0	0.1353	0.2881	0.3035	0.1693	0.0832	. . .	1.8411	1.8411	1.5077	1.5077
3.0	0.0498	0.1590	0.2475	0.2322	0.1779	. . .	2.7919	2.7919	2.3400	2.3400
5.0	0.0067	0.0359	0.0919	0.1523	0.1963	. . .	4.6971	4.6973	3.9918	3.9929

3. Gamma Distribution

When *f*(*t*) follows the Gamma distribution, given by p.d.f. $f(t) = \frac{t^{\alpha-1}e^{-t/\beta}}{\beta^{\alpha}\Gamma(\alpha)}$, the LT is $\tilde{f}(s) = \frac{1}{(1+\beta s)^{\alpha}}$, where α and β are the shape and scale parameters, respectively.

From setting the parameters $\alpha = 0.55$ and $\beta = 1$, the Padé approximation function [4/5], where [4/5] denotes a numerator of degree 4 and a denominator of degree 5 in $\tilde{f}(s)$, is given by

$$\tilde{f}(s) = \frac{1 + 1.97778s + 1.27938s^2 + 0.29852s^3 + 0.01804s^4}{1 + 2.52778s + 2.24340s^2 + 0.81724s^3 + 0.10556s^4 + 0.00232s^5}.$$

For details regarding Padé approximation, refer to Appendix A.3. Further, the $P_n(t)$, mean, and variance for various values of *t* are then determined and are presented along with the mean and variance from Baxter et al. (1982) in Table 3.

Table 3: $P_n(t)$ for the Gamma distribution with α = 0.55 and β = 1

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.1	0.6871	0.2385	0.0602	0.0119	0.0019	. . .	0.4040	0.4040	0.4623	0.4623
0.4	0.4071	0.3088	0.1677	0.0743	0.0283	. . .	1.0545	1.0545	1.3970	1.3970
1.25	0.1291	0.1951	0.2050	0.1730	0.1249	. . .	2.6650	2.6663	4.0370	4.0486

4. Truncated Normal Distribution

When *f*(*t*) follows the Truncated Normal Distribution, given by p.d.f. $f(t) = \frac{1}{a\sigma\sqrt{2\pi}} e^{-(t-\mu)^2/2\sigma^2}$, where $a = 1 - \Phi(\frac{-\mu}{\sigma})$, with $\Phi(\frac{-\mu}{\sigma})$ being the standard normal distribution function. From setting the parameters $\mu = 0$ and $\sigma = 1$, the Padé approximation [5/6] of $\tilde{f}(s)$ is given by

$$\tilde{f}(s) = \frac{1 + 0.7707s + 0.3130s^2 + 0.0704s^3 + 0.0089s^4 + 0.0005s^5}{1 + 1.5686s + 1.0645s^2 + 0.4015s^3 + 0.0891s^4 + 0.0111s^5 + 0.0006s^6}.$$

The $P_n(t)$, mean, and variance for various values of *t* are then determined and are presented along with the mean and variance from Baxter et al. (1982) in Table 4.

Table 4: $P_n(t)$ for the Truncated Normal distribution with μ = 0 and σ = 1

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.15	0.8807	0.1121	0.0068	0.0003	0.0000	. . .	0.1267	0.1267	0.1261	0.1261
0.45	0.6527	0.2849	0.0548	0.0068	0.0006	. . .	0.4178	0.4178	0.4023	0.4023
1.0	0.3173	0.4118	0.1978	0.0583	0.0124	. . .	1.0443	1.0443	0.8995	0.8997
1.25	0.2113	0.4003	0.2567	0.0981	0.0268	. . .	1.3507	1.3507	1.0997	1.1006

5. Inverse Gaussian Distribution

$f(t)$ follows the Inverse Gaussian Distribution, given by p.d.f. $f(t)=\sqrt{\dfrac{\lambda}{2\pi t^3}}e^{-\lambda(t-\mu)^2/2\mu^2 t}$, with LT $\tilde{f}(s)=e^{\phi\left(1-\sqrt{1+2s/\gamma}\right)}$. From setting the parameters $\mu=0.75$ and $\lambda=0.5625$, where $\phi=\dfrac{\lambda}{\mu}$ and $\gamma=\dfrac{\mu^2}{\lambda}$, the LT becomes $\tilde{f}(s)=e^{0.75-0.75\sqrt{1+2s}}$ and the Padé approximation [4/7] of $\tilde{f}(s)$ is given by

$$\tilde{f}(s)=\frac{1+4.9575s+8.7086s^2+6.3286s^3+1.5707s^4}{1+5.7075s+12.3330s^2+12.5594s^3+6.1046s^4+1.2729s^5+0.0928s^6+0.0038s^7}.$$

The $P_n(t)$, mean, and variance for various values of t are then determined and are presented along with the mean and variance from Baxter et al. (1982) in Table 5.

Table 5: $P_n(t)$ for the Inverse Gaussian distribution with μ = 0.75 and λ = 0.5625

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.25	0.7445	0.2442	0.0112	0.0001	0.0000	. . .	0.2669	0.2669	0.2188	0.2188
0.70	0.3390	0.4042	0.2062	0.0457	0.0046	. . .	0.9736	0.9736	0.7732	0.7732
1.25	0.1623	0.2869	0.2867	0.1762	0.0683	. . .	1.7635	1.7635	1.5294	1.5294

6. Poisson (Exponential) Distribution

$f(t)$ follows the exponential distribution with p.d.f. $f(t)=\lambda e^{-\lambda t}$ and LT $\tilde{f}(s)=\dfrac{\lambda}{s+\lambda}$, where the parameter $1/\lambda$ is equal to both the mean and standard deviation. Here, $\mathrm{Y}(t)$ follows the compound Poisson process. From considering the case where the parameter $\lambda = 0.5$, the numerical values for $P_n(t)$, the mean, and the variance for various values of *t* are determined. These results are presented in Table 6.

Table 6: $P_n(t)$ for the Exponential distribution with λ = 0.5

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.5	0.7788	0.1947	0.0243	0.0020	0.0001	. . .	0.2500	0.2500	0.2500	0.2500
1.0	0.6065	0.3033	0.0758	0.0126	0.0016	. . .	0.5000	0.5000	0.5000	0.5000
2.0	0.3679	0.3679	0.1839	0.0613	0.0153	. . .	1.0000	1.0000	1.0000	1.0000
3.0	0.2231	0.3347	0.2510	0.1255	0.0471	. . .	1.5000	1.5000	1.5000	1.5000
5.0	0.0821	0.2052	0.2565	0.2138	0.1336	. . .	2.5000	2.5000	2.5000	2.5000
10.0	0.0067	0.0337	0.0842	0.1404	0.1755	. . .	5.0000	5.0000	5.0000	5.0000

The numerical results for the exponential distribution can be likewise obtained from first inverting the expression $\tilde{p}(z,s) = \dfrac{1}{s-\lambda(1-z)}$, a result from $\tilde{P}(z,s) = \dfrac{1-\tilde{f}(s)}{s - s\tilde{f}(s)P_X(z)}$, where $P_X(z) = z$ for the case of single arrivals. Next, expand the expression as a Taylor's series and collect the coefficients of z^n to determine the values for $P_n(t)$.

7.2 Bulk Arrivals

Now assume that $N(t)$ renewals can occur in groups of size X_i, such that the total number of renewals is $Y(t) = \sum_{i=1}^{N(t)} X_i$. By using the $P_n(t)$ from the single arrival case, expanding p.g.f. $P(z,t) = \sum_{n=0}^{\infty} \left(P_X(z)\right)^n P_n(t)$ as a Taylor's series, and then collecting the coefficients of z^n, the distribution of the number of renewals in the bulk process is obtained for various values of *t*. These $P\left(Y(t) = n\right)$ probabilities, henceforth referred to as $P_n(t)$ for simplicity, can likewise be used to obtain the mean and variance for the various values of *t*.

7.3 Poisson Group Size

The following results were obtained by assuming that the group size distribution is Poisson with p.g.f. $P_X(z) = ze^{-\mu(1-z)}, z < 1$. Let $\mu = 0.5$ such that $P_X(z) = ze^{-0.5+0.5z}$. This distribution was presented in Brown (2008) and was used to determine the following results:

Table 7: $P_n(t)$ for the Mixed Generalized Erlang distribution with $c_1 = c_5 = 0.25$, $c_{10} = 0.5$, and $\lambda = 1$

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.5	0.9000	0.0570	0.0306	0.0093	0.0023	. . .	0.1594	0.1594	0.2965	0.2965
1.0	0.8433	0.0849	0.0481	0.0165	0.0050	. . .	0.2621	0.2621	0.5002	0.5002
2.0	0.7720	0.1150	0.0697	0.0276	0.0101	. . .	0.4084	0.4084	0.8103	0.8103
3.0	0.7130	0.1372	0.0871	0.0377	0.0153	. . .	0.5402	0.5402	1.0916	1.0917
5.0	0.5969	0.1755	0.1200	0.0589	0.0274	. . .	0.8238	0.8239	1.6680	1.6688
10.0	0.2355	0.2580	0.2113	0.1329	0.0774	. . .	1.8937	1.8948	3.0954	3.1069

Table 8: $P_n(t)$ for the non-phase-type Matrix Exponential distribution $f(t) = (1+1/4\pi^2)(1-\cos(2\pi t))e^{-t}$

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.5	0.6373	0.2101	0.1110	0.0323	0.0074	. . .	0.5689	0.5689	0.7945	0.7945
1.0	0.3679	0.2375	0.1980	0.1143	0.0528	. . .	1.3482	1.3482	1.9760	1.9761
2.0	0.1353	0.1747	0.1990	0.1713	0.1274	. . .	2.7608	2.7616	4.3055	4.3128
3.0	0.0498	0.0964	0.1392	0.1549	0.1493	. . .	4.1639	4.1879	6.5080	6.6609

Table 9: $P_n(t)$ for the Gamma distribution with $\alpha = 0.55$ and $\beta = 1$

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.1	0.6871	0.1447	0.0945	0.0429	0.0183	. . .	0.6059	0.6059	1.2418	1.2420
0.4	0.4071	0.1873	0.1554	0.1017	0.0634	. . .	1.5790	1.5817	3.6382	3.6706
1.25	0.1291	0.1183	0.1346	0.1288	0.1150	. . .	3.8295	3.9994	9.0965	10.4426

Table 10: $P_n(t)$ for the Truncated Normal distribution with μ = 0 and σ = 1

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.15	0.8807	0.0680	0.0365	0.0111	0.0028	. . .	0.1900	0.1900	0.3471	0.3471
0.45	0.6527	0.1728	0.1066	0.0433	0.0161	. . .	0.6267	0.6267	1.1141	1.1141
1.0	0.3173	0.2498	0.1977	0.1170	0.0628	. . .	1.5662	1.5664	2.5446	2.5464
1.25	0.2113	0.2428	0.2158	0.1467	0.0887	. . .	2.0254	2.0260	3.1452	3.1518

Table 11: $P_n(t)$ for the Inverse Gaussian distribution with μ = 0.75 and λ = 0.5625

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.25	0.7445	0.1481	0.0782	0.0226	0.0052	. . .	0.4003	0.4003	0.6257	0.6257
0.70	0.3390	0.2452	0.1984	0.1167	0.0590	. . .	1.4603	1.4603	2.2265	2.2266
1.25	0.1623	0.1740	0.1925	0.1665	0.1246	. . .	2.6414	2.6452	4.2987	4.3229

Table 12: $P_n(t)$ for the Exponential distribution with λ = 0.5

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.5	0.7788	0.1181	0.0680	0.0242	0.0076	. . .	0.3750	0.3750	0.6875	0.6875
1.0	0.6065	0.1839	0.1199	0.0537	0.0222	. . .	0.7500	0.7500	1.3750	1.3750
2.0	0.3679	0.2231	0.1792	0.1092	0.0611	. . .	1.5000	1.5000	2.7500	2.7500
3.0	0.2231	0.2030	0.1938	0.1457	0.0988	. . .	2.2500	2.2500	4.1250	4.1250
5.0	0.0821	0.1245	0.1566	0.1576	0.1394	. . .	3.7500	3.7500	6.8750	6.8750
10.0	0.0067	0.0204	0.0412	0.0649	0.0866	. . .	7.5000	7.5000	13.750	13.750

7.4 1-3-5 Group Size

The following results were obtained by assuming that the group size distribution has a p.g.f. $P_X(z) = 0.5z + 0.4z^3 + 0.1z^5$. This suggests that the group size can be either 1, 3, or 5 with 50%, 40%, and 10% probabilities, respectively. This distribution was presented in Chaudhry, Samanta, and Pacheco (2012) and was used to determine the following results:

Table 13: $P_n(t)$ for the Mixed Generalized Erlang distribution with $c_1 = c_5 = 0.25$, $c_{10} = 0.5$, and λ = 1

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.5	0.9000	0.0470	0.0014	0.0376	0.0023	. . .	0.2338	0.2338	0.7104	0.7106
1.0	0.8433	0.0700	0.0038	0.0561	0.0062	. . .	0.3844	0.3845	1.1941	1.1955
2.0	0.7720	0.0948	0.0083	0.0764	0.0133	. . .	0.5983	0.5990	1.9198	1.9295
3.0	0.7130	0.1131	0.0126	0.0916	0.0202	. . .	0.7903	0.7923	2.5683	2.5949
5.0	0.5969	0.1447	0.0219	0.1184	0.0353	. . .	1.2011	1.2084	3.8718	3.9657
10.0	0.2355	0.2127	0.0559	0.1804	0.0911	. . .	2.7218	2.7790	6.9687	7.5479

Table 14: $P_n(t)$ for non-phase-type Matrix Exponential distribution $f(t)= (1+1/4\pi^2)(1-\cos(2\pi t))e^{-t}$

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.5	0.8194	0.1243	0.0434	0.0105	0.0020	. . .	0.2528	0.2528	0.3723	0.3723
1.0	0.6290	0.2165	0.1019	0.0369	0.0146	. . .	0.5992	0.5992	0.9008	0.9008
2.0	0.3926	0.2688	0.1743	0.0922	0.0430	. . .	1.2274	1.2274	1.8974	1.8974
3.0	0.2442	0.2505	0.2064	0.1385	0.0812	. . .	1.8611	1.8613	2.8997	2.9013

Table 15: $P_n(t)$ for the Gamma distribution with α = 0.55 and β = 1

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.1	0.6871	0.1193	0.0151	0.0969	0.0242	. . .	0.8858	0.8887	2.9100	2.9483
0.4	0.4071	0.1544	0.0419	0.1328	0.0689	. . .	2.2314	2.3199	7.6063	8.6176
1.25	0.1291	0.0976	0.0513	0.0997	0.0898	. . .	4.5014	5.8659	14.3037	24.2881

Table 16: $P_n(t)$ for the Truncated Normal distribution with μ = 0 and σ = 1

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.15	0.9402	0.0409	0.0145	0.0036	0.0007	. . .	0.0845	0.0845	0.1476	0.1476
0.45	0.8144	0.1188	0.0474	0.0143	0.0038	. . .	0.2785	0.2785	0.4573	0.4573
1.0	0.5897	0.2288	0.1126	0.0448	0.0161	. . .	0.6962	0.6962	1.0917	1.0960
1.25	0.4998	0.2595	0.1403	0.0619	0.0246	. . .	0.9003	0.9005	1.3891	1.3896

Table 17: $P_n(t)$ for the Inverse Gaussian distribution with μ = 0.75 and λ = 0.5625

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.25	0.7445	0.1221	0.0028	0.0977	0.0045	. . .	0.5871	0.5871	1.5286	1.5286
0.70	0.3390	0.2021	0.0515	0.1674	0.0828	. . .	2.1363	2.1418	5.3976	5.4560
1.25	0.1623	0.1435	0.0717	0.1368	0.1189	. . .	3.7480	3.8796	9.4947	10.506

Table 18: $P_n(t)$ for the Exponential distribution with λ = 0.5

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	. . .	μ_{NUM}	μ_{ANA}	σ^2_{NUM}	σ^2_{ANA}
0.5	0.7788	0.0974	0.0061	0.0781	0.0097	. . .	0.5500	0.5500	1.6500	1.6500
1.0	0.6065	0.1516	0.0190	0.1229	0.0304	. . .	1.1000	1.1000	3.3000	3.3000
2.0	0.3679	0.1839	0.0460	0.1548	0.0745	. . .	2.2000	2.2000	6.6000	6.6000
3.0	0.2231	0.1673	0.0628	0.1496	0.1034	. . .	3.3000	3.3000	9.9000	9.9000
5.0	0.0821	0.1026	0.0641	0.1088	0.1110	. . .	5.5000	5.5000	16.500	16.500
10.0	0.0067	0.0168	0.0211	0.0310	0.0447	. . .	10.996	11.000	32.923	33.000

7.5 Program Validation

Although the use of software such as MAPLE enabled the successful programming of an algorithm to invert the required LTs and then produce the numerical results, in the case of bulk arrivals the distributions could only be calculated with an extremely high degree of accuracy for relatively small values of *t*. Certain distributions, such as the exponential, mixed generalized Erlang, and matrix exponential, have properties that enabled accurate computations over greater ranges of *t*, but in general, each of the trials was limited by the availability of computational power and the ability to collect a finite number of coefficients from the p.g.f.s using a crude, semi-manual coding procedure. Nevertheless, in each of the trials the asymptotic results derived in Chapter 5 for the case of single arrivals and in Chapter 6 for the cases of bulk arrivals were employed to illustrate that the computed distributions had first and second moments that matched the asymptotic results for higher values of *t*.

The *plot* function was used in each MAPLE program iteration to illustrate the degree of precision of the data sets. Program validation was based on a comparison of the analytical, asymptotic, and numerical results for the first and second moments of each distribution. As a result of the limitation of calculating a finite number of $P_n(t)$ values, the numerical results for greater values of *t* begin to diverge from the analytical and asymptotic results. A compromise was made between achieving a high degree of precision and limiting the programming and computational time, so this verification was particularly important to confirm the validity of the results within an acceptable range of *t* values.

A comparison of Figure 1 and Figure 2 suggests that higher precision for a greater range of *t* is achieved from using the 1-3-5 group size distribution versus the Poisson group size distribution for arrival times based on the matrix exponential distribution. In both cases, however, high precision was achieved at $t = 3$, a relatively high value in the context of renewal theory. This is further evidenced by the convergence of the analytic and asymptotic results at a value of approximately t = 1.5. A secondary method of program validation included the summation of all calculated $P_n(t)$ values to verify how closely they approached the expected value of 1. These methods are evident in the sample MAPLE printout in Appendix C.

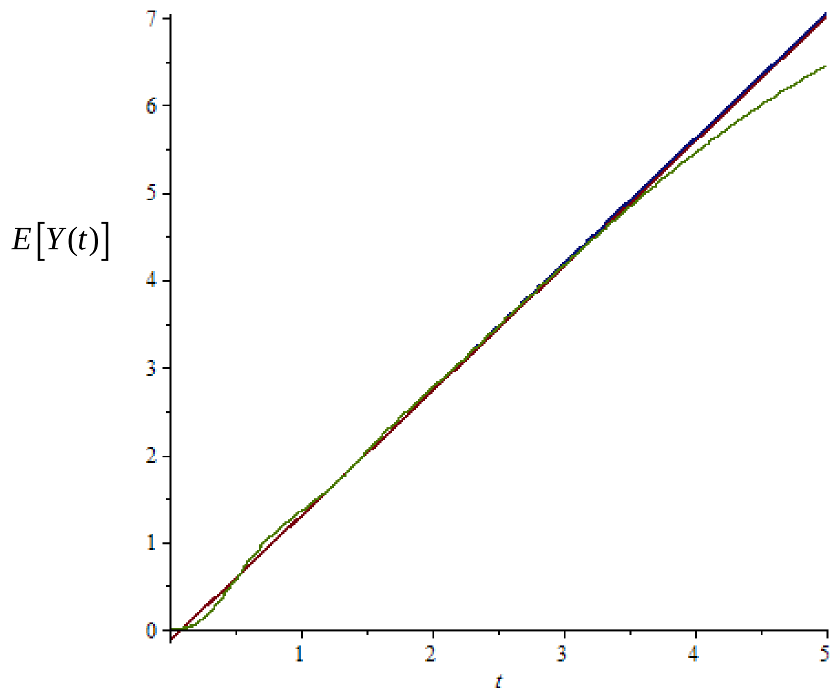

Figure 1: Comparison of the analytical (red), asymptotic (blue), and numerical (green) results for the first moment for the matrix exponential distribution with Poisson arrivals.

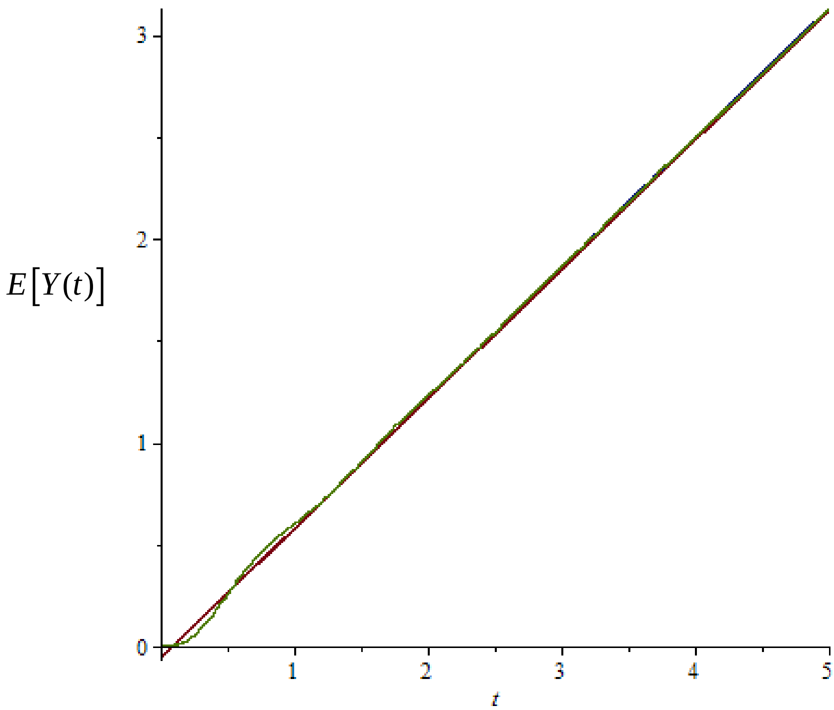

Figure 2: Comparison of the analytical (red), asymptotic (blue), and numerical (green) results for the first moment for the matrix exponential distribution with 1-3-5 arrivals.

8 HIGHER ORDER MOMENTS

Chapters 4 – 6 illustrate the development of a new derivation for some asymptotic results in renewal theory. In discrete time, these results are based on using generating functions to model the first and second moments, whereas in continuous time, Laplace transforms are used. This chapter presents the development of the g.f.s and LTs for the first five moments, illustrating that similar derivations for higher order moments can be achieved along similar lines, provided similar assumptions hold true. The discrete time moments match the results published by Brown (2008), but they are derived using a much simpler procedure. This approach is employed to derive similar results in continuous time.

8.1 Discrete Time

Consider the definition of the renewal function:

$$\begin{aligned} M_n^{(l)} &= E[N_n^l] \\ &= E\left[E[N_n^l \,|\, T = k] \right] \\ &= \sum_{k=0}^{n} E[(1+N_{n-k})^l] P(T=k). \end{aligned}$$

Its corresponding g.f. is:

$$\begin{aligned} M^{(l)}(z) &= \sum_{n=0}^{\infty} M_n^{(l)} z^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{l} \binom{l}{j} E[N_{n-k}^j] P(T=k) z^{n-k} z^k \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{n} P(T=k) z^k + \sum_{n=k}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{l} \binom{l}{j} E[N_{n-k}^j] z^{n-k} z^k \end{aligned}$$

$$M^{(l)}(z) = \frac{f(z)}{1-z} + \sum_{j=1}^{l} \binom{l}{j} M^{(j)}(z) f(z).$$

As a result, the appropriate g.f. can be derived for the first l^{th} moments:

l = 1:

$$M^{(1)}(z) = \frac{f(z)}{1-z} + M^{(1)}(z) f(z)$$
$$= \frac{f(z)}{(1-z)\left(1-f(z)\right)};$$

l = 2:

$$M^{(2)}(z) = \frac{f(z)}{1-z} + 2M^{(1)}(z) f(z) + M^{(2)}(z) f(z)$$
$$= \frac{f(z)+f^{2}(z)}{(1-z)\left(1-f(z)\right)^{2}};$$

l = 3:

$$M^{(3)}(z) = \frac{f(z)}{1-z} + 3M^{(1)}(z) f(z) + 3M^{(2)}(z) f(z) + M^{(3)}(z) f(z)$$
$$= \frac{f(z)+4f^{2}(z)+f^{3}(z)}{(1-z)\left(1-f(z)\right)^{3}};$$

l = 4:

$$M^{(4)}(z) = \frac{f(z)}{1-z} + 4M^{(1)}(z) f(z) + 6M^{(2)}(z) f(z) + 4M^{(3)}(z) f(z) + M^{(4)}(z) f(z)$$
$$= \frac{f(z)+11f^{2}(z)+11f^{3}(z)+f^{4}(z)}{(1-z)\left(1-f(z)\right)^{4}}; \text{ and}$$

l = 5:

$$M^{(5)}(z) = \frac{f(z)}{1-z} + 5M^{(1)}(z)\,f(z) + 10M^{(2)}(z)\,f(z) + 10M^{(3)}(z)\,f(z) + 5M^{(4)}(z)\,f(z) + M^{(5)}(z)\,f(z)$$
$$= \frac{f(z) + 26f^2(z) + 66f^3(z) + 26f^4(z) + f^5(z)}{(1-z)\left(1-f(z)\right)^5}.$$

8.2 Continuous Time

A similar approach can be taken to determine the LTs for higher moments in continuous time. The LT for the l^{th} moment is as follows:

$$\tilde{M}^{(l)}(s) = \frac{\tilde{f}(s)}{s} + \sum_{j=1}^{l} \binom{l}{j} \tilde{M}^{(j)}(s)\,\tilde{f}(s).$$

As a result, the appropriate LT can be derived for the first l^{th} moments:

l = 1:

$$\tilde{M}^{(1)}(s) = \frac{\tilde{f}(s)}{s} + \tilde{M}^{(1)}(s)\,\tilde{f}(s)$$
$$= \frac{\tilde{f}(s)}{s\left(1-\tilde{f}(s)\right)};$$

l = 2:

$$\tilde{M}^{(2)}(s) = \frac{\tilde{f}(s)}{s} + 2\tilde{M}^{(1)}(s)\,\tilde{f}(s) + \tilde{M}^{(2)}(s)\,\tilde{f}(s)$$
$$= \frac{\tilde{f}(s) + \tilde{f}^2(s)}{s\left(1-\tilde{f}(s)\right)^2};$$

l = 3:

$$\tilde{M}^{(3)}(s) = \frac{\tilde{f}(s)}{s} + 3\tilde{M}^{(1)}(s)\tilde{f}(s) + 3\tilde{M}^{(2)}(s)\tilde{f}(s) + \tilde{M}^{(3)}(s)\tilde{f}(s)$$
$$= \frac{\tilde{f}(s) + 4\tilde{f}^{2}(s) + \tilde{f}^{3}(s)}{s\left(1 - \tilde{f}(s)\right)^{3}};$$

l = 4:

$$\tilde{M}^{(4)}(s) = \frac{\tilde{f}(s)}{s} + 4\tilde{M}^{(1)}(s)\tilde{f}(s) + 6\tilde{M}^{(2)}(s)\tilde{f}(s) + 4\tilde{M}^{(3)}(s)\tilde{f}(s) + \tilde{M}^{(4)}(s)\tilde{f}(s)$$
$$= \frac{\tilde{f}(s) + 11\tilde{f}^{2}(s) + 11\tilde{f}^{3}(s) + \tilde{f}^{4}(s)}{s\left(1 - \tilde{f}(s)\right)^{4}};\text{ and}$$

l = 5:

$$\tilde{M}^{(5)}(s) = \frac{\tilde{f}(s)}{s} + 5\tilde{M}^{(1)}(s)\tilde{f}(s) + 10\tilde{M}^{(2)}(s)\tilde{f}(s) + 10\tilde{M}^{(3)}(s)\tilde{f}(s) + 5\tilde{M}^{(4)}(s)\tilde{f}(s) + \tilde{M}^{(5)}(s)\tilde{f}(s)$$
$$= \frac{\tilde{f}(s) + 26\tilde{f}^{2}(s) + 66\tilde{f}^{3}(s) + 26\tilde{f}^{4}(s) + \tilde{f}^{5}(s)}{s\left(1 - \tilde{f}(s)\right)^{5}}.$$

9 CONCLUSION

9.1 Future Extensions

Some of the results presented in this thesis have already inspired further graduate work in renewal theory, particularly in discrete time, but there are several other areas of consideration where more time and study could provide additional contributions to the field. Chapter 8 presents the development of generating functions and Laplace transforms for higher order moments in discrete time and continuous time, respectively. The algorithm used in deriving the results presented in Chapters 4 – 6 could likewise be applied to produce explicit results for each of the corresponding higher order moments. Since the literature had previously failed to provide a constant term for the second moment in single arrival renewal theory, deriving such precise results of the higher moments could prove most useful.

The development of numerical results in bulk renewal theory was another important contribution of this thesis, but the MAPLE program that was created to provide the results could certainly be improved. The latest version of this program does use Padé approximation when the LT of a distribution is not rational, but it does require the function to be of closed form. There are countless other distributions with varying characteristics that could have been included, but in the interest of practicality, six renewal time distributions and two bulk group size distributions were selected for this course of study.

The numerical results in bulk renewal theory are also limited by the value of t that is used. This is a result of the requirement to collect coefficients from the Taylor series expansion of the generating function for the distribution of group size using a semi-manual procedure and the very strong effect that increasing precision had on execution times. Due to a limited amount of time available to complete this research, there was a practical limitation for the number of coefficients that could be collected, and this lead to an important decision of creating a MAPLE code including only 13 such coefficients. With this program, a single iteration of a particular renewal time and group size distribution had execution times ranging between 11.5 - 26 minutes, although the computing resources available at the time were admittedly limited to an outdated Acer Aspire 5920G notebook with an Intel Core 2 Duo processor. Expanding the MAPLE program past 13 coefficients to produce precise results for a greater range of t would have required further coding time and considerably greater execution time. As a result, it was decided that the final code yielded sufficient results to make a useful contribution to the field, particularly when supported by the asymptotic values resulting from these new and elegant derivations.

9.2 Summary

Several new results in renewal theory have been presented in this thesis. Simple and elegant solutions for the asymptotic results for the first and second moments have been derived in both discrete and continuous time. In the latter case, the derivations were taken several steps further to provide similar results that account for bulk arrivals. Further, MAPLE was used in order to produce accurate numerical results in both single arrival and bulk arrival renewal theory in continuous time. These results have led to either the publication or submission of separate academic papers in journals related to the field, as well as to the continuation of research by other graduate students.

9.3 Thesis Contributions

The following is a list of contributions that are presented in this thesis:

- A new derivation for the asymptotic results of the first and second moments, including constant terms, in discrete time renewal theory
- A new derivation for the asymptotic results of the first and second moments, including constant terms, in continuous time renewal theory
- A new derivation for the asymptotic results of the first and second moments, including constant terms, in continuous time bulk renewal theory
- A MAPLE program producing numerical results in both single arrival and bulk renewal theory for a variety of renewal time and group size distributions
- A new derivation for the generating functions of higher order moments in discrete time for future research
- A new derivation for the Laplace transforms of higher order moments in continuous time for future research

REFERENCES

Abate, J., Whitt, W., 2006. A Unified Framework for Numerically Inverting Laplace Transforms. INFORMS Journal on Computing. 18(4), 408-421.

Abate, J., Whitt, W., 1995. Numerical inversion of Laplace transforms of probability distributions. ORSA Journal on Computing. 7, 36-43.

Baker Jr., G.A., Graves-Morris, P., 1996. Padé Approximants Second Edition (Encyclopedia of Mathematics and its Applications), Cambridge University Press.

Baxter, L.A., Scheuer, E.M., McConalogue, D.J., Blischke, W.R., 1982. On the Tabulation of the Renewal Function. Technometrics. 24, 151-156.

Beichelt, F.E., 2006. Stochastic Processes in Science, Engineering, and Finance. Chapman & Hall, Boca Raton.

Brown, G.W., 2008. Higher order moments of renewal processes and Eulerian polynomials, Statistics and Probability Letters. 78, 3008-3013.

Chaudhry, M.L., Templeton, J.G.C., 1983. A first course in bulk queues. John Wiley & Sons, New York.

Chaudhry, M.L. 1995. On computations of the mean and variance of the number of renewals: A unified approach. The Journal of the Operations Research Society. 46, 1352-1364.

Chaudhry, M.L., Fisher, B., 2012. Simple and elegant derivations for some asymptotic results in the discrete-time renewal process. Statistics and Probability Letters. 83(1), 315-319.

Chaudhry, M.L., Samanta, S.K., Pacheco, A., 2012. Analytically explicit results for the GI/C-MSP/1/∞ queueing system using roots. Probability in the Engineering and Informational Sciences. 26, 221-244.

Chaudhry, M.L., Yang, X., Ong, B., 2013. Computing the Distribution Function of the Number of Renewals. American Journal of Operations Research. 3(3), 380-386.

Chaudhry, M.L., Kim, J., 2013. Asymptotic Results for the First and Second Moments of Discrete-Time Bulk-Renewal Process. Journal of Mathematics and System Science (accepted Dec 2013).

Cox, D.R., 1962. Renewal Theory. Spottiswoode Ballantyne & Co Ltd, London.

Fackrell, M., 2004. Characterization of Matrix-exponential distributions. Department of Mathematics and Statistics. Accessed 25 November, 2012. http://www.or.ms.unimelb.edu.au/Talk_Slides_2004/talk_ORSUM_2004_PDF.pdf.

Feller, W., 1949. Fluctuation theory of recurrent events. Transactions of the American Mathematical Society. 67, 98-119.

Feller, W., 1968. An Introduction to Probability Theory and its Applications Volume 1. John Wiley & Sons, New York.

Fisher, B., Chaudhry, M.L., 2014. Computing the distribution for the number of renewals with bulk arrivals. INFORMS Journal on Computing (accepted Mar 2014).

Fisher, B., 2010. Computation Techniques for Finding Distributions and Performance Measures for the Qu,eueing System GI/G/1. Honours Mathematics Senior Project, Royal Military College of Canada, Kingston, ON.

Harris, C.M., Marchal, W.G., 1998. Distribution estimation using Laplace transform. INFORMS Journal on Computing. 10, 448-458.

Heyman, D.P., Sobel, M.J., 1982. Stochastic Models in Operations Research Volume 1:

Stochastic Processes and Operating Characteristics, McGraw-Hill Book Company, New York.

Hunter, J., 1983. Mathematical Techniques of Applied Probability Volume 1: Discrete Time Models: Basic Theory. Academic Press, New York.

Karlin, S., Taylor, H.M., 1975. A First Course in Stochastic Processes, second ed. Academic Press, New York.

Kohlas, J., 1982. Stochastic methods of operations research. Cambridge University Press, Cambridge.

Parzen, E., 1962. Stochastic Processes. Holden-Day, Inc, San Francisco.

Polya, G., 1954. Mathematics and Plausible Reasoning, Vol. 1: Induction and Analogy in Mathematics. Princeton University Press, Princeton.

Rackwitz, R., 2001 Optimizing systematically renewed structures. Reliability Engineering and System Safety. 73, 269-279.

Tijms, H., 2003. A First Course in Stochastic Models, Wiley, New York.

Van der Weide, J.A.M., Pandey, M.D., Noortwijk, J.M., 2007. A conceptual interpretation of the renewal theorem with applications, in: Aven, T., Vinnem, J.E. (Eds.), Risk, Reliability and Societal Safety. Taylor & Francis Group, London, pp. 477-484.

Van Noortwijk, J.M., 2003. Explicit formulas for the variance of discounted life-cycle cost. Reliability Engineering and System Safety. 80, 185-195.

APPENDIX A ELEMENTARY CONCEPTS

All definitions and descriptions assume the use of *lattice random variables* (r.v.s), which has all values that are integer multiples of a number. Mathematically, if X is a lattice r.v., then $P(X = x) = 0$ if $x \neq nd$, $d \geq 0$, and $n = 0, 1, 2, \ldots$

The descriptions of elementary concepts, unless otherwise noted, are based off the work of Chaudhry and Templeton (1983).

A.1 Generating Function

The generating function (g.f.) is a power series, typically expressed in closed form, that encodes information about a sequence of numbers. If a sequence of numbers composes a probability distribution function, then the corresponding g.f. is called a probability generating function (p.g.f.). Both g.f.s and p.g.f.s have been compared with a bag that allows a user to carry multiple pieces of information regarding a specific sequence while only dealing with one piece (Polya, 1954, and Chaudhry and Templeton, 1983).

Let X be a nonnegative lattice r.v. assuming integral values 0, 1, 2, ..., such that $P(X = n) = p_X(n)$. The sequence of probabilities $\{p_X(n), n \geq 0\}$ is proper if $p_X(n) \geq 0$ and $\sum_{n=0}^{\infty} p_X(n) = 1$. It is generally assumed that all probability distributions are proper. The p.g.f. of the random variable X is defined by Chaudhry and Templeton (1983) as follows:

$$P_X(z) = \sum_{n=0}^{\infty} p_X(n) z^n = E(z^X).$$

$P_X(z)$ is an analytic function of z, a complex variable, and as a result the theory of analytic functions can be used to obtained results concerning $\{p_X(n)\}$. $P_X(z)$ is absolutely convergent for $|z| \le 1$ since $P_X(1) = 1$.

The mean and variance of a probability distribution are easy to derive from a p.g.f. as illustrated by the following properties:

$$m_X = E(X) = \sum_{n=0}^{\infty} n p_X(n) = P^{(1)}(1),$$

where

$$P^{(r)}(1) = \left.\frac{d^r P(z)}{dz^r}\right|_{z=1}, \quad r = 1, 2, 3, \ldots$$

Further, since

$$E\left(X(X-1)\right) = \sum_{n=0}^{\infty} n(n-1) p_X(n) = P^{(2)}(1),$$

the variance σ_x^2 is given by

$$\sigma_x^2 = P^{(2)}(1) + P^{(1)}(1) - \left(P^{(1)}(1)\right)^2.$$

Higher order moments may be obtained similarly.

A.2 Laplace Transform

The Laplace transform (LT) is employed in a very similar manner as the generating function. LTs provide a bag to collect random variables just as p.g.f.s provide a bag to collect integer valued r.v.s. As such, LTs are used in continuous time renewal theory whereas p.g.f.s are used when working in discrete time.

Assuming $f(t)$ is a probability density function (p.d.f.) of a non-negative r.v. *T*, then the LT $\tilde{f}(s)$ is defined as

$$\tilde{f}(s) = E\left[e^{-st}\right] = \int_{t=0}^{\infty} e^{-st} f(t)dt.$$

$\tilde{f}(0) = 1$ and $\tilde{f}(s)$ is an analytic function in the half-place, $\mathrm{Re}(s) > s_0$, where $s_0 \leq 0$, since $0 \leq \tilde{f}(s) \leq 1$ for all $s \geq 0$. This characteristic justifies several of the important derivations made using LTs.

Since the LT is a moment generating function, the moments of the distribution $f(t)$ can be easily obtained by the alternate form:

$$\tilde{f}(s) = \sum_{r=0}^{\infty} \frac{(-1)^r s^r \mu'^r}{r!}.$$

As a result, the first moment, or mean, is

$$E\left[T\right] = -\tilde{f}\,'(s)$$

and the second moment is $E\left[T^2\right] = \tilde{f}\,''(s)$ such that the variance is

$$\sigma_T^2 = \tilde{f}\,''(0) - \left(\tilde{f}\,'(0)\right)^2.$$

There is a more general form of the LT, called the Laplace-Stieltjes transform (LST), defined as

$$\tilde{f}(s) = \int_{t=0}^{\infty} e^{-st} dF_T(t).$$

It can be advantageous to use the LST since it is even more closely related to the p.g.f. than the LT. In fact, these latter two only differ by the change in variable $z = e^{-s}$.

A.3 Inversion of Laplace Transforms and the Padé Method

If $\tilde{f}(s)$ is the LT of $f(t)$, then $f(t)$ is the inverse LT of $\tilde{f}(s)$. The inversion of LTs is required to derive both the asymptotic results in renewal theory as well as the distribution of the number of renewals. The inverse of $\tilde{f}(s)$ is defined as

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\tilde{f}(s)ds,$$

where the contour is any vertical line $s = a$ so $\tilde{f}(s)$ has no singularities on, or to the right of it (Abate and Whitt, 1995). This type of integration is not always easily accomplished, even with powerful computing software such as MAPLE.

$\tilde{f}(s)$ should be rational in order to be easily inverted. Many common distributions are not rational, however, so a rational function $\hat{f}^*(s)$ is used to approximate $\tilde{f}(s)$. As illustrated in Chaudhry, Yang, and Ong (2013), a useful method of rational approximation is the Padé method, where $\tilde{f}(s)$ can be expanded as

$$\tilde{f}(s) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n!}M_n s^n.$$

In this expression, $M_n = \int_0^{\infty} x^n dF(x)$ is the n^{th} moment of the inter-renewal time. As a result, a rational approximation function can be expressed as

$$\hat{f}^*(s) = \frac{\tilde{N}(s)}{\tilde{D}(s)} = \frac{\sum_{n=0}^{K} b_n s^n}{\sum_{n=0}^{L} a_n s^n},$$

where $\tilde{N}(s)$ and $\tilde{D}(s)$ are polynomials of degree K and L, respectively, with undetermined coefficients b_n and a_n, such that the first $K + L$ moments of $\tilde{f}(s)$ are equal to those of $\hat{f}^*(s)$. It is from these moments that the Padé approximation notation $[K/L]$ originates. The Padé method requires high numerical precision in computations since the coefficients b_n and a_n are uniquely determined once a_0 is set to 0, *K* and *L* are selected through trial and error, and the process of equating the moments and formulating the simultaneous equations is completed. For details, see Harris and Marchal (1998) or Baker Jr. and Graves-Morris (1996).

The primary goal of inversion in this thesis is to determine values for $P_n(t)$, the probability of *n* renewals occurring between $[0,t)$. Since the Padé method can be used to approximate any non-rational function as rational, the Partial Fraction method of inversion can be used for the inversion of all closed-form probability distributions of interest.

By assuming that

$$\tilde{f}(s) = \frac{\tilde{N}(s)}{\tilde{D}(s)}$$

and considering the expression

$$\frac{\tilde{p}_n(s)}{s} = \frac{\tilde{f}^n(s)\left(1-\tilde{f}(s)\right)}{s},$$

the following rational function is derived:

$$\frac{\tilde{p}_n(s)}{s} = \frac{\tilde{N}^n(s)\left[\tilde{D}(s)-\tilde{N}(s)\right]}{s\tilde{D}^{n+1}(s)}. \tag{A.1}$$

It is assumed that the equation $\tilde{D}(s)=0$ has *k* distinct roots $s_1, s_2, \ldots, s_k$, and since $\tilde{f}(0)=1$, $\tilde{N}(s)$ and $\tilde{D}(s)$ have identical constant terms,

$$\tilde{D}^{n+1}(s) = \left(s - s_1\right)^{n+1}\left(s - s_2\right)^{n+1} \ldots \left(s - s_k\right)^{n+1}.$$

Eq. (A.1) can be expressed in partial fractions as

$$\frac{\tilde{p}_n(s)}{s} = \sum_{j=1}^{k}\sum_{i=1}^{n+1}\frac{A_{j,i}}{(s - s_j)^i},$$

where the constant coefficient $A_{j,i}$ is given by

$$A_{j,i} = \frac{1}{(i-1)!}\frac{d^{(i-1)}}{ds^{i-1}}\left[\frac{\tilde{p}_n(s)}{s}\left(s - s_j\right)^i\right]_{s=s_j}.$$

Consequently, the final inversion can be written as

$$P_n(t) = \sum_{j=1}^{k}\sum_{i=1}^{n+1}\frac{A_{j,i}}{(i-1)!}t^{i-1}e^{s_j t}.$$

The case where $\tilde{D}(s) = 0$ has repeated roots can be dealt with similarly.

A.4 Asymptotic Theory

Asymptotic results in renewal theory are concerned with the behaviour of a function as *t* tends to infinity. These are often called limiting distributions, and they were the focus of attention throughout Chapters 4 – 6. Asymptotic distributions are related to the notion of an asymptotic function, whereby a curve approaches a constant value, or asymptote, but never actually reaches it.

Clearly, for larger values of *t*, portions of a function concerning *t* will have a greater effect on a solution than constant terms, thus explaining why limiting distributions are of such great interest. Figure 1 and Figure 2 in Chapter 7 illustrate how the behaviour of distribution functions in renewal theory are not correlated with the limiting distributions for small values of *t*, but as *t* increases the two values begin to converge.

Asymptotic results are often easier to manipulate and to use, as illustrated by the *Elementary Renewal Theorem*:

$$\frac{M(t)}{t} \to \frac{1}{\mu} \text{ as } t \to \infty,$$

where the renewal process $m(t)$ goes to infinity at an average rate of $\frac{1}{\mu}$ (Chaudhry and Templeton, 1983).

APPENDIX B PROBABILITY DISTRIBUTIONS

A probability density function (p.d.f.) is assumed to be zero for negative values of *x* or *t* in discrete time and continuous time, respectively.

B.1 Poisson

The Poisson distribution is among the most common of discrete probability distributions. It provides the probability of a number of independent events occurring in an interval of time based on an average rate of occurrence. Parzen (1962) argues that events are said “to occur randomly in time if they are occurring in accord with a Poisson process” (p. 15). The p.d.f. of the Poisson distribution is

$$f(x) = e^{-\lambda}\frac{\lambda^x}{x!}, \qquad x = 0,1,\ldots$$

with mean λ and variance λ.

The probability generating function (p.g.f.) of the Poisson distribution is

$$P_X(z) = e^{-\lambda(1-z)}.$$

In continuous time, the Poisson distribution is also related with the exponential distribution, with p.d.f.

$$f(t) = \lambda e^{-\lambda t}, \qquad t > 0$$

and Laplace Transform (LT)

$$\tilde{f}(s) = \frac{\lambda}{s+\lambda}.$$

The Poisson distribution is used as the group size distribution in this thesis, whereas the exponential distribution is used for the renewal time distribution.

B.2 Binomial

The binomial distribution is another common discrete probability distribution. It provides the probability of a number of events occurring from a set number of independent trials or attempts, based on the probability of a success or failure each time. The p.d.f. of the binomial distribution is

$$f(x) = \binom{m}{x} p^x q^{m-x}, \qquad x = 0,1,\ldots,m$$

with mean np and variance npq, where $n = 1,2,\ldots$ and $0 \le p,q \le 1$, $p+q=1$.

The p.g.f. of the binomial distribution is

$$P_X(z) = \left(pz+q\right)^m.$$

The binomial distribution in its general form cannot be used as a p.g.f. for group size for computing the distribution of the number of renewals. This is an important point to consider since it is a common distribution that would appear ideal for use as a group size. With that being said, it would be quite simple to restructure the p.g.f. for a specific example of a binomially distributed group size into an appropriate form. For example, for a binomial distribution where p = 0. 5, q = 0. 5, and m = 2, the p.g.f. would be $P_X(z) = 0.25 + 0.5z + 0.25z^2$.

B.3 Mixed Generalized Erlang

The Erlang distribution is used in continuous time and is an extension of the exponential distribution. Whereas the number of events that can incur in a time interval are modelled by an exponential distribution, the waiting times between a number of occurrences of an event are modelled by the Erlang distribution. As such, the Erlang distribution is the sum of independent events having an exponential distribution. The p.d.f. of the Erlang distribution is

$$f(t)=\frac{\lambda^k t^{k-1}e^{-\lambda t}}{(k-1)!},\qquad t\geq 0$$

with shape parameter k, mean $\frac{k}{\lambda}$, and variance $\frac{k}{\lambda^2}$, where $k=1,2,\ldots$ and $\lambda>0$.

The LT of the Erlang distribution is

$$\tilde{f}(s)=\left(\frac{\lambda}{s+\lambda}\right)^k.$$

The Mixed Generalized Erlang distribution differs from the more general Erlang distribution since different sets of occurrence are given greater priority over others through the use of weighting coefficients. The p.d.f. of the Mixed Generalized Erlang distribution is

$$f(t)=\sum_{j=1}^{k}c_j\frac{\lambda^j t^{j-1}e^{-\lambda t}}{(j-1)!},$$

where $\sum_{j=1}^{k}c_j=1$. As a result, the LT of the Mixed Generalized Erlang distribution is

$$\tilde{f}(s)=\sum_{j=1}^{k}c_j\left[\frac{\lambda}{\lambda+s}\right]^j.$$

The Mixed Generalized Erlang distribution is used solely for the renewal time distribution in this thesis.

B.4 Matrix Exponential

Although it does not have a simple probabilistic interpretation, the matrix exponential distribution is a specific class of exponential distributions that is versatile, dense, and algorithmically tractable (Fackrell, 2004). Its p.d.f. is

$$f(t) = \beta e^{Mt} m,$$

where M is a state space Matrix $M = \{0,1,\ldots,p\}$ of a continuous-time Markov chain. Its LT is

$$\tilde{f}(s) = \beta\left(sI - M\right)^{-1} m + f(0).$$

As there are no restrictions on parameters β, M, or m, other than that they correspond to a distribution, there are a wide variety of applications for matrix exponential distributions in fields such as queueing theory, telecommunications, control risk, and insurance risk (Fackrell, 2004). The matrix exponential distribution is used solely for the renewal time distribution in this thesis.

B.5 Gamma

The gamma distribution is also used in continuous time, and provides the waiting time required to observe a set number of occurrences of a specified event. It is a common two-parameter distribution, with p.d.f.

$$f(t) = \frac{t^{\alpha-1} e^{-t/\beta}}{\beta^{\alpha}\Gamma(\alpha)}, \qquad t > 0,$$

where $\alpha > 0$ is the α^{th} occurrence of an event under consideration, $\frac{1}{\beta} > 0$ is the mean rate at which events occur, and $\Gamma(\alpha)$ is the gamma function, defined as $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. α and β are commonly referred to as the shape and scale parameters, respectively. The mean is $\alpha\beta$ and variance is $\alpha\beta^2$. The LT of the gamma distribution is

$$\tilde{f}(s) = \frac{1}{(1+\beta s)^\alpha}.$$

The gamma distribution is used solely for the renewal time distribution in this thesis.

B.6 Truncated Normal

The normal distribution is perhaps the most practical of all distributions, as a great deal of natural phenomena closely follow it. A perfectly symmetrical distribution, the normal distribution is shaped like a bell, with the mean, median, and mode of the distribution all occurring at the centre. Its p.d.f. is

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-\mu)^2/2\sigma^2}, \qquad -\infty < t < \infty,$$

where $\mu > 0$ is the mean and $\sigma^2 > 0$ is the variance.

The truncated normal distribution is very similar to the normal distribution except that its value can be bounded. This p.d.f. is

$$f(t) = \frac{1}{a\sigma\sqrt{2\pi}} e^{-(t-\mu)^2/2\sigma^2}, \qquad t > 0,$$

where $a = 1 - \Phi(\frac{-\mu}{\sigma})$, with $\Phi(\cdot)$ being the standard normal distribution function. The truncated normal distribution requires Padé approximation and is used solely for the renewal time distribution in this thesis.

B.7 Inverse Gaussian

The inverse Gaussian, or Wald distribution, is another two-parameter distribution used in continuous time. Its p.d.f. is

$$f(t) = \sqrt{\frac{\lambda}{2\pi t^3}} e^{-\lambda(t-\mu)^2/2\mu^2 t}, \qquad t > 0,$$

where $\mu > 0$ is the mean and $\lambda > 0$ is the shape parameter. Its variance is $\frac{\mu^3}{\lambda}$, and as $\lambda \to 0$, the inverse Gaussian distribution takes on the shape of the normal distribution. The inverse Gaussian distribution can also be expressed in terms of $\phi = \frac{\lambda}{\mu}$ and $\gamma = \frac{\mu^2}{\lambda}$, its shape and scale parameters, respectively. These parameters are used in the most common expression of the LT of the distribution, which is

$$\tilde{f}(s) = e^{\phi\left(1-\sqrt{1+2s/\gamma}\right)}.$$

The inverse Gaussian distribution is used solely for the renewal time distribution in this thesis.

APPENDIX C SAMPLE MAPLE PRINTOUT

The following is a printout of the algorithm programmed in MAPLE to produce the numerical results presented in Chapter 7:

```
>
  # NUMERICAL RESULTS IN RENEWAL THEORY
      CALCULATOR - SINGLE AND BULK ARRIVALS

      # BRENT FISHER - BASED ON PRIOR RESEARCH BY
      XIAOFENG YANG

  # ROYAL MILITARY COLLEGE OF CANADA, 2012-2013

  # RENEWAL DISTRIBUTION : GAMMA
  #    PADE METHOD: [4-5]
  # GROUP SIZE DISTRIBUTION : SAMANTA

> restart;
>
  with(inttrans) :
  with(LinearAlgebra) :
  with(numapprox) :
  with(Statistics) :

> #PRECISION PARAMETERS Digits := 200;

> # NUMBER OF RENEWALS - USED IN DETERMINING Pn(t)'s

  NN := 13;

>
> # GROUP SIZE PARAMETERS - USED FOR BULK ARRIVALS
> # assume number of renewals inside each group follow the distribution
      studied by Samanta:
```

> $PxZ := 0.5\,z + 0.4\,z^3 + 0.1z^5$; *# test with single arrival pgf#PxZ:=z*

> $\frac{d}{dz}\,PxZ;\ \frac{d}{dz}\left(\frac{d}{dz}\,PxZ\right);$

> $dPx1 := simplify\left(subs\left(z = 1, \frac{d}{dz}\,PxZ\right)\right);$

$ddPx1 := simplify\left(subs\left(z = 1, \frac{d}{dz}\left(\frac{d}{dz}\,PxZ\right)\right)\right);$

```
> # RENEWAL DISTRIBUTION: f(t)
```

```
> # we will work on GAMMA DISTRIBUTION  function defined below as
      ftorig:
```

>

$$alfa := 0.55;$$

$$ftorig := \frac{t^{(alfa-1)}}{\Gamma(alfa)} \exp(-t);$$

$$Ftorig := \int_0^t ftorig \,\mathrm{d}t;$$

$$\mathrm{meantest} := \mathrm{convert}\left(\int_0^{20} t \cdot ftorig \,\mathrm{d}t, float\right);$$

$$secondmoment := convert\left(\int_0^{20} t^2 \cdot ftorig \,\mathrm{d}t, float\right);$$

$$sigmatest := secondmoment - meantest^2;$$

> $plot(ftorig, t = 0.1 ..3);$

> $plot(Ftorig, t = 0 ..10);$

> *# we need up to Nth moments*

> $N := 32;$

>

$M := Vector_{column}(N);$

for i **from** 1 **to** N **do**

$$M[i] := evalf\left(\int_0^{\infty} t^{i-1} \cdot ftorig \,\mathrm{d}t\right);$$

end do;

>

$unassign('i');$

$$tmpf := i \rightarrow \frac{(-1)^{i-1} \cdot s^{i-1}}{(i-1)!};$$

$tmpvector := Vector_{column}(N, tmpf);$

 # Fs:=tmpvector.M;

$Fs1 := 0;$

$unassign('i');$

for i **from** 1 **to** N **do**

$Fs1 := Fs1 + tmpvector_i \cdot M_i;$

end do;

> $Fs := pade(Fs1, s, [4, 5]);$

> $ft := invlaplace(Fs, s, t);\; Ft := \int_0^t ft \,\mathrm{d}t;$

```
> plot([ft,ftorig], t = 0 ..4);
  plot([ft,ftorig], t = 0 ..2);
  plot([ft,ftorig], t = 0 ..0.2);

> plot([ft,ftorig], t = 3 ..5);
  plot([Ft, Ftorig], t = 0 ..2);
  plot([Ft, Ftorig], t = 0.01 ..0.2);

> taylor(Fs, s, 20);
>

  # SINGLE ARRIVAL ANALYTIC RESULTS - MEAN AND VARIANCE:
```

> $sENs := \frac{Fs}{s\cdot(1-Fs)}$; $simplify(sENs)$;

> $sENt := invlaplace(sENs, s, t)$;

> $sVNs1 := \frac{Fs\cdot(1+Fs)}{s\cdot(1-Fs)^2}$; $simplify(sVNs1)$;

> $sVNt1 := invlaplace(sVNs1, s, t)$;

> $sVNt := sVNt1 - sENt^2$;

> simplify($sVNt$);

>

> $evalf(subs(t = 10, sENt))$;

> $evalf(subs(t = 0.5, sVNt))$;

>

> $spns := \frac{Fs^n\cdot(1-Fs)}{s}$; $simplify(spns)$;

> $spnt := invlaplace(spns, s, t)$;

> # PREPARE pns FOR LOOP

> $spns := \frac{Fs^n\cdot(1-Fs)}{s}$; $simplify(spns)$;

>

```
  # CALCULATE Pn(t)'s FOR SINGLE ARRIVAL CASE:

  spnstmp := spns;
  for l from 0 to NN do
  spnstmp := subs(n = l, spns) :
  pnt[l] := invlaplace(spnstmp, s, t);
  #pntvalue:=subs(n = l, t = 4, pnt) :
  #sxx[l] := evalf(pntvalue);
  end do;
```

> M1:=-subs(s=0, diff(Fs, s));

```
M2:=subs(s=0, diff(diff(Fs, s),s))/2;
> u := M1;
usquare := M1^2;
sigmasquare :=M2*2 - usquare;
> smt := t/u + (sigmasquare - usquare)/2/usquare;
> unassign(q);
>

  # NUMERICAL MEAN

  sMENt := 0;
  for q to NN do
  sMENt := sMENt + pnt[q]·q
  end do

>
>


     # GRAPHICAL COMPARISON OF ANALYTIC (sENt),
     ASYMPTOTIC (smt), AND NUMERICAL (sMENt) RESULTS


> plot([smt,sENt], t = 0 ..2);
> plot([smt, sENt], t = 0 ..5);
> plot([smt, sENt], t = 0 ..20);
>
> plot([smt, sENt, sMENt], t = 0 ..2);
> plot([smt, sENt, sMENt], t = 0 ..5);
> plot([smt, sENt, sMENt], t = 0 ..10);
> plot([smt, sENt, sMENt], t = 0 ..20);
> plot([smt, sENt, sMENt], t = 0 ..30);
>
  # NUMERICAL RESULTS FOR SINGLE ARRIVALS - VARIOUS
     VALUES OF t
```

\>

$steval := 0.1;$
for c **from** 0 **to** NN **do**
$sxx[c] := evalf(subs(t = steval, pnt[c]));$
end do;
$unassign('j');$

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

$evalf(subs(t = steval, sENt));$

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

$evalf(subs(t = steval, sVNt));$

\>

$steval := 0.15;$
for c **from** 0 **to** NN **do**
$sxx[c] := evalf(subs(t = steval, pnt[c]));$
end do;
$unassign('j');$

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

$evalf(subs(t = steval, sENt));$

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

$evalf(subs(t = steval, sVNt));$

```
>
steval := 0.25;
 for c from 0 to NN do
sxx[c] := evalf(subs( t = steval, pnt[c]));
end do;
 unassign('j');
```

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

$evalf(subs(t = steval, sENt));$

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

$evalf(subs(t = steval, sVNt));$

```
>
steval := 0.4;
 for c from 0 to NN do
sxx[c] := evalf(subs( t = steval, pnt[c]));
end do;
 unassign('j');
```

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

$evalf(subs(t = steval, sENt));$

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

$evalf(subs(t = steval, sVNt));$

```
>
steval := 0.45;
 for c from 0 to NN do
sxx[c] := evalf(subs( t = steval, pnt[c]));
end do;
 unassign('j');
```

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

```
 evalf(subs(t = steval, sENt));
```

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

```
evalf(subs(t = steval, sVNt));
```

```
>
steval := 0.5;
 for c from 0 to NN do
sxx[c] := evalf(subs( t = steval, pnt[c]));
end do;
 unassign('j');
```

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

```
 evalf(subs(t = steval, sENt));
```

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

```
evalf(subs(t = steval, sVNt));
```

```
>
steval := 0.6;
 for c from 0 to NN do
sxx[c] := evalf(subs( t = steval, pnt[c]));
end do;
 unassign('j');
```

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

```
 evalf(subs(t = steval, sENt));
```

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

```
evalf(subs(t = steval, sVNt));
```

```
>
steval := 0.7;
 for c from 0 to NN do
sxx[c] := evalf(subs( t = steval, pnt[c]));
end do;
 unassign('j');
```

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

```
 evalf(subs(t = steval, sENt));
```

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

```
evalf(subs(t = steval, sVNt));
```

```
>
steval := 0.75;
for c from 0 to NN do
sxx[c] := evalf(subs( t = steval, pnt[c]));
end do;
unassign('j');
```

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

$evalf(subs(t = steval, sENt));$

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

$evalf(subs(t = steval, sVNt));$

```
>
steval := 1;
for c from 0 to NN do
sxx[c] := evalf(subs( t = steval, pnt[c]));
end do;
unassign('j');
```

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

$evalf(subs(t = steval, sENt));$

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

$evalf(subs(t = steval, sVNt));$

\>
$steval := 1.25;$
for c **from** 0 **to** NN **do**
sxx[c] := $evalf(subs(\mathrm{t} = steval, \mathrm{pnt}[\mathrm{c}]));$
end do;
$unassign('j');$

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

$evalf(subs(t = steval, sENt));$

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

$evalf(subs(t = steval, sVNt));$

\>
$steval := 2;$
for c **from** 0 **to** NN **do**
sxx[c] := $evalf(subs(\mathrm{t} = steval, \mathrm{pnt}[\mathrm{c}]));$
end do;
$unassign('j');$

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

$evalf(subs(t = steval, sENt));$

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

$evalf(subs(t = steval, sVNt));$

>

$steval := 5;$
for c **from** 0 **to** NN **do**
$\mathrm{sxx}[\mathrm{c}] := evalf(subs(\mathrm{t} = steval, \mathrm{pnt}[\mathrm{c}]));$
end do;
$unassign('j');$

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

$evalf(subs(t = steval, sENt));$

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

$evalf(subs(t = steval, sVNt));$

>

$steval := 10;$
for c **from** 0 **to** NN **do**
$\mathrm{sxx}[\mathrm{c}] := evalf(subs(\mathrm{t} = steval, \mathrm{pnt}[\mathrm{c}]));$
end do;
$unassign('j');$

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

$evalf(subs(t = steval, sENt));$

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

$evalf(subs(t = steval, sVNt));$

>

```
>
steval := 20;
for c from 0 to NN do
sxx[c] := evalf(subs( t = steval, pnt[c]));
end do;
unassign('j');
```

$$sxxsum := \sum_{j=0}^{NN} sxx[j];$$

$$sxxmean := \sum_{j=0}^{NN} j \cdot sxx[j];$$

$evalf(subs(t = steval, sENt));$

$$sxxvar := \sum_{j=0}^{NN} j^2 \cdot sxx[j] - sxxmean^2;$$

$evalf(subs(t = steval, sVNt));$

\> # BULK ARRIVALS:

\> # ANALYTIC RESULTS - MEAN AND VARIANCE:

\> $ENs := \frac{Fs}{s \cdot (1 - Fs)} \cdot dPx1;$ $simplify(ENs);$

\> $ENt := invlaplace(ENs, s, t);$

\> $VNs1 := \frac{Fs}{s \cdot (1 - Fs)} \cdot \left(ddPx1 + \frac{2 \cdot Fs \cdot (dPx1)^2}{(1 - Fs)} + dPx1 \right);$
$simplify(VNs1);$

\> $VNt1 := invlaplace(VNs1, s, t);$

\> $VNt := VNt1 - ENt^2;$

\> simplify(VNt);

\>

\> $evalf(subs(t = 10, ENt));$

\> $evalf(subs(t = 0.5, VNt));$

\>

\> # PREPARE pns FOR LOOP

\>

\> #FIND COEFFICIENTS FOR USE IN BULK PGF

```
> 
PXZ1 := taylor(PxZ, z, 14);
z0100 := coeff(PXZ1, z, 0);
z0101 := coeff(PXZ1, z, 1);
z0102 := coeff(PXZ1, z, 2);
z0103 := coeff(PXZ1, z, 3);
z0104 := coeff(PXZ1, z, 4);
z0105 := coeff(PXZ1, z, 5);
z0106 := coeff(PXZ1, z, 6);
z0107 := coeff(PXZ1, z, 7);
z0108 := coeff(PXZ1, z, 8);
z0109 := coeff(PXZ1, z, 9);
z0110 := coeff(PXZ1, z, 10);
z0111 := coeff(PXZ1, z, 11);
z0112 := coeff(PXZ1, z, 12);
z0113 := coeff(PXZ1, z, 13);
```

```
> 
PXZ2 := taylor(PxZ^2, z, 14);
z0200 := coeff(PXZ2, z, 0);
z0201 := coeff(PXZ2, z, 1);
z0202 := coeff(PXZ2, z, 2);
z0203 := coeff(PXZ2, z, 3);
z0204 := coeff(PXZ2, z, 4);
z0205 := coeff(PXZ2, z, 5);
z0206 := coeff(PXZ2, z, 6);
z0207 := coeff(PXZ2, z, 7);
z0208 := coeff(PXZ2, z, 8);
z0209 := coeff(PXZ2, z, 9);
z0210 := coeff(PXZ2, z, 10);
z0211 := coeff(PXZ2, z, 11);
z0212 := coeff(PXZ2, z, 12);
z0213 := coeff(PXZ2, z, 13);
```

```
> 
PXZ3 := taylor(PxZ^3, z, 14);
z30 := coeff(PXZ3, z, 0);
z31 := coeff(PXZ3, z, 1);
z32 := coeff(PXZ3, z, 2);
z33 := coeff(PXZ3, z, 3);
z34 := coeff(PXZ3, z, 4);
z35 := coeff(PXZ3, z, 5);
z36 := coeff(PXZ3, z, 6);
z37 := coeff(PXZ3, z, 7);
z38 := coeff(PXZ3, z, 8);
z39 := coeff(PXZ3, z, 9);
z310 := coeff(PXZ3, z, 10);
z311 := coeff(PXZ3, z, 11);
z312 := coeff(PXZ3, z, 12);
z313 := coeff(PXZ3, z, 13);
```

\>

$PXZ4 := taylor(PxZ^4, z, 14);$
$z40 := coeff(PXZ4, z, 0);$
$z41 := coeff(PXZ4, z, 1);$
$z42 := coeff(PXZ4, z, 2);$
$z43 := coeff(PXZ4, z, 3);$
$z44 := coeff(PXZ4, z, 4);$
$z45 := coeff(PXZ4, z, 5);$
$z46 := coeff(PXZ4, z, 6);$
$z47 := coeff(PXZ4, z, 7);$
$z48 := coeff(PXZ4, z, 8);$
$z49 := coeff(PXZ4, z, 9);$
$z410 := coeff(PXZ4, z, 10);$
$z411 := coeff(PXZ4, z, 11);$
$z412 := coeff(PXZ4, z, 12);$
$z413 := coeff(PXZ4, z, 13);$

\>

$PXZ5 := taylor(PxZ^5, z, 14);$
$z50 := coeff(PXZ5, z, 0);$
$z51 := coeff(PXZ5, z, 1);$
$z52 := coeff(PXZ5, z, 2);$
$z53 := coeff(PXZ5, z, 3);$
$z54 := coeff(PXZ5, z, 4);$
$z55 := coeff(PXZ5, z, 5);$
$z56 := coeff(PXZ5, z, 6);$
$z57 := coeff(PXZ5, z, 7);$
$z58 := coeff(PXZ5, z, 8);$
$z59 := coeff(PXZ5, z, 9);$
$z510 := coeff(PXZ5, z, 10);$
$z511 := coeff(PXZ5, z, 11);$
$z512 := coeff(PXZ5, z, 12);$
$z513 := coeff(PXZ5, z, 13);$

\>

$PXZ6 := taylor(PxZ^6, z, 14);$
$z60 := coeff(PXZ6, z, 0);$
$z61 := coeff(PXZ6, z, 1);$
$z62 := coeff(PXZ6, z, 2);$
$z63 := coeff(PXZ6, z, 3);$
$z64 := coeff(PXZ6, z, 4);$
$z65 := coeff(PXZ6, z, 5);$
$z66 := coeff(PXZ6, z, 6);$
$z67 := coeff(PXZ6, z, 7);$
$z68 := coeff(PXZ6, z, 8);$
$z69 := coeff(PXZ6, z, 9);$
$z610 := coeff(PXZ6, z, 10);$
$z611 := coeff(PXZ6, z, 11);$
$z612 := coeff(PXZ6, z, 12);$
$z613 := coeff(PXZ6, z, 13);$

>

$PXZ7 := taylor(PxZ^7, z, 14);$
$z70 := coeff(PXZ7, z, 0);$
$z71 := coeff(PXZ7, z, 1);$
$z72 := coeff(PXZ7, z, 2);$
$z73 := coeff(PXZ7, z, 3);$
$z74 := coeff(PXZ7, z, 4);$
$z75 := coeff(PXZ7, z, 5);$
$z76 := coeff(PXZ7, z, 6);$
$z77 := coeff(PXZ7, z, 7);$
$z78 := coeff(PXZ7, z, 8);$
$z79 := coeff(PXZ7, z, 9);$
$z710 := coeff(PXZ7, z, 10);$
$z711 := coeff(PXZ7, z, 11);$
$z712 := coeff(PXZ7, z, 12);$
$z713 := coeff(PXZ7, z, 13);$

>

$PXZ8 := taylor(PxZ^8, z, 14);$
$z80 := coeff(PXZ8, z, 0);$
$z81 := coeff(PXZ8, z, 1);$
$z82 := coeff(PXZ8, z, 2);$
$z83 := coeff(PXZ8, z, 3);$
$z84 := coeff(PXZ8, z, 4);$
$z85 := coeff(PXZ8, z, 5);$
$z86 := coeff(PXZ8, z, 6);$
$z87 := coeff(PXZ8, z, 7);$
$z88 := coeff(PXZ8, z, 8);$
$z89 := coeff(PXZ8, z, 9);$
$z810 := coeff(PXZ8, z, 10);$
$z811 := coeff(PXZ8, z, 11);$
$z812 := coeff(PXZ8, z, 12);$
$z813 := coeff(PXZ8, z, 13);$

>

$PXZ9 := taylor(PxZ^9, z, 14);$
$z90 := coeff(PXZ9, z, 0);$
$z91 := coeff(PXZ9, z, 1);$
$z92 := coeff(PXZ9, z, 2);$
$z93 := coeff(PXZ9, z, 3);$
$z94 := coeff(PXZ9, z, 4);$
$z95 := coeff(PXZ9, z, 5);$
$z96 := coeff(PXZ9, z, 6);$
$z97 := coeff(PXZ9, z, 7);$
$z98 := coeff(PXZ9, z, 8);$
$z99 := coeff(PXZ9, z, 9);$
$z910 := coeff(PXZ9, z, 10);$
$z911 := coeff(PXZ9, z, 11);$
$z912 := coeff(PXZ9, z, 12);$
$z913 := coeff(PXZ9, z, 13);$

```
>
PXZ10 := taylor(PxZ^10, z, 14);
z100 := coeff(PXZ10, z, 0);
z101 := coeff(PXZ10, z, 1);
z102 := coeff(PXZ10, z, 2);
z103 := coeff(PXZ10, z, 3);
z104 := coeff(PXZ10, z, 4);
z105 := coeff(PXZ10, z, 5);
z106 := coeff(PXZ10, z, 6);
z107 := coeff(PXZ10, z, 7);
z108 := coeff(PXZ10, z, 8);
z109 := coeff(PXZ10, z, 9);
z1010 := coeff(PXZ10, z, 10);
z1011 := coeff(PXZ10, z, 11);
z1012 := coeff(PXZ10, z, 12);
z1013 := coeff(PXZ10, z, 13) ;
```

```
>
PXZ11 := taylor(PxZ^11, z, 14);
z110 := coeff(PXZ11, z, 0);
z111 := coeff(PXZ11, z, 1);
z112 := coeff(PXZ11, z, 2);
z113 := coeff(PXZ11, z, 3);
z114 := coeff(PXZ11, z, 4);
z115 := coeff(PXZ11, z, 5);
z116 := coeff(PXZ11, z, 6);
z117 := coeff(PXZ11, z, 7);
z118 := coeff(PXZ11, z, 8);
z119 := coeff(PXZ11, z, 9);
z1110 := coeff(PXZ11, z, 10);
z1111 := coeff(PXZ11, z, 11);
z1112 := coeff(PXZ11, z, 12);
z1113 := coeff(PXZ11, z, 13);
```

```
>
PXZ12 := taylor(PxZ^12, z, 14);
z120 := coeff(PXZ12, z, 0);
z121 := coeff(PXZ12, z, 1);
z122 := coeff(PXZ12, z, 2);
z123 := coeff(PXZ12, z, 3);
z124 := coeff(PXZ12, z, 4);
z125 := coeff(PXZ12, z, 5);
z126 := coeff(PXZ12, z, 6);
z127 := coeff(PXZ12, z, 7);
z128 := coeff(PXZ12, z, 8);
z129 := coeff(PXZ12, z, 9);
z1210 := coeff(PXZ12, z, 10);
z1211 := coeff(PXZ12, z, 11);
z1212 := coeff(PXZ12, z, 12);
z1213 := coeff(PXZ12, z, 13);
```

```
> PXZ13 := taylor(PxZ^13, z, 14);
  z130 := coeff(PXZ13, z, 0);
  z131 := coeff(PXZ13, z, 1);
  z132 := coeff(PXZ13, z, 2);
  z133 := coeff(PXZ13, z, 3);
  z134 := coeff(PXZ13, z, 4);
  z135 := coeff(PXZ13, z, 5);
  z136 := coeff(PXZ13, z, 6);
  z137 := coeff(PXZ13, z, 7);
  z138 := coeff(PXZ13, z, 8);
  z139 := coeff(PXZ13, z, 9);
  z1310 := coeff(PXZ13, z, 10);
  z1311 := coeff(PXZ13, z, 11);
  z1312 := coeff(PXZ13, z, 12);
  z1313 := coeff(PXZ13, z, 13);

> # CREATE EQUATIONS FOR DETERMINING Pn(t)'s FOR BULK
     ARRIVALS - USE SINGLE ARRIVAL Pn(t)'s AND
     COEFFICIENTS OF z^n FROM SERIES EXPANSION

> bpnt[0] := simplify(pnt[0] + z0100 pnt[1] + z0200 pnt[2]
     + z30 pnt[3] + z40 pnt[4] + z50 pnt[5] + z60 pnt[6]
     + z70 pnt[7] + z80 pnt[8] + z90 pnt[9] + z100 pnt[10]
     + z110 pnt[11] + z120 pnt[12] + z130 pnt[13]);

> bpnt[1] := simplify(z0101 pnt[1] + z0201 pnt[2] + z31 pnt[3]
     + z41 pnt[4] + z51 pnt[5] + z61 pnt[6] + z71 pnt[7]
     + z81 pnt[8] + z91·pnt[9] + z101 pnt[10] + z111 pnt[11]
     + z121 pnt[12] + z131 pnt[13]);

> bpnt[2] := simplify(z0102 pnt[1] + z0202 pnt[2] + z32 pnt[3]
     + z42 pnt[4] + z52 pnt[5] + z62 pnt[6] + z72 pnt[7]
     + z82 pnt[8] + z92 pnt[9] + z102 pnt[10] + z112 pnt[11]
     + z122 pnt[12] + z132 pnt[13]);

> bpnt[3] := simplify(z0103 pnt[1] + z0203 pnt[2] + z33 pnt[3]
     + z43pnt[4] + z53 pnt[5] + z63·pnt[6] + z73 pnt[7] + z83
     ·pnt[8] + z93 pnt[9] + z103 pnt[10] + z113·pnt[11]
     + z123 pnt[12] + z133 pnt[13]);

> bpnt[4] := simplify(z0104 pnt[1] + z0204 pnt[2] + z34 pnt[3]
     + z44 pnt[4] + z54 pnt[5] + z64 pnt[6] + z74 pnt[7]
     + z84 pnt[8] + z94 pnt[9] + z104 pnt[10] + z114 pnt[11]
     + z124 pnt[12] + z134 pnt[13]);

> bpnt[5] := simplify(z0105 pnt[1] + z0205 pnt[2] + z35 pnt[3]
     + z45 pnt[4] + z55 pnt[5] + z65 pnt[6] + z75 pnt[7]
     + z85 pnt[8] + z95 pnt[9] + z105 pnt[10] + z115 pnt[11]
     + z125 pnt[12] + z135 pnt[13]);
```

> $bpnt[6] := simplify(z0106\,pnt[1] + z0206\,pnt[2] + z36\,pnt[3] + z46\,pnt[4] + z56\,pnt[5] + z66\,pnt[6] + z76\,pnt[7] + z86\,pnt[8] + z96\,pnt[9] + z106\,pnt[10] + z116\,pnt[11] + z126\,pnt[12] + z136\,pnt[13]);$

> $bpnt[7] := simplify(z0107\,pnt[1] + z0207\,pnt[2] + z37\,pnt[3] + z47\,pnt[4] + z57\,pnt[5] + z67\,pnt[6] + z77\,pnt[7] + z87\,pnt[8] + z97\,pnt[9] + z107\,pnt[10] + z117 \cdot pnt[11] + z127\,pnt[12] + z137\,pnt[13]);$

> $bpnt[8] := simplify(z0108\,pnt[1] + z0208\,pnt[2] + z38\,pnt[3] + z48\,pnt[4] + z58\,pnt[5] + z68\,pnt[6] + z78\,pnt[7] + z88\,pnt[8] + z98\,pnt[9] + z108\,pnt[10] + z118\,pnt[11] + z128\,pnt[12] + z138\,pnt[13]);$

> $bpnt[9] := simplify(z0109\,pnt[1] + z0209\,pnt[2] + z39\,pnt[3] + z49\,pnt[4] + z59\,pnt[5] + z69\,pnt[6] + z79\,pnt[7] + z89\,pnt[8] + z99\,pnt[9] + z109\,pnt[10] + z119\,pnt[11] + z129\,pnt[12] + z139\,pnt[13]);$

> $bpnt[10] := simplify(z0110\,pnt[1] + z0210\,pnt[2] + z310\,pnt[3] + z410\,pnt[4] + z510\,pnt[5] + z610 \cdot pnt[6] + z710\,pnt[7] + z810\,pnt[8] + z910\,pnt[9] + z1010\,pnt[10] + z1110\,pnt[11] + z1210\,pnt[12] + z1310\,pnt[13]);$

> $bpnt[11] := simplify(z0111\,pnt[1] + z0211\,pnt[2] + z311\,pnt[3] + z411\,pnt[4] + z511 \cdot pnt[5] + z611 \cdot pnt[6] + z711\,pnt[7] + z811\,pnt[8] + z911\,pnt[9] + z1011\,pnt[10] + z1111\,pnt[11] + z1211\,pnt[12] + z1311\,pnt[13]);$

> $bpnt[12] := simplify(z0112\,pnt[1] + z0212\,pnt[2] + z312\,pnt[3] + z412\,pnt[4] + z512 \cdot pnt[5] + z612 \cdot pnt[6] + z712\,pnt[7] + z812 \cdot pnt[8] + z912\,pnt[9] + z1012\,pnt[10] + z1112\,pnt[11] + z1212\,pnt[12] + z1312\,pnt[13]);$

> $bpnt[13] := simplify(z0113\,pnt[1] + z0213\,pnt[2] + z313\,pnt[3] + z413\,pnt[4] + z513 \cdot pnt[5] + z613 \cdot pnt[6] + z713\,pnt[7] + z813 \cdot pnt[8] + z913\,pnt[9] + z1013\,pnt[10] + z1113\,pnt[11] + z1213\,pnt[12] + z1313\,pnt[13]);$

> *# NUMERICAL RESULTS FOR BULK RENEWALS - VARIOUS VALUES OF t*

> *#NN counter for bulk Pn(t) calculations*

> $NN := 13$

```
>
teval := 0.1;
for l from 0 to NN do
xx[l] := evalf(subs( t = teval, bpnt[l]));
end do;
unassign('i');
```

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

$evalf(subs(t = teval, ENt));$

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

$evalf(subs(t = teval, VNt));$

```
>
teval := 0.15;
for l from 0 to NN do
xx[l] := evalf(subs( t = teval, bpnt[l]));
end do;
unassign('i');
```

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

$evalf(subs(t = teval, ENt));$

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

$evalf(subs(t = teval, VNt));$

```
>
teval := 0.25;
for l from 0 to NN do
xx[l] := evalf(subs( t = teval, bpnt[l]));
end do;
unassign('i');
```

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

$evalf(subs(t = teval, ENt));$

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

$evalf(subs(t = teval, VNt));$

```
>
teval := 0.4;
for l from 0 to NN do
xx[l] := evalf(subs( t = teval, bpnt[l]));
end do;
unassign('i');
```

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

$evalf(subs(t = teval, ENt));$

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

$evalf(subs(t = teval, VNt));$

>

```
teval := 0.45;
for l from 0 to NN do
xx[l] := evalf(subs( t = teval, bpnt[l]));
end do;
unassign('i');
```

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

```
evalf(subs(t = teval, ENt));
```

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

```
evalf(subs(t = teval, VNt));
```

>

```
teval := 0.5;
for l from 0 to NN do
xx[l] := evalf(subs( t = teval, bpnt[l]));
end do;
unassign('i');
```

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

```
evalf(subs(t = teval, ENt));
```

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

```
evalf(subs(t = teval, VNt));
```

>

$teval := 0.6;$
for l **from** 0 **to** NN **do**
$xx[l] := evalf(subs(t = teval, bpnt[l]));$
end do;
$unassign('i');$

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

$evalf(subs(t = teval, ENt));$

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

$evalf(subs(t = teval, VNt));$

>

$teval := 0.7;$
for l **from** 0 **to** NN **do**
$xx[l] := evalf(subs(t = teval, bpnt[l]));$
end do;
$unassign('i');$

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

$evalf(subs(t = teval, ENt));$

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

$evalf(subs(t = teval, VNt));$

```
>
teval := 0.75;
for l from 0 to NN do
xx[l] := evalf(subs( t = teval, bpnt[l]));
end do;
unassign('i');
```

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

$evalf(subs(t = teval, ENt));$

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

$evalf(subs(t = teval, VNt));$

```
>
teval := 1;
for l from 0 to NN do
xx[l] := evalf(subs( t = teval, bpnt[l]));
end do;
unassign('i');
```

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

$evalf(subs(t = teval, ENt));$

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

$evalf(subs(t = teval, VNt));$

```
>
teval := 1.25;
for l from 0 to NN do
xx[l] := evalf(subs( t = teval, bpnt[l]));
end do;
unassign('i');
```

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

$evalf(subs(t = teval, ENt));$

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

$evalf(subs(t = teval, VNt));$

```
>
teval := 2;
for l from 0 to NN do
xx[l] := evalf(subs( t = teval, bpnt[l]));
end do;
unassign('i');
```

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

$evalf(subs(t = teval, ENt));$

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

$evalf(subs(t = teval, VNt));$

\>

$teval := 3;$
for l **from** 0 **to** NN **do**
$xx[l] := evalf(subs(\ t = teval, bpnt[l]));$
end do;
$unassign('i');$

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

$evalf(subs(t = teval, ENt));$

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

$evalf(subs(t = teval, VNt));$

\>

$teval := 5;$
for l **from** 0 **to** NN **do**
$xx[l] := evalf(subs(\ t = teval, bpnt[l]));$
end do;
$unassign('i');$

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

$evalf(subs(t = teval, ENt));$

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

$evalf(subs(t = teval, VNt));$

```
>
teval := 10;
for l from 0 to NN do
xx[l] := evalf(subs( t = teval, bpnt[l]));
end do;
unassign('i');
```

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

```
evalf(subs(t = teval, ENt));
```

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

```
evalf(subs(t = teval, VNt));
```

```
>
teval := 20;
for l from 0 to NN do
xx[l] := evalf(subs( t = teval, bpnt[l]));
end do;
unassign('i');
```

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

```
evalf(subs(t = teval, ENt));
```

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

```
evalf(subs(t = teval, VNt));
```

```
>
teval := 50;
for l from 0 to NN do
xx[l] := evalf(subs( t = teval, bpnt[l]));
end do;
unassign('i');
```

$$xxsum := \sum_{i=0}^{NN} xx[i];$$

$$xxmean := \sum_{i=0}^{NN} i \cdot xx[i];$$

```
evalf(subs(t = teval, ENt));
```

$$xxvar := \sum_{i=0}^{NN} i^2 \cdot xx[i] - xxmean^2;$$

```
evalf(subs(t = teval, VNt));
```

```
> interface(displayprecision = 6);
> Fs;
> # ASYMPTOTIC MOMENTS AND PARAMETERS
```

$$> \; Mt := \frac{dPx1 \cdot t}{\mathrm{mu}} + dPx1 \cdot \left(\frac{\sigma^2 - \mu^2}{2 \cdot \mu^2} \right);$$

```
> dPx1;
>
```

$$\mathrm{meantest} := \mathrm{convert}\left(\int_0^{20} t \cdot ftorig \, \mathrm{d}t, float \right);$$

$$secondmoment := convert\left(\int_0^{20} t^2 \cdot ftorig \, \mathrm{d}t, float \right);$$

$$sigmatest := secondmoment - meantest^2;$$

$$thirdmoment := convert\left(\int_0^{20} t^3 \cdot ftorig \, \mathrm{d}t, float \right);$$

$$> \; \mathrm{mu} := subs\left(s = 0, -\frac{\mathrm{d}}{\mathrm{d}s} Fs \right);$$

$$> \; \mathrm{sigma} := subs\left(s = 0, \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{\mathrm{d}}{\mathrm{d}s} Fs \right) \right) - \mu^2;$$

```
> # FIRST MOMENT
```

$$> \; Mt := \frac{dPx1 \cdot t}{\mathrm{mu}} + dPx1 \cdot \left(\frac{\sigma^2 - \mu^2}{2 \cdot \mu^2} \right);$$

```
> # SECOND MOMENT
```

> $$M2t := \frac{dPx1^2 \cdot t^2}{\mu^2} + \left(\frac{ddPx1 - 2 \cdot dPx1^2 + dPx1}{\mu^1} + \frac{2 \cdot \sigma^2 \cdot dPx1^2}{\mu^3} \right) \cdot t + \frac{\sigma^2 \cdot ddPx1}{2 \cdot \mu^2} + \frac{\sigma^2 \cdot dPx1}{2 \cdot \mu^2} - \frac{dPx1}{2} - \frac{2 \cdot thirdmoment \cdot dPx1^2}{3 \cdot \mu^3} + \frac{3 \cdot \sigma^4 \cdot dPx1^2}{2 \cdot \mu^4} + \frac{\sigma^2 \cdot dPx1^2}{\mu^2} + \frac{3\, dPx1^2}{2} - \frac{ddPx1}{2};$$

> *# NUMERICAL MEAN*

> $MENt := 0;$
for w **from** 1 **to** NN **do**
$MENt := MENt + bpnt[w] \cdot w;$
end do;

>

 # GRAPHICAL COMPARISON OF ANALYTIC (ENt), ASYMPTOTIC (Mt), AND NUMERICAL (MENt) RESULTS

> $plot(Mt,\ t = 0 .. 2);$

> $plot([ENt, Mt],\ t = 0 .. 2);$

> $plot([ENt, Mt], t = 0 ..20);$

>

> $plot([Mt, ENt, MENt], t = 0 ..1);$

> $plot([Mt, ENt, MENt], t = 0 ..5);$

> $plot([Mt, ENt, MENt], t = 0 ..10);$

> $plot([Mt, ENt, MENt], t = 0 ..20);$

> *# ASYMPTOTIC NUMERICAL RESULTS - FIRST AND SECOND MOMENTS - VARIOUS VALUES OF t*

> $t := 2;$

> $$Mt := \frac{dPx1 \cdot t}{\mathrm{mu}} + dPx1 \cdot \left(\frac{\sigma^2 - \mu^2}{2 \cdot \mu^2} \right);$$

> $$M2t := \frac{dPx1^2 \cdot t^2}{\mu^2} + \left(\frac{ddPx1 - 2 \cdot dPx1^2 + dPx1}{\mu^1} + \frac{2 \cdot \sigma^2 \cdot dPx1^2}{\mu^3} \right) \cdot t + \frac{\sigma^2 \cdot ddPx1}{2 \cdot \mu^2} + \frac{\sigma^2 \cdot dPx1}{2 \cdot \mu^2} - \frac{dPx1}{2} - \frac{2 \cdot thirdmoment \cdot dPx1^2}{3 \cdot \mu^3} + \frac{3 \cdot \sigma^4 \cdot dPx1^2}{2 \cdot \mu^4} + \frac{\sigma^2 \cdot dPx1^2}{\mu^2} + \frac{3\, dPx1^2}{2} - \frac{ddPx1}{2};$$

> $ASYVARIANCE := M2t - Mt^2;$

> $t := 5;$

> $$Mt := \frac{dPx1 \cdot t}{\mathrm{mu}} + dPx1 \cdot \left(\frac{\sigma^2 - \mu^2}{2 \cdot \mu^2} \right);$$

$$> M2t := \frac{dPx1^2 \cdot t^2}{\mu^2} + \left(\frac{ddPx1 - 2 \cdot dPx1^2 + dPx1}{\mu^1} + \frac{2 \cdot \sigma^2 \cdot dPx1^2}{\mu^3} \right) \cdot t + \frac{\sigma^2 \cdot ddPx1}{2 \cdot \mu^2} + \frac{\sigma^2 \cdot dPx1}{2 \cdot \mu^2} - \frac{dPx1}{2} - \frac{2 \cdot thirdmoment \cdot dPx1^2}{3 \cdot \mu^3} + \frac{3 \cdot \sigma^4 \cdot dPx1^2}{2 \cdot \mu^4} + \frac{\sigma^2 \cdot dPx1^2}{\mu^2} + \frac{3\, dPx1^2}{2} - \frac{ddPx1}{2};$$

$> ASYVARIANCE := M2t - Mt^2;$

$>$

$> t := 20;$

$$> Mt := \frac{dPx1 \cdot t}{\mathrm{mu}} + dPx1 \cdot \left(\frac{\sigma^2 - \mu^2}{2 \cdot \mu^2} \right);$$

$$> M2t := \frac{dPx1^2 \cdot t^2}{\mu^2} + \left(\frac{ddPx1 - 2 \cdot dPx1^2 + dPx1}{\mu^1} + \frac{2 \cdot \sigma^2 \cdot dPx1^2}{\mu^3} \right) \cdot t + \frac{\sigma^2 \cdot ddPx1}{2 \cdot \mu^2} + \frac{\sigma^2 \cdot dPx1}{2 \cdot \mu^2} - \frac{dPx1}{2} - \frac{2 \cdot thirdmoment \cdot dPx1^2}{3 \cdot \mu^3} + \frac{3 \cdot \sigma^4 \cdot dPx1^2}{2 \cdot \mu^4} + \frac{\sigma^2 \cdot dPx1^2}{\mu^2} + \frac{3\, dPx1^2}{2} - \frac{ddPx1}{2};$$

$> ASYVARIANCE := M2t - Mt^2;$

$> t := 40;$

$$> Mt := \frac{dPx1 \cdot t}{\mathrm{mu}} + dPx1 \cdot \left(\frac{\sigma^2 - \mu^2}{2 \cdot \mu^2} \right);$$

$$> M2t := \frac{dPx1^2 \cdot t^2}{\mu^2} + \left(\frac{ddPx1 - 2 \cdot dPx1^2 + dPx1}{\mu^1} + \frac{2 \cdot \sigma^2 \cdot dPx1^2}{\mu^3} \right) \cdot t + \frac{\sigma^2 \cdot ddPx1}{2 \cdot \mu^2} + \frac{\sigma^2 \cdot dPx1}{2 \cdot \mu^2} - \frac{dPx1}{2} - \frac{2 \cdot thirdmoment \cdot dPx1^2}{3 \cdot \mu^3} + \frac{3 \cdot \sigma^4 \cdot dPx1^2}{2 \cdot \mu^4} + \frac{\sigma^2 \cdot dPx1^2}{\mu^2} + \frac{3\, dPx1^2}{2} - \frac{ddPx1}{2};$$

$> ASYVARIANCE := M2t - Mt^2;$

> *# END OF PROGRAM*

CURRICULUM VITAE

CURRICULUM VITAE

BRENT DAVID FISHER

Born 4 August 1988 in Kingston, ON

Contact Information:
1001-327 Maitland St., Victoria, BC, V9A 7G7
fisher24712@gmail.com

POST SECONDARY EDUCATION

May 2012 – Present **Royal Military College of Canada**

- Completing Masters of Science in Mathematics as a part-time student
- Concentrating on stochastic processes and renewal theory
- Thesis entitled “Renewal Theory: Simple and Elegant Derivations”

September 2010 – May 2012 **Royal Military College of Canada**

- Received a Masters in Business Administration while on a scholarship from Defence Research and Development Canada
- Program of study concentrated on management science
- Finished top of graduating class with a program academic average of 91 percent
- Successfully defended thesis entitled “A Decision Support Tool for Adopting Non Strategic Projects in the Royal Canadian Navy”

September 2006 – May 2010 **Royal Military College of Canada**

- Received an Honours Bachelor of Science in Mathematics (minor in Physics) with first class distinction
- Finished top of graduating class with a final-year academic average of 96 percent
- Simultaneously received an undergraduate diploma from the Canadian Operational Research Society

WORK EXPERIENCE

June 2006 – Present **Member of the Canadian Forces Regular Force as a Maritime Surface and Sub-surface Officer**

- Graduated from the Naval Officer Training Centre as top student
- Completed five-month posting as an Exchange Officer with the Armada de Chile
- Currently a Bridge Watch-keeper Under Training with HMCS WINNIPEG

June 2011 – August 2011 **Research Assistant at the Naval Postgraduate School Operations Research Center**

- Created a preliminary decision support tool to optimize the scheduling of Explosive Ordnance Disposal pre-deployment training for the U.S. Navy
- Remotely coordinated with senior naval officers and civilian officials to implement the model’s formulation in a timely yet accurate manner

May 2011 – May 2012 **Volleyball Tendency Software Developer**

- Developed software to track and summarize various opponent tendencies in volleyball
- Used by the Canadian National University team at the 26th Summer Universiade in China
- Advanced version leased to the RMC and local Pegasus Volleyball programs at no cost

May 2010 – September 2010 **Research Assistant in the RMC Department of Business Administration**

- Conducted support and lead research resulting in the publication of two articles in 2012 and the presentation of one other at the *Defence and Security Economics Workshop 2010*

OTHER EXPERIENCE

January 2008 – August 2012 **Volunteer with the RMC Club of Canada**

- Performed secretarial duties for the Board of Directors
- Assisted staff in promotional programs to improve club membership
- Published numerous human interest articles for alumni newsletters
- Escorted several WWII veterans at various events

September 2006 – May 2010 **Student Leader at the Royal Military College**

- Accepted responsibility for the well-being of seventy-five officer cadet peers as Cadet Squadron Leader
- Led and supervised projects relating to the training and discipline of more than 1000 officer cadet peers as Deputy Cadet Wing Training Officer
- Trained, mentored, and evaluated the initial progress of seven First Year officer cadets as a Cadet Section Commander

PRIZES AND AWARDS

- *Chief of the Maritime Staff Award* for graduating as top candidate in MARS training in the Royal Canadian Navy (2013)
- *Canadian Forces Logistics Branch Medal of Academic Excellence* for graduating as top student in the RMC MBA program (2012)
- *Governor General's Gold Medal Finalist* for achieving the highest overall academic average in any program of study at the masters degree level (2012)
- *Rhodes Scholarship Finalist* in Ontario (2010)
- *Sword of Honour* for attaining the highest standard of proficiency among the graduating students across all components of the RMC course of study (2010)
- *Governor General's Silver Medal* for achieving the highest overall academic average in the Fourth Year of study (2010)
- *Victor Van Der Smissen-Ridout Memorial Award* for having been voted as the top graduate morally, intellectually, and physically by the student body (2010)
- *General Douglas MacArthur Leadership Award* for demonstrating the highest potential based on the credo Duty-Honour-Country and potential for future service in the profession of arms (2010)
- *Canadian Operational Research Society Diploma* for undergraduate work (2010)
- *Canadian Interuniversity Sport Academic All-Canadian* in volleyball (2007 – 2011)
- *Public Service Commission Second Language Certification* for achieving advanced standing (C) in reading, writing, and oral communication examinations