Locating a Minisum Annulus
Partial Coverage Distance Models

Repérage d'un espace annulaire selon le critère mini-somme
Des modèles à distance avec couverture partielle

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Abstract

The problem is to find the best location in the plan of a minisum annulus with given width using the partial coverage distance model. The concept of partial coverage distance is that given demand points in the covering area are covered at no cost, while for uncovered demand points there will be additional costs proportional to their distances to the covering area. The objective of the problem is to locate the annulus such that the sum of distances from the uncovered demand points to the annulus (covering area) is minimized. The distance is measured by the Euclidean norm. We discuss the cases where the radius of the inner circle of the annulus is variable and given. For the variable radius, we prove that at least two demand points must be on the boundary of any optimal annulus. Based on that, an algorithm to solve the problem is introduced. In the case of given radius, we introduce the model and show its usefulness in locating undesirable facilities.

Résumé

Le problème est de repérer le meilleur emplacement dans le plan d'un espace annulaire avec des données de la largeur en utilisant le critère mini-somme et un modèle à distance avec couverture partielle. La notion de distance avec couverture partielle est que les points dans la zone de couverture sont servis sans pénalité, mais il y aura, pour la découverte des points, un surcoût proportionnel à leur distance de la zone de couverture. L'objectif du problème est de localiser l'espace annulaire tel que la somme des distances comprises entre les points non couverts par l'espace annulaire (superficie) est réduit au minimum. La distance est mesurée par la norme euclidienne. Nous discutons les cas où le rayon du cercle intérieur de l'espace annulaire est variable et donné. Pour le rayon variable, nous avons prouvé qu'au moins deux points devaient être sur la limite d'un anneau optimale. Sur cette base, un algorithme pour résoudre le problème est introduit. Dans le cas d'un rayon donné, nous introduisons le modèle et nous prouvons son utilité en localisant des installations indésirables.

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Chapter 1

Introduction

1.1 Introducing Facility Location Problems

The importance of choosing the location of a new facility has led to the development of the field of research known as facilities location theory (or location problems). The study of location problems dates back to the seventeenth century when Pierre de Fermat posed the question of where to locate a point in the plane such that the sum of its distances to three given points is minimized. Fermat, Evagelistica Torricelli, Battista Cavallieri, and others suggested ways to solve the problem. In 1909, Alfred Weber reintroduced the problem converting it to how to locate a plant so the transportation cost to the customers, who make different demands, is minimized. Not only did Weber generalize the problem and present it as a practical problem instead of a mathematical one, but he introduced the idea of assigned weights to each demand point. That was the origin of the famous Weber problem. In 1964, the contribution to location theory increased dramatically when Hakimi introduced the p-Median problem. Since then, location theory has interested researchers from many different fields such as mathematics, engineering, computer science, and management science.

Location problems are composed basically of four elements: one or more **new facilities** being located with respect to **existing facilities** in a given **space** such that a specific **objective** (e.g., the sum of the distances between the new facility and the existing ones) is optimized. The existing facilities can be points or dimensional facilities. We will only consider the the case in which existing facilities are points and will refer to them as demand points ("customers" and "given or fixed points" are other synonyms used in different references). Also the new facility (or facilities) to be located can take different forms such as a point, a line, a circle. The new facility studied in this thesis takes an annulus (a ring) shape, which consists of two circles sharing the same centre and the area between them. The facility itself may be represented by a point or a circle, but will treat it as an annulus to find its location. Before explaining the problem of locating an annulus, a general idea about the different categories of location problems and some of the common location problems will be presented.

Even though the area of location problems has been widely investigated, there is no agreed classification scheme. Therefore, location problems can be classified, arguably, in many ways based on the four elements. For instance, location problems can be categorized based on the given space whether it is a **network** or a subset of **real space**. Another categorization is subject to the nature of the new facility, i.e., if it is desirable (**pull** problems) or undesirable (**push** problems). Furthermore, depending on the location availability of the new facility, two categories arise: **continuous**, in the case that a feasible area is available to locate the new facility, and **discrete**, when the new facility can only be located in one of a finite number of given locations. Eiselt and Marianov [14] provide a short history of the development of location theory and its categories. In this thesis, we will concentrate on continuous location problems in which the space is a subset of the two-dimensional real space $\Re^2$.

The variety of facility location objectives results in numerous location problems. Two of the most common location problems are the **minisum** (or median) problem and the **minimax** (or centre) problem. In minisum problems, the objective is to minimize the sum of distances[1] between demand points and the new facility. This model is usually applied in the business sector as the priority is minimizing the cost of travel. Minimax models, on the other hand, minimize the maximum distance between demand points and the new facility. Since the public sector (e.g. emergency services) tends to satisfy all customers (demand points), it uses more applications of minimax models than the business sector. The following example illustrates the difference between minisum and minimax models.

Example 1.1. *Suppose there are three demand points located along a line at coordinates* 1,4*, and* 10*. We want to find the best location for a new point facility with*

[1] The distance function will be introduced in Chapter 2.

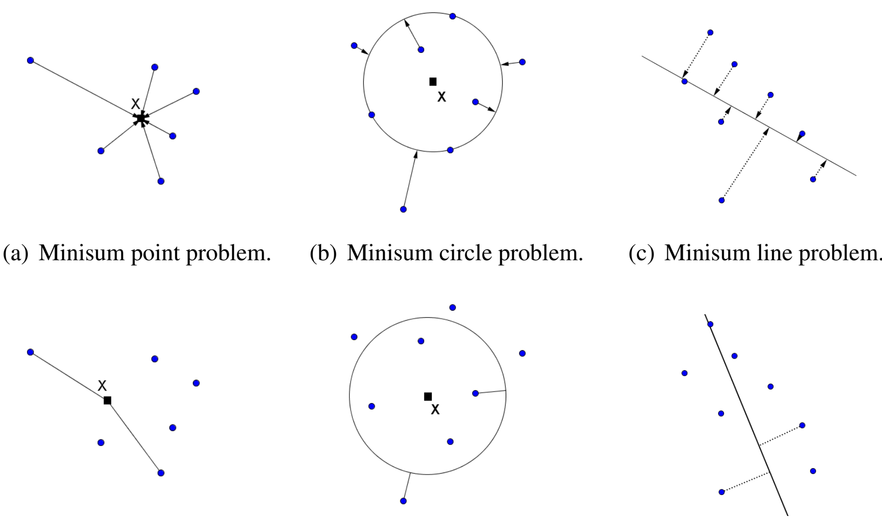
(a) The minisum model (b) The minimax model

Figure 1.1: An illustration of Example 1.1.

respect to these demand points. If the objective is to minimize the sum of distances between the new point and demand points, then the minisum model is used. The optimal location is at $x = 4$*, coinciding with one of the demand points, and the objective value is* $|4-1|+|4-4|+|4-10| = 9$*. However, the minimax model leads to a different optimal location since it aims at minimizing the maximum distance. The maximum distance between the new point and demand points is minimized to a value of* 4.5 *when the new point is located at* $x = 5.5$*. Note that the maximum distance between the new point and demand points in the minisum model is* 6*. On the other hand, the sum of distances in the minimax model is* 10.5 *(see Figure 1.1). Therefore, the minisum model results in the minimum cost, but one of the demand points (the point at* 10*) would not be as pleased with the service as other points. The minimax model, though, considers the convenience of all demand points even if that leads to more costs.*

The most common form of the new facility is the point facility, as in example 1.1. The circle facility is another form that the new facility might take. In this case, the distance is measured as the closest distance between a demand point and the circle circumference. This is equivalent to the absolute value of the difference between the distance from the demand point to the centre point of the circle and its radius. Another form of the new facility is the line facility, where the shortest distances between demand points and the line are considered. Figure 1.2 shows different forms of new facilities that are located using minisum and minimax models.

Beside minisum and minimax problems, **covering** problems have also been well-studied in facility location theory. In covering problems, demand points that are located within a specific range of the new facility are served (covered), otherwise, they



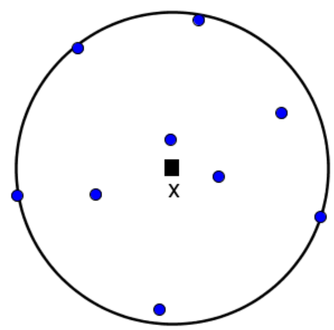
(a) Minisum point problem. (b) Minisum circle problem. (c) Minisum line problem.

(d) Minimax point problem. (e) Minimax circle problem. (f) Minimax line problem.

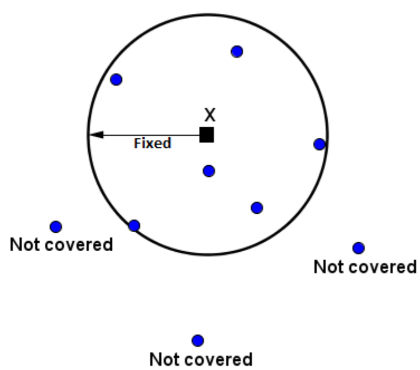
Figure 1.2: Different forms of new facilities in minisum and minimax location problems.

are not served (not covered). The most prominent problem in this area is the covering disc problem. A demand point is covered by the disc only when the distance between the point and the centre of the disc is less than or equal to the radius length. Unlike minisum and minimax models, where all demand points are served after locating the new facility, some of the demand points might not be served after locating the new facility using a covering model. When all demand points must be covered by the new facility (e.g. the case of locating a fire station), then we have the category of **full** covering (or set covering) problems. The objective of full covering models is to locate a disc facility covering all demand points such that the disc's radius is minimized. This is equivalent to the minimax point problem. The other category of covering problems is the **maximal** covering (some references call it partial covering). In this case, the radius of the new disc facility is fixed, and the goal is to locate the centre of the disc to cover as many demand points as possible. Examples of full covering and maximal covering are shown in Figure 1.3. Plastria [25] gives a detailed overview of continuous covering problems.

The disc facility is not the only form the new facility can take in covering problems. Another form of a new facility in covering problems is the annulus facility.



(a) Full covering.



(b) Maximal covering.

Figure 1.3: Full covering and maximal covering circles assigned to the same set of demand points.

Locating an annulus is equivalent to locating two circles with the same centre point. Demand points are covered only when they are located between the two circles or on their circumferences. The annulus facility is the form of the new facility that will be studied in this thesis. However, it is neither a full covering nor maximal covering annulus; instead it is a **partial coverage distance** annulus, which will be introduced in the following section.

1.2 Partial Coverage Distance for an Annulus.

One of the unpleasant consequences of using maximal covering models is not being able to serve (cover) some customers. Many models have been introduced to avoid this situation (see Berman *et al.* [2] for example). One of these models is the partial coverage distance model which was introduced recently in Brimberg *et al.* [4]. Note that the term *partial coverage distance* is different from the term *partial covering* that is used in some references to refer to the maximal covering. The objective of partial coverage distance models is similar to the objective of maximal covering models in which a new facility is located to cover demand points. In maximal covering models the goal is maximizing the number of covered demand points. On the other hand, the objective of partial coverage distance models is to locate a facility to cover demand points such that the sum of distances between the uncovered demand points and the covering area is minimized, which is the case of a minisum model. Also a minimax model can be used to minimize the maximum distance between uncovered demand points and the covering area. Thus, using partial coverage distance models, all demand points will be served, though there might be additional costs for serving demand points outside the covering range depending on their distances to the covering area. Compared to maximal models, it is likely that the partial coverage distance models result in more uncovered demand points, although they will be served at additional cost. For example, consider locating a new restaurant to serve customers within a three miles range. If the maximal covering model is used to locate the restaurant, then only customers within the covering range (3 miles) will be considered. That means if a customer is 3 miles away from the restaurant, it will be able to get its delivery orders while a customer at 3.1 miles away will not be able to. This is not a realistic situation. Thus, models such as the partial coverage distance

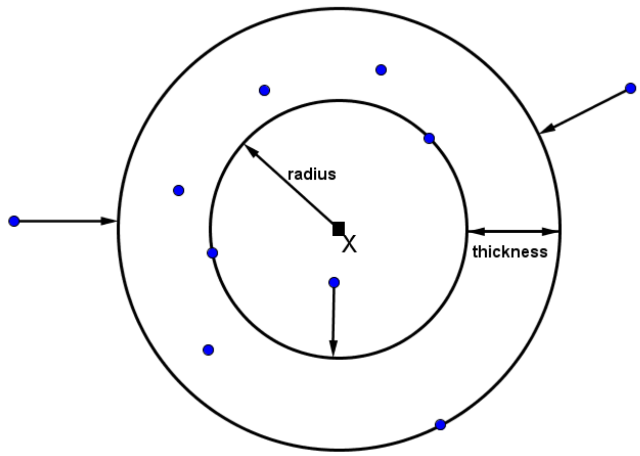


Figure 1.4: An example of a partial coverage distance annulus. The points inside the inner circle and the points outside the outer circle are covered at additional costs.

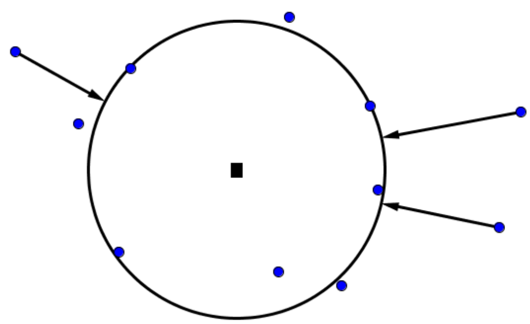
may give more sensible and practical results. These models aim at keeping customers within the covering range in addition to minimizing the cost of delivery to customers outside the covering range.

Brimberg *et al.* [4] discuss locating a covering disc with a given radius using both minisum and minimax partial coverage distance models. In this thesis, we use a minisum partial coverage distance model to find the best location of an annulus. The goal is to find the centre of an annulus (a common centre for the inner and outer circle) to cover demand points such that the sum of (weighted) distances between the inner circle and the uncovered points inside it, and between the outer circle and the points outside of it, is minimized. The width of the annulus is given but the radius of the inner circle can be either variable or given. The case in which the radius is variable will be discussed in more detail (see Figure 1.4).

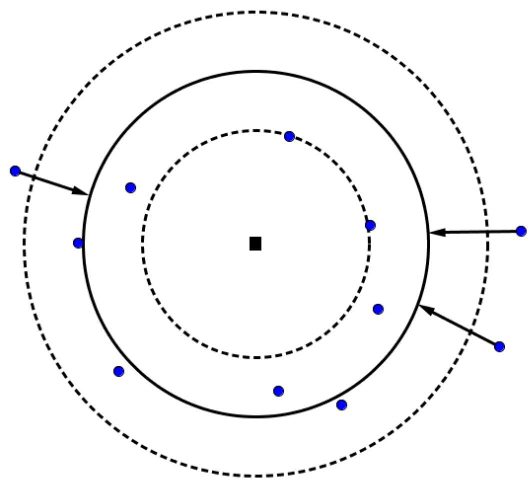
The problem of locating an annulus has been studied in other contexts. For example, full covering annulus models have been studied and developed in the past three decades. The objective of these models is to locate an annulus covering all demand points at minimum width; in this case the width and the inner circle radius are both unknown. This problem is equivalent to the problem of finding a minimax circle. Comprehensive studies about full covering annuli and their models and applications can be found in many references (e.g. [1, 16, 22, 23, 27, 8, 17]).

Using the partial coverage distance model to locate an annulus facility can be beneficial to many applications. Locating an annulus using this model can be a useful generalization of the circle location problem and most of its applications. For instance, consider finding the optimal location of a ring road serving customers. Using minisum circle location models, the road will be located based on the distances between every customer and the road. It is likely that the majority of customers will be very close to the road while a few customers will be inconveniently far from the road. However, if a customer is close enough to the road, then it would not be important for the customer if the distance to the road is a little longer or shorter. Therefore, a threshold distance to the road might be considered as the **comfort zone** (comfortable range for the customers to be close enough to the road). That means, the distances to the road for customers who are within the comfort zone will not affect the decision of locating the road. Consequently, the problem will be transformed from a minisum circle location problem to a minisum partial coverage distance annulus problem. After the problem has changed, a better location for the road can be found by moving the centre point of the ring road or changing the length of its radius, so customers who are far from the road will be closer. On the other hand, the distances to the road for customers who are very close to it might be increased but they will still be in the comfort zone. In Figure 1.5 we show a ring road being located with respect to a set of demand points using a minisum circle model and a minisum partial coverage distance model. There are three demand points that will be far from the road if the road is located using the minisum circle model; however, if the road is located using the partial coverage distance model, the distances from the same three points and the road will be decreased.

Another application that motivates the study of locating a partial coverage distance annulus is the problem of locating undesirable facilities. Models to solve undesirable facilities location problems were first introduced in 1978 by Church and Garfinkel [9] when they introduced the **maxisum model**. Other push models such as **maximin**, **empty covering**, and **minimal covering** have been used to solve these problems. The objectives of these models are the opposite of the minisum, minimax, full covering, and maximal covering models, respectively. A good literature review with more details about these models and undesirable facilities location problems in



(a) A minisum circle model.



(b) A partial coverage distance model.

Figure 1.5: Locating a ring road using a minisum circle model and a partial coverage distance model.

general can be found in Melachrinoudis [24]. Although the objective of these models is to locate undesirable facilities as far as possible from customers, the location will not be pleasant for some customers (if not the majority) if the facility is extremely far, because they still have to travel to the facility. For example, nuclear and chemical plants, airports, and landfills are not desirable near demand points; however, plant workers, passengers, etc., would not be pleased if they need to spend hours reaching their destinations. Thus, constraints and bounds, such as upper and lower bounds for the distance between the new facility and demand points, are added to models that solve such undesirable facility problems. It may be difficult, however, to find a feasible solution in these cases. A good alternative would be using the unconstrained partial coverage distance model to locate an annulus, where the undesirable facility is the centre point. For example, the objective of the minimal covering model is to locate a disc with a given radius such that the number of covered demand points is minimized. This model suggests that if only one point is covered by the disc and the undesirable facility will be very close to this point, then this solution is better than any solution that will result in more than one point covered by the disc, even if another solution results in two covered points located near the edge of the disc. The importance of the distances to the new facility is ignored not only from covered points, but also it is ignored from uncovered points, so the minimal covering model may result in many distant demand points. To avoid similar circumstances, the problem can be solved by locating an annulus with given radius and width using the minisum partial coverage distance model. Demand points within the annulus (between the inner and outer circle) will be at perfect distances to the facility (the annulus centre), far enough from the possible harm of the facility and close enough to be in a realistic travel range. The annular zone may be called the comfort zone as before. Points that are inside the inner circle will be as far from the centre point as possible since the model minimizes the sum of their distances to the inner circle; also because the model minimizes the distances to the outer circle for points on the outside, the points will not be very far from the facility.

1.3 Overview

Given a set of demand points (customers), we use the partial coverage distance model to locate an annulus with fixed width. The objective is to find an annulus such that the distances from demand points to the annulus are minimized.

In Chapter 2, we introduce a model to locate a minisum annulus with variable inner radius. We prove some properties that an optimal annulus must satisfy. The most important result in the chapter is that the optimal annulus satisfies an incidence property. The search for the centre point of an optimal annulus is thus reduced to a search only in the bisectors and defined hyperbolas between pairs of demand points.

Based on the incidence property, an exact algorithm to find an optimal annulus is introduced and used to solve examples in Chapter 3. The results from solved examples demonstrate some interesting characteristics of an optimal annulus. One of the characteristics is that the optimal annulus often has three demand points on its boundary, which allows us to use a heuristic to solve the problem in much faster time.

Finding the optimal minisum annulus with fixed radius is discussed in Chapter 4. We introduce the model but do not discuss solution methods. The main point of the chapter is to show how the model can be useful to locate undesirable facilities. A comparison is made between this model and other common models used to solve undesirable facilities location problems.

In Chapter 5, we summarize the results and suggest some directions of future work related to the partial coverage distance annulus.

Chapter 2

The Minisum Covering Annulus of Variable Radius and Given Width

2.1 Notation

The model we propose in this chapter will locate an annulus of given width in order to minimize the sum of weighted distances from a set of given points to the annulus. The distance between an area $\mathscr{S}$ and a point $P = (p_1, p_2)$ is the shortest distance between P and $\mathscr{S}$, denoted by

$$d(\mathscr{S}, P) = \min\{d(Y, P) : Y = (y_1, y_2) \in \mathscr{S}\},$$

where $d(Y, P)$ is a given distance function which measures the distance between any two points $Y, P \in \Re^2$. The distance function we are studying is the Euclidean distance (the distance given by the Euclidean norm). Before discussing some definitions and properties, the notation required to describe the model will be presented.

a) *Parameters*

Let

- D_i be the i^{th} demand point (or existing facility), for $i = 1, \ldots, n$,
- (x_i, y_i) be the given coordinates (location) of D_i,
- t be the width (thickness) of the desired annulus, and

- $w_i > 0$ be the weight (or demand) at D_i.

b) *Decision Variables*
Let

- r be the inner circle radius, or the annulus radius (because the outer circle radius depends on it), and
- $X = (x,y)$ be the location of the centre of the annulus.

The radius of the outer circle is a variable depending on r and it can be denoted as $r+t$.

c) *Objective Function*
Let

- $d_i(X) := d(D_i,X) = \sqrt{(x_i - x)^2 + (y_i - y)^2}$ be the distance between the demand point D_i and the point X,
- $A(X,r)$ denote the annulus with centre X and radius r, where $A(X,r) = \{Y \in \Re^2 : r \leq d(X,Y) \leq r+t\}$,
- $J^- = \{i : d_i(X) < r\}$; and $J^+ = \{i : d_i(X) > r+t\}$, and
- $d_i(A(X,r)) := d(D_i,A(X,r))$ be the shortest distance between the demand point D_i and the annulus A, which is defined by

$$d_i(A(X,r)) = \begin{cases} r - d_i(X), & \text{if } d_i(X) < r \\ 0, & \text{if } r \leq d_i(X) \leq r+t \\ d_i(X) - (r+t), & \text{if } r+t < d_i(X), \end{cases} \tag{2.1}$$

or equivalently

$$d_i(A(X,r)) = \max \{d_i(X) - (r+t),\, r - d_i(X),\, 0\}. \tag{2.2}$$

Now the objective is to choose a location of the annulus centre and an inner radius length such that the sum of weighted distances between the demand points and the annulus is minimized. The objective is thus given by

$$\min f(A(X,r)) = \sum_{i=1}^{n} w_i d_i(A(X,r)) \tag{2.3}$$

$$= \sum_{i\in J^-} w_i(r - d_i(X)) + \sum_{i\in J^+} w_i(d_i(X) - (r+t)). \tag{2.4}$$

When $d_i(A(X,r)) = 0$, i.e. $r \le d_i(X) \le r+t$, then the demand point D_i is covered by the annulus $A(X,r)$. Demand points inside the inner circle or outside the outer circle are not covered by the annulus. Figure 2.1 shows an annulus of radius r and width t and a group of demand points. Also when we say the boundary of the annulus, we mean both the inner and outer circles.

2.2 Properties

For the case $n \le 3$, the optimal solution to the problem is any annulus that contains all the points. Since any three non-collinear points can be contained in a circle, then obviously they can be contained in an annulus. Moreover, for $n = 3$, the optimal solution to the circle problem is unique. However, in the annulus problem the number of solutions is infinite.Thus, we will assume that $n \ge 4$ in the remainder of the discussion.

A special case of the partial coverage distance annulus $A(X,r)$ occurs when the given width $t = 0$. In this case, the annulus transforms to a circle, presenting the corresponding circle problem. Since the annulus problem is a general case of the circle problem, then the circle properties and lemmas can be generalized. In fact, most of the lemmas and theorems in this section are generalized from the circle case. A study of the problem of locating a circle and its properties is given in Brimberg *et al.* [6] for the Euclidean distance case, and in Brimberg *et al.* [5] when the distance is measured by an arbitrary norm.

When the inner radius of the annulus $A(X,r)$ is given by $r = 0$, then the partial coverage distance annulus $A(X,0)$ becomes a partial coverage distance disc of radius t. The partial coverage distance for a disc was treated in Brimberg *et al.* [4]. However, when r is unknown, then it must be positive for any optimal solution under some circumstances, as will be shown in Lemma 2.1. On the other hand, as $r \to \infty$, the

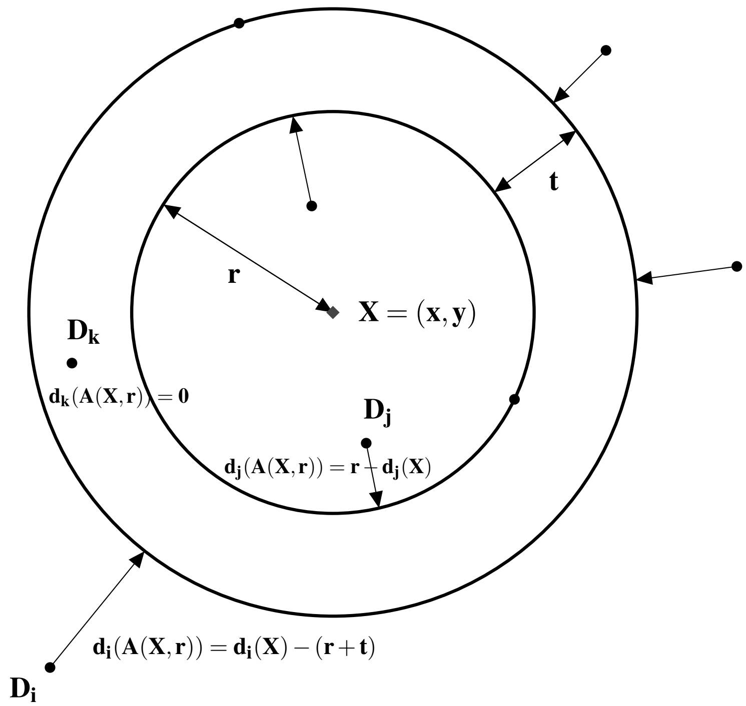


Figure 2.1: Distances to an annulus from a group of demand points.

annulus inner and outer circles transform to two parallel lines. The two parallel lines and the area between them form a "strip", and we may denote it by S. The strip is another form of new facility in location theory. A classical location problem involving a strip requires computing the minimum width of a set of points, which is also equivalent to finding a minimax line, in which the objective is to cover a set of points at the minimum width (see Houle [20] for the problem investigation).

Also in Lemma 2.2, we prove that there are sufficient conditions for the optimal r to be finite ($r < \infty$). Thus, as it was proven in the circle problem [6] that the radius of an optimal circle must be positive and finite, we get the same results in the problem of finding the annulus $A(X,r)$, under some conditions.

Another important case regarding the full covering annulus should be considered. As mentioned in section 1.2, the full covering annulus has a variable radius and width, and the objective is to cover all demand points at minimum width. Let $A'(X',r',t')$ denote an optimal full covering annulus with a minimum width t'. It follows that for the partial coverage distance annulus $A(X,r)$ with given width $t > t'$, $A'(X',r')$ is an optimal annulus, and furthermore, the problem will have an infinite number of solutions with objective value of 0. Therefore, we are only interested in the case that $t \leq t'$.

In conclusion, we will assume that $n \geq 4$, $0 \leq t \leq t'$, and $0 < r < \infty$ in the remainder of the discussion.

Lemma 2.1. *If the demand points cannot be fully covered by a disc of radius t, then any optimal solution must have a positive radius* $(r > 0)$.

Proof. Consider a disk of radius t, or equivalently, an annulus of radius 0, $A(X,0)$ and width t. The objective function is

$$f(A(X,0)) = \sum_{i=1}^{n} w_i d_i(A(X,0)) > 0.$$

Now we have two cases:

(i) X **does not coincide with a demand point:** Construct another annulus with the same centre and different radius, $A(X,r_1)$, such that $r_1 = \min d_i(X)$, i.e., we increase the radius of $A(X,0)$ until the inner circle intersects with the first demand point, say D_j. As a result,$\forall i$, $d_i(A(X,r_1)) \leq d_i(A(X,0))$, and $\exists i$ where the inequality is strict. Thus

$$f(A(X,r_1)) = \sum_{i=1}^{n} w_i d_i(A(X,r_1)) < \sum_{i=1}^{n} w_i d_i(A(X,0)) = f(A(X,0)).$$

(ii) X **coincides with a demand point,** D_k**:**

Now consider another annulus $A(X_2,r_2)$ such that $0 < 2r_2 \leq \min d_i(X_2)$, $i \neq k$, and $X = D_k$ intersects with the inner circle. Now, as in the first case, we have $\forall i$, $d_i(A(X_2,r_2)) \leq d_i(A(X,0))$. In fact the inequality holds strictly for at least one i, except in the case that all demand points outside the outer circle are on the ray through X from X_2. Subsequently,

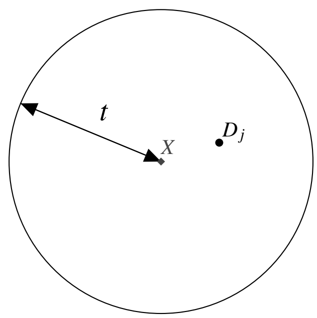
$$f(A(X_2,r_2)) = \sum_{i=1}^{n} w_i d_i(A(X_2,r_2)) < \sum_{i=1}^{n} w_i d_i(A(X,0)) = f(A(X,0)).$$

Therefore, in both cases there exists an annulus with a positive radius which is better than the annulus of radius zero. Figure 2.2 shows the two cases and the three annuli. □

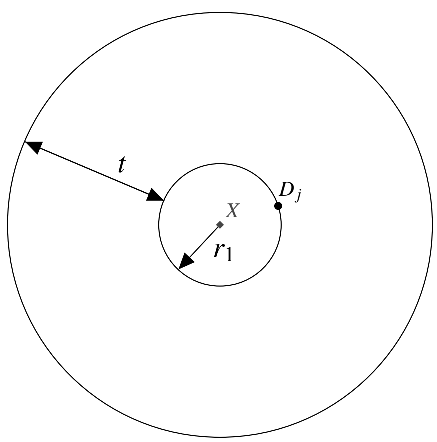
Lemma 2.1 shows that any optimal annulus must have a positive radius. In some cases, it may have an infinite radius ($r \to \infty$), which means that the optimal solution is a strip. However, the following lemma gives sufficient conditions for any optimal annulus to have a finite radius.

Lemma 2.2. *Let S^* be any optimal median (minisum) strip of width t. If three or more demand points lie outside S^*, and are not all on the same line, and no three points are on the boundary of S^*, then any optimal annulus must have a finite radius* $(r < \infty)$.

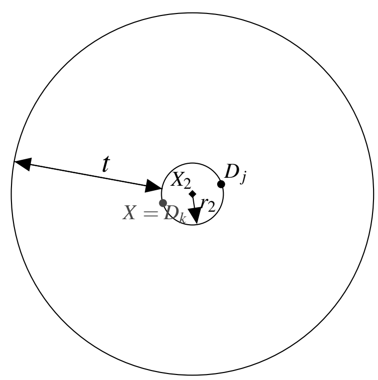
Proof. Suppose that the strip S^* with width of t is an optimal solution to the problem. Then by Brimberg *et al.*[7], there must be at least two points on its boundary (both on one side or one on each). By the conditions in the lemma, S^* will contain exactly two points on its boundary. Assume, without loss of generality, that the two points are D_1, D_2. Now construct two annuli $A_a = A(X_a,r), A_b = A(X_b,r)$, both with the same radius r and width of t. The centre points are on the bisector between the projections of the two points in one of the lines of the strip S^* with one centre point on each



(a) The annulus $A(X,0)$.



(b) The annulus $A(X,r_1)$ (case *i*).



(c) The annulus $A(X_2,r_2)$ (case *ii*).

Figure 2.2: Lemma 2.1 illustration.

side of the strip. Let the inner circle of each annulus intersect with the closer line of the strip S^* in D_1 and D_2 (or their projections on each line). Now both D_1 and D_2 are contained in both annuli A_a and A_b. Making r large enough will lead to all other demand points being inside the inner circle of either A_a or A_b and the points covered by S^* will be covered by the annuli. Now, let S_r be the right line of the strip S^* and S_l be the left line and assume that A_a is the annulus on the left hand side. Also define C_a and C_b to be the inner circles of A_a and A_b, respectively. Let L_i be the perpendicular line to S^* through D_i, and let $a_i, b_i, l_i,$ and r_i denote the intersection points closest to S^* between L_i and $C_a, C_b, S_l,$ and S_r, respectively. As a result of symmetry, we have $d(a_i, l_i) = d(b_i, r_i) = \delta_i$ (see Figure 2.3). Now define:

$$\mathscr{A}_a^- = \{D_i : d(D_i, a_i) \leq d(D_i, S_l)\},\ \mathscr{A}_a^+ = \{D_i : d(D_i, a_i) > d(D_i, S_l)\}$$

$$\mathscr{A}_b^- = \{D_i : d(D_i, b_i) < d(D_i, S_r)\},\ \mathscr{A}_b^+ = \{D_i : d(D_i, b_i) \geq d(D_i, S_r)\}$$

$$\varepsilon_i = \begin{cases} \delta_i & \text{if } D_i \in \mathscr{A}_a^+ \\ -\delta_i & \text{if } D_i \in \mathscr{A}_b^+ \end{cases}$$

.

Note. $\mathscr{A}_a^+ = \mathscr{A}_b^-$ *and* $\mathscr{A}_a^- = \mathscr{A}_b^+$.

By the definitions and notation above, we get:

$$d(D_i, C_a) \leq d(D_i, S_l) + \varepsilon_i. \tag{2.5}$$

If D_i is inside C_a we obtain:

$$d(D_i, A_a) \leq d(D_i, S^*) + \varepsilon_i. \tag{2.6}$$

Note. *If D_i is inside C_b we will get the same result by subtracting t from both sides of (2.5).*

Also we have:

$$d(D_i, C_b) \leq d(D_i, S_r) - \varepsilon_i, \tag{2.7}$$

and similarly:

$$d(D_i, A_b) \leq d(D_i, S^*) - \varepsilon_i. \tag{2.8}$$

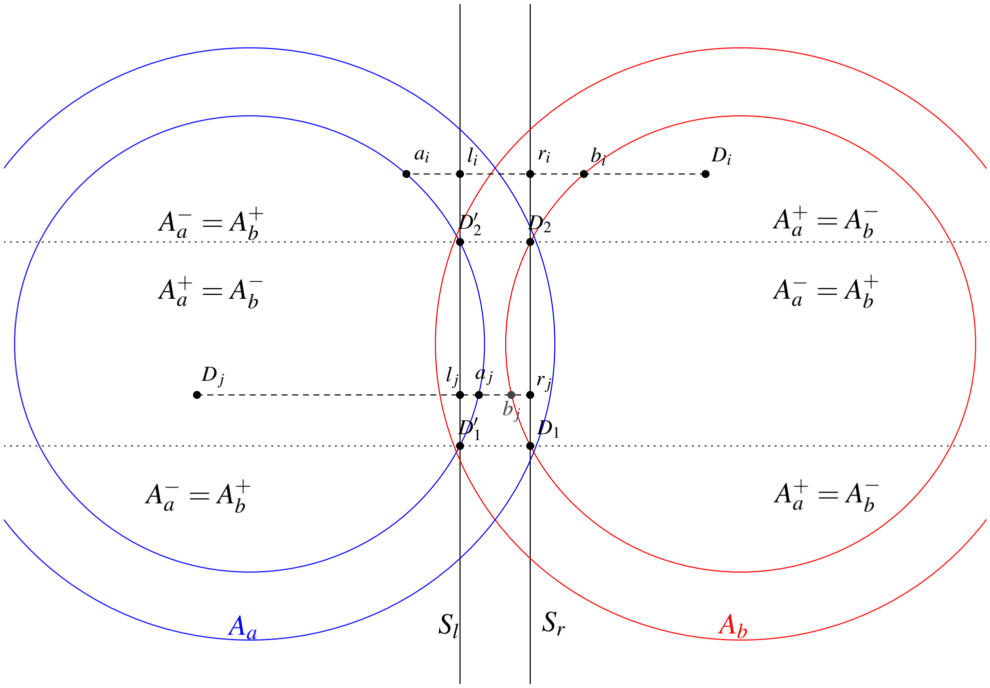


Figure 2.3: An illustration of the proof of Lemma 2.2.

Since not all demand points outside S^* are on the same line, then the inequalities (2.6) and (2.8) must be strict for at least one demand point. In fact, it is sufficient here to identify one demand point that is not on the line through the centre points X_a and X_b. Now if S^* is optimal we obtain:

$$f(S^*) \leq f(A_a) = \sum_{i=0}^{n} w_i d(D_i, A_a) \underbrace{<}_{\text{by (2.6)}} \sum_{i=0}^{n} w_i (d(D_i, S^*) + \varepsilon_i),$$

and

$$f(S^*) \leq f(A_b) = \sum_{i=0}^{n} w_i d(D_i, A_b) \underbrace{<}_{\text{by (2.8)}} \sum_{i=0}^{n} w_i (d(D_i, S^*) - \varepsilon_i).$$

Since the assumption that S^* is optimal leads to a contradiction, then the optimal annulus must have a finite radius. □

Lemma 2.3. *There always exists an optimal annulus with at least one demand point on its boundary.*

Proof. Suppose that $A(X,r)$ is an optimal annulus that has no demand points on its boundary. In the discussion below, the sets J^- and J^+ will refer respectively to the set of inner and outer demand points of the annulus $A(X,r)$. With regard to the sum of demand point weights inside the inner circle and outside the outer circle, we have three cases:

(i) $\sum_{i\in J^-} w_i < \sum_{i\in J^+} w_i$:
In this case, increase the radius until the first demand point intersects with either the outer or the inner circle. Let $r+\delta$ be the new radius. Now consider another annulus $A(X,r+\delta)$ of radius $r+\delta$ and with the same centre X. We have

$$\begin{aligned}
f(A(X,r)) - f(A(X,r+\delta)) = \\
&\sum_{i\in J^-} w_i(r-d_i(X)) + \sum_{i\in J^+} w_i(d_i(X)-(r+t)) \\
&-(\sum_{i\in J^-} w_i(r+\delta-d_i(X)) + \sum_{i\in J^+} w_i(d_i(X)-(r+\delta+t))), \\
&= \sum_{i\in J^-} w_i r - \sum_{i\in J^-} w_i d_i(X) + \sum_{i\in J^+} w_i d_i(X) - \sum_{i\in J^+} w_i(r+t) \\
&-(\sum_{i\in J^-} w_i r + \sum_{i\in J^-} w_i\delta - \sum_{i\in J^-} w_i d_i(X) \\
&\quad + \sum_{i\in J^+} w_i d_i(X) - \sum_{i\in J^+} w_i(r+t) - \sum_{i\in J^+} w_i\delta), \\
&= \sum_{i\in J^+} w_i\delta - \sum_{i\in J^-} w_i\delta > 0, \\
&\quad \Rightarrow f(A(X,r)) > f(A(X,r+\delta)).
\end{aligned}$$

Thus $A(X,r)$ is not optimal.

(ii) $\sum_{i\in J^-} w_i > \sum_{i\in J^+} w_i$:
Similarly, by decreasing the radius until the first demand point intersects with the boundary of the annulus, we can construct a better annulus.

(iii) $\sum_{i\in J^-} w_i = \sum_{i\in J^+} w_i$:
Since the sum of weights inside the inner circle equals the sum outside, then increasing or decreasing the radius will not result in any change in value of the objective function as long the sum of weights inside and outside does not change. Therefore if $A(X,r)$ is optimal, another optimal annulus can be found

by changing r (without changing the sum of weights inside and outside) until a demand point intersects the boundary.

From the three cases above, we conclude that there always exists an optimal annulus with at least one demand point intersecting its boundary. □

In the circle problem, the objective function has a complex shape, and without the use of some simplifying properties, finding a solution would be difficult. The *incidence property* helps to reduce the search for the optimal circle since the centre point of any optimal circle must be on one of the bisectors between any two demand points. On the other hand, the objective function of the annulus problem is more complicated and finding a solution is harder. The following theorem extends the incidence property to the annulus problem but it is not guaranteed that two demand points will be in the same circle. However, the property is still able to significantly reduce the search for the optimal solution as the centre point of an optimal annulus will lie on either the bisector or a defined hyperbola between two demand points.

Note. *In the case $t = t'$, the optimal annulus must have at least four demand points on its boundary, two points on each circle (see Rivlin [26]). Therefore the following theorem only considers the case $t < t'$.*

Theorem 2.1. *At least two demand points must lie on the boundary of any optimal annulus.*

Proof. Since an optimal strip must contain at least two demand points on its boundary (see Brimberg *et al.* [7]), then only the case with $0 < r < \infty$ will be discussed.
Suppose that the annulus $A(X_0, r')$ with no demand points on its boundary is optimal. Assume that the annulus $A(X_0, r)$ is also optimal and contains exactly one demand point, say D_s (we know from Lemma 2.3 that such an annulus exists). Two cases are considered:

(i) X_0 **does not coincide with any demand point:**
Consider a perturbation of the solution about $A(X_0, r)$ such that D_s remains on the same circle (inner or outer) of $A(X, r)$; i.e., we force r to be dependent on X such that the resulting annulus still contains D_s on its boundary and J^+, J^- do not change. Then the objective function $f(A(X_0, r))$ is differentiable in a small

enough neighborhood of X_0 and can be rewritten as a function of only X:

$$f(X) = \sum_{i\in J^-} w_i(r - d_i(X)) + \sum_{i\in J^+} w_i(d_i(X) - (r+t)). \tag{2.9}$$

Now we have two cases:

(a) D_s is on the inner circle. Then we have:

$$\begin{aligned} f(X) &= \sum_{i\in J^-} w_i(d_s(X) - d_i(X)) + \sum_{i\in J^+} w_i(d_i(X) - (d_s(X)+t)) \\ &= \sum_{i\in J^-} w_i(d_s(X) - d_i(X)) + \sum_{i\in J^+} w_i(d_i(X) - (d_s(X)) - \sum_{i\in J^+} w_i t. \end{aligned}$$

(b) D_s is on the outer circle. Then we have:

$$\begin{aligned} f(X) &= \sum_{i\in J^-} w_i((d_s(X) - t) - d_i(X)) + \sum_{i\in J^+} w_i(d_i(X) - d_s(X)) \\ &= \sum_{i\in J^-} w_i(d_s(X) - d_i(X)) + \sum_{i\in J^+} w_i(d_i(X) - (d_s(X)) - \sum_{i\in J^-} w_i t. \end{aligned}$$

By taking the second derivatives of f, both cases (a) and (b) will have the same results. Define θ_i to be the angle shown in Figure 2.4. Also by knowing that:

$$\begin{aligned} & d_i(X) = \sqrt{(x_i - x)^2 + (y_i - y)^2} \\ \Rightarrow \quad & \frac{\partial d_i}{\partial x} = -\frac{x_i - x}{d_i(X)} = -\cos\theta_i, \quad \text{and} \quad \frac{\partial d_i}{\partial y} = -\frac{y_i - y}{d_i(X)} = -\sin\theta_i \\ \Rightarrow \quad & \frac{\partial^2 d_i}{\partial x^2} = \frac{(\sin\theta_i)^2}{d_i(X)}, \quad \text{and} \quad \frac{\partial^2 d_i}{\partial y^2} = \frac{(\cos\theta_i)^2}{d_i(X)}, \end{aligned}$$

we will be able to write the second derivatives of f as:

$$\frac{\partial^2 f}{\partial x^2} = \sum_{i\in J^-} w_i\left(\frac{(\sin\theta_s)^2}{d_s(X)} - \frac{(\sin\theta_i)^2}{d_i(X)}\right) + \sum_{i\in J^+} w_i\left(\frac{(\sin\theta_i)^2}{d_i(X)} - \frac{(\sin\theta_s)^2}{d_s(X)}\right), \tag{2.10}$$

$$\frac{\partial^2 f}{\partial y^2} = \sum_{i\in J^-} w_i\left(\frac{(\cos\theta_s)^2}{d_s(X)} - \frac{(\cos\theta_i)^2}{d_i(X)}\right) + \sum_{i\in J^+} w_i\left(\frac{(\cos\theta_i)^2}{d_i(X)} - \frac{(\cos\theta_s)^2}{d_s(X)}\right). \tag{2.11}$$

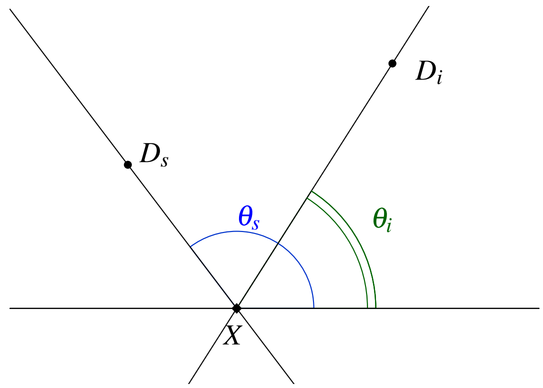


Figure 2.4: The angle θ_i used in the proof of Theorem 2.1.

By the sum of (2.10) and (2.11), we obtain:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \sum_{i \in J^-} w_i \left(\frac{1}{d_s(X)} - \frac{1}{d_i(X)} \right) + \sum_{i \in J^+} w_i \left(\frac{1}{d_i(X)} - \frac{1}{d_s(X)} \right) < 0.$$

Note. *The sum is negative because*
$d_s(X) > d_i(X), \forall i \in J^-$ *and* $d_s(X) < d_i(X), \forall i \in J^+$.

It follows that X_0 cannot be a local minimum since at least one of the second-order derivatives (2.10) or (2.11) is negative. Thus, there exists a strictly better annulus $A^*(X^*, r^*)$ than the annulus $A(X_0, r)$ (and then $A(X_0, r')$ since it has the same objective value).

(ii) **X_0 coincides with a demand point, say D_j:**
Now the objective function written in (2.9) is no longer differentiable; hence we get a new formula by separating the term where $i = j$:

$$f(X) = w_j(r - d_j(X)) \tag{2.12a}$$
$$+ \sum_{i \in J^- \setminus \{j\}} w_i(r - d_i(X)) + \sum_{i \in J^+} w_i(d_i(X) - (r + t)). \tag{2.12b}$$

We know from the previous case that the part contained in (2.12b) does not result in the minimum value of the objective function of the problem without the point D_j. Furthermore, for the part in (2.12a), we have from the triangle inequality for any X:

$$\begin{aligned} & d(D_s,X) \leq d(D_s,D_j) + d(D_j,X) \\ \Leftrightarrow \quad & d_s(X) - d_j(X) \leq d_s(D_j) - d_j(D_j). \end{aligned}$$

By subtracting t from both sides of the inequality, we conclude if D_s is on the inner or outer circle, the part $r - d_j(X)$ is locally maximized when $X = D_j$.

Therefore, from the first and second case, we conclude that at least two demand points must lie on the boundary of any optimal annulus. □

Now the question is: in general, can we prove that an optimal annulus must have three demand points on its boundary? The answer is no. In fact the following example presents a case in which an annulus that has two boundary points is better than any annulus that has more than two demand points on its boundary.

Example 2.1. *Suppose we wish to find the optimal location of an annulus of given width* $t = 1$ *to serve four demand points. The locations and the corresponding weights of the demand points are given in Table 2.1. Obviously, the optimal annulus is the annulus* $A((0,0),5)$*, which covers demand points* D_1 *and* D_2 *on its outer circle. The objective function value is* $f(A((0,0),5) = 2$*. Other annuli that cover* D_3 *or* D_4 *on the boundary in addition to* D_1 *and* D_2 *will result in higher objective values. Three different annuli covering points* D_1*,* D_2*, and* D_3 *on its boundary in different cases are shown in Table 2.2. Other annuli covering* 3 *demand points on the boundary either are similar to what is shown in Table 2.2, or clearly will result in higher objective values. The optimal annulus in addition to the three annuli are illustrated in the figures Figure 2.5 to Figure 2.8.*

Theorem 2.1 shows that an optimal annulus must have two boundary points. That means the centre point of the optimal annulus must be on the bisector between the two boundary points, if they are on the same circle, or on a defined hyperbola between the two points, if they are on different circles. This is the most important property in the thesis, as it forms the foundation of the solution procedure proposed in the next chapter.

Table 2.1: The Locations and the Weights of Demand Points in Example 2.1.

Demand point	Location	Weight
D_1	(-6,0)	50
D_2	(6,0)	50
D_3	(0,4)	1
D_4	(0,-4)	1

Table 2.2: Three Solutions to Example 2.1 with their Objective Values.

Annulus	Objective value
((0,-1.1),5.1)	2.2
((-0.52,-1.72),5.74)	3.4
((0,-2.47),5.47)	3.94

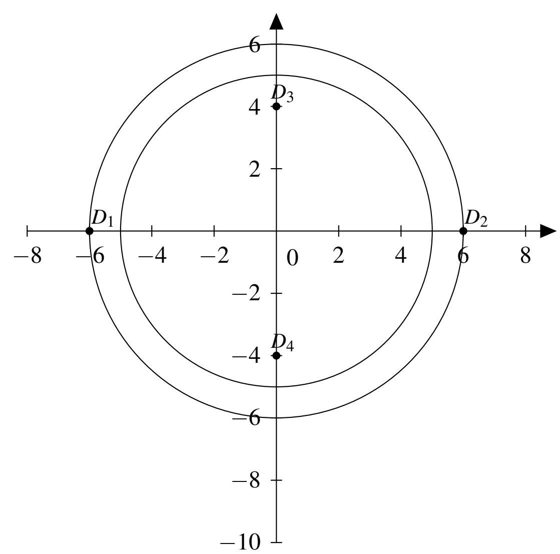


Figure 2.5: The optimal annulus for example 2.1.

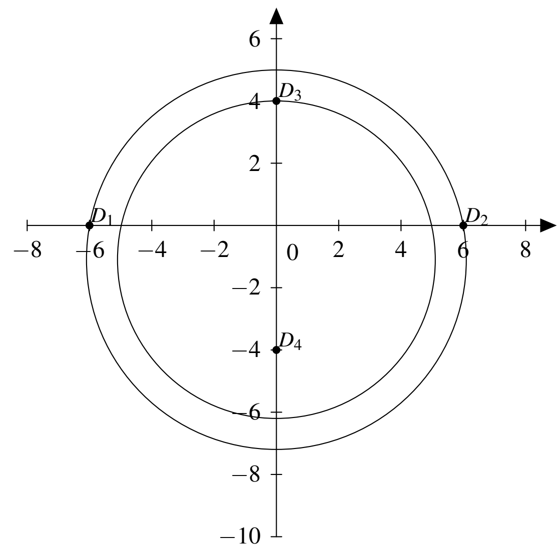


Figure 2.6: $A((0,-1.1),5.1)$ in example 2.1.

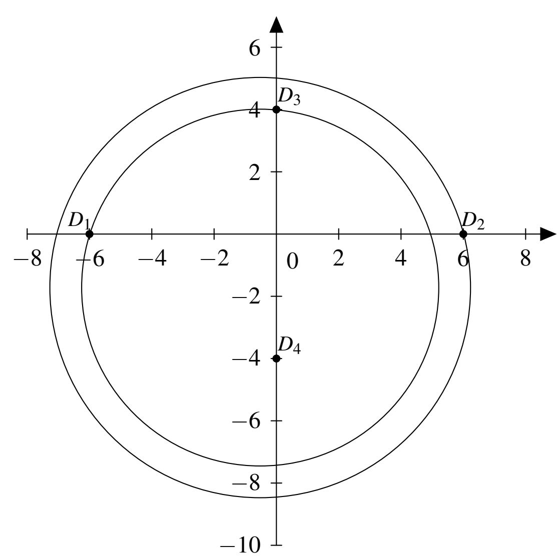


Figure 2.7: $A((-0.52,-1.72),5.74)$ in example 2.1.

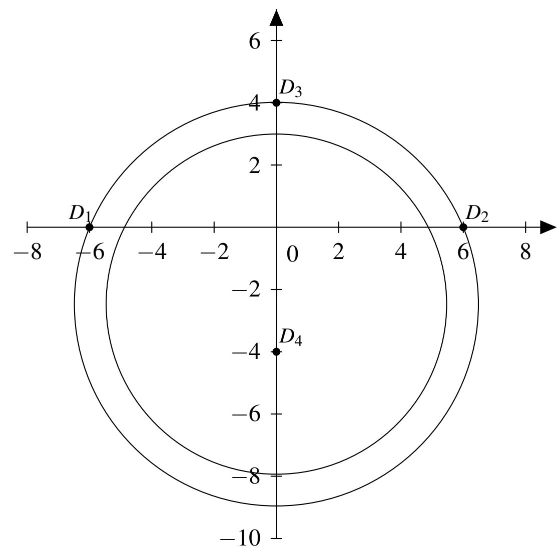


Figure 2.8: $A((0,-2.47),5.47)$ in example 2.1.

Chapter 3

Finding The Optimal Annulus

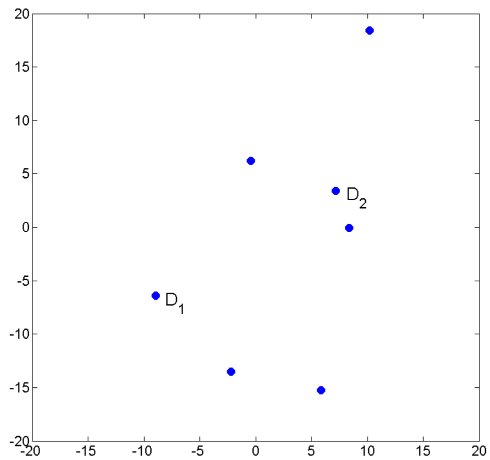
3.1 An Exact Algorithm to Find an Optimal Solution

The objective function $f(A(X,r)) = \sum_{i=1}^{n} w_i d_i(A(X,r))$ is a function of three variables: x and y (the coordinates of the centre point), and the inner radius r. To reduce the complexity of the problem, we proved some properties in Chapter 2, most important of which is the incidence property (Theorem 2.1). Given that at least two demand points must be on the boundary of an optimal annulus, then we know that the centre point of an optimal annulus must be on either a bisector or a defined hyperbola between two demand points. To find an optimal annulus, we analyze different cases for each pair of demand points. There are four different cases for two demand points, say D_1 and D_2, to be on the boundary of an annulus. Before we show the cases, for the sake of simplicity, the axes will be rotated and translated with respect to the two demand points D_1 and D_2. The new coordinates of the pair of demand points will be

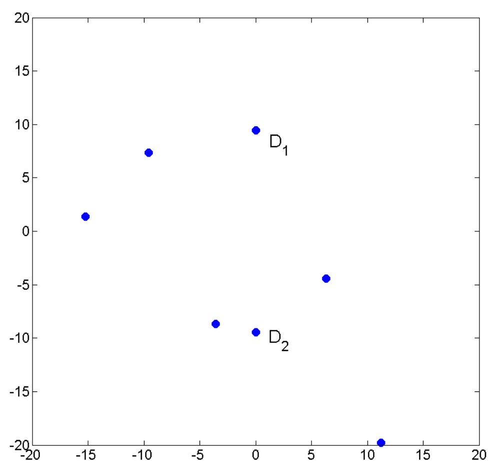
$$D_1 = (0, y_1) \quad \text{, and} \quad D_2 = (0, -y_1).$$

The remaining demand points will also have new coordinates. The Euclidean distance will not be affected by the rotation and translation, so the distances between demand points will remain unchanged. Figure 3.1 illustrates the rotation and translation with respect to a pair of points.

After the rotation and translation of the axes with respect to the pair of points, D_1 and D_2 can be on the boundary of an annulus in four different cases (all cases are



(a) Demand points before the rotation and the translation.



(b) Demand points after the rotation and the translation with respect to the points D_1 and D_2.

Figure 3.1: A set of demand points before and after the rotation and the translation.

shown in Figure 3.2 and Figure 3.3):

(i) **D_1 and D_2 are both on the inner circle:**
In this case the centre point (X) of the annulus will lie on the bisector between the two demand points, and the radius r is the distance to X from the point D_1 or D_2. Note that now the radius r is a function of x only. Thus, we have:

$$X = (x,0), \quad \text{and} \quad r = d_1(X) = d_2(X) = \sqrt{x^2 + y_1^2}.$$

Also the distance between a demand point D_i and the centre point X is given by

$$d_i(X) = \sqrt{(x - x_i)^2 + y_i^2}.$$

Therefore the objective function $f(A(X,r))$ can be rewritten as a function of one variable x as the following:

$$\begin{aligned} f(x) = &\sum_{i \in J^-} w_i(\sqrt{x^2 + y_1^2} - \sqrt{(x - x_i)^2 + y_i^2}) \\ &+ \sum_{i \in J^+} w_i(\sqrt{(x - x_i)^2 + y_i^2} - (\sqrt{x^2 + y_1^2} + t)). \end{aligned}$$

(ii) **D_1 and D_2 are both on the outer circle:**
This case is similar to the previous case. The centre point will be on the bisector between the pair of demand points. The only difference is that the radius r equals the distance to the centre point from either one of the pair of demand points minus the width of the annulus. We have

$$X = (x,0),$$

$$r = d_1(X) - t = d_2(X) - t = \sqrt{x^2 + y_1^2} - t,$$

$$d_i(X) = \sqrt{(x - x_i)^2 + y_i^2}.$$

The objective function is

$$f(x) = \sum_{i \in J^-} w_i((\sqrt{x^2+y_1^2} - t) - \sqrt{(x-x_i)^2+y_i^2})$$
$$+ \sum_{i \in J^+} w_i(\sqrt{(x-x_i)^2+y_i^2} - \sqrt{x^2+y_1^2}).$$

(iii) **D_1 is on the inner circle and D_2 is on the outer circle:**

Since the two demand points are on different circles of the same annulus, then the difference of the distances from the two points to the centre point of the annulus is constant and given by t. Thus we need to search for the centre point among the points satisfying that. In fact, the curve containing all of these candidate centre points is a defined hyperbola between the demand points D_1 and D_2. Because D_1 is on the positive side of the y-axis, then the search for a centre point in this case will be on the positive branch of the hyperbola. The equation form of the hyperbola between the points D_1 and D_2, when t is the annulus width, is

$$\frac{x^2}{(t/2)^2 - y_1^2} + \frac{y^2}{(t/2)^2} = 1.$$

This was derived from the equation of the hyperbola given by Goodman [18].

The y coordinate of the centre point is a function of x, instead of zero as in the previous cases. Also the radius and the distances to the centre points are functions of x only. They are given by

$$X = (x, y), \quad \text{where } y(x) = \frac{t}{2}\sqrt{(1 - \frac{x^2}{(t/2)^2 - y_1^2})},$$

$$r = d_1(X) = \sqrt{x^2 + (\frac{t}{2}\sqrt{(1 - \frac{x^2}{(t/2)^2 - y_1^2})} - y_1)^2},$$

$$d_i(X) = \sqrt{(x-x_i)^2 + (\frac{t}{2}\sqrt{(1 - \frac{x^2}{(t/2)^2 - y_1^2})} - y_i)^2}.$$

The objective function now can be given by

$$f(x) = \sum_{i \in J^-} w_i\left(\sqrt{x^2 + (y(x) - y_1)^2} - \sqrt{(x - x_i)^2 + (y(x) - y_i)^2}\right)$$
$$+ \sum_{i \in J^+} w_i\left(\sqrt{(x - x_i)^2 + (y(x) - y_i)^2} - \left(\sqrt{x^2 + (y(x) - y_1)^2} + t\right)\right).$$

(iv) **D_2 is on the inner circle and D_1 is on the outer circle:**
Similarly, the centre point must be on the hyperbola between D_1 and D_2. The only difference from the previous case is that the search will be on the negative branch of the hyperbola, since D_2, which is in the negative side, is on the inner circle. We obtain

$$X = (x, y), \quad \text{where } y(x) = -\frac{t}{2}\sqrt{\left(1 - \frac{x^2}{(t/2)^2 - y_1^2}\right)},$$

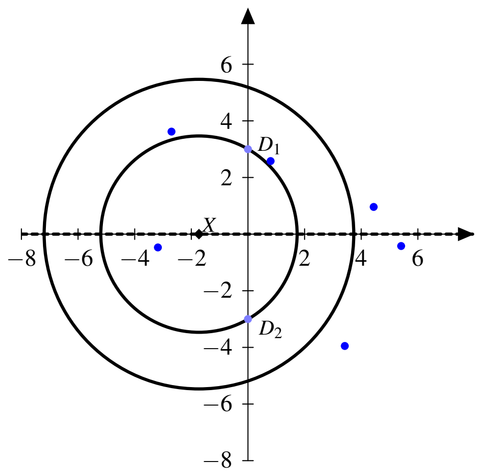
$$r = d_1(X) = \sqrt{x^2 + \left(-\frac{t}{2}\sqrt{\left(1 - \frac{x^2}{(t/2)^2 - y_1^2}\right)} - (-y_1)\right)^2},$$

$$d_i(X) = \sqrt{(x - x_i)^2 + \left(-\frac{t}{2}\sqrt{\left(1 - \frac{x^2}{(t/2)^2 - y_1^2}\right)} - y_i\right)^2}.$$

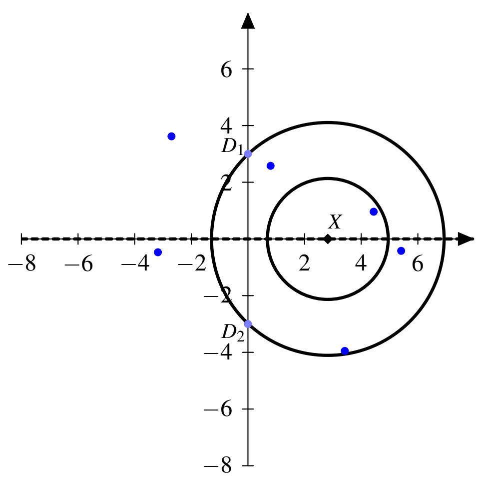
Also the objective function is similar:

$$f(x) = \sum_{i \in J^-} w_i\left(\sqrt{x^2 + (y(x) + y_1)^2} - \sqrt{(x - x_i)^2 + (y(x) - y_i)^2}\right)$$
$$+ \sum_{i \in J^+} w_i\left(\sqrt{(x - x_i)^2 + (y(x) - y_i)^2} - \left(\sqrt{x^2 + (y(x) + y_1)^2} + t\right)\right).$$

Note. *If $d(D_i, D_j) < t$, then these two demand points cannot be on different circles of the annulus. Thus, only cases (i) and (ii) are considered. If $d(D_i, D_j) \geq t$, then the four cases will be considered. For the special case $d(D_i, D_j) = t$, these points will be on different circles only when the centre point of the annulus is on the straight line through D_i and D_j and not on the line segment D_iD_j (see Figure 3.4). The objective function in this case also is a function of one variable.*

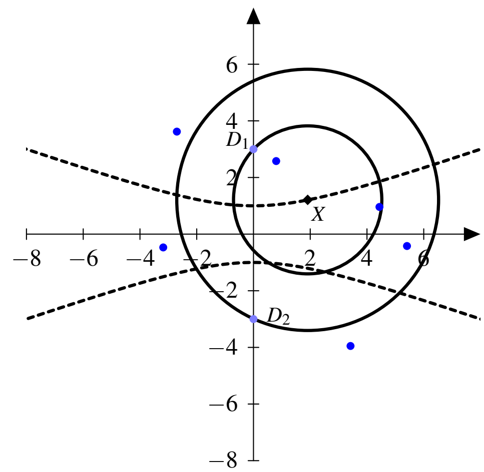


(a) Case i.

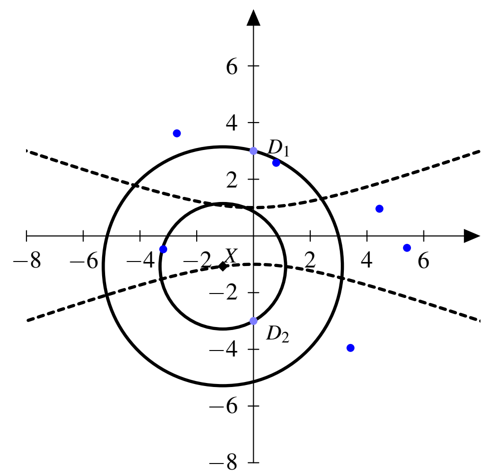


(b) Case ii.

Figure 3.2: The cases that the pair of demand points is in one circle of an annulus.



(a) Case iii.



(b) Case iv.

Figure 3.3: The cases that two demand points are in different circles of an annulus.

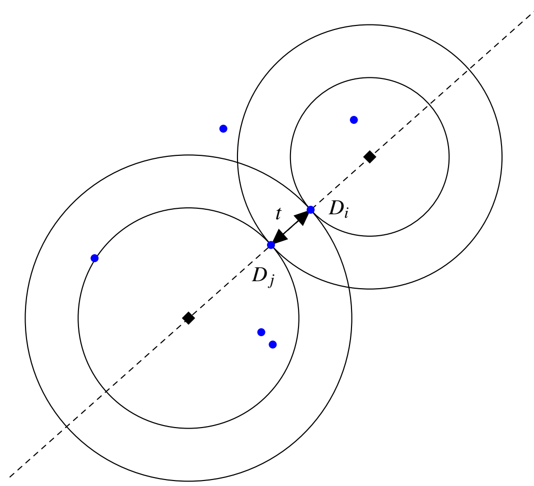


Figure 3.4: The line containing the best annulus that has demand points D_i and D_j on its boundary (one on each circle).

An enumeration algorithm may be implemented by performing the following steps for each pair of demand points, and retaining the best candidate as the optimal annulus:

step 1 Rotate and translate the axes so that the pair of demand points will lie on the vertical axis and the origin will be exactly in the middle between the pair.

step 2 Search for the centre point X of the annulus by minimizing the objective functions given in the cases above. If the distance between the pair is less than or equal to the given width, minimize the objective functions considering what was mentioned in the preceding note.

step 3 Identify the X that gives the minimum value among up to four resulting objective values.

Repeating the steps above for every pair of n demand points will result in $\binom{n}{2}$ objective values and candidate centre points X. The location of the centre point that results in the minimum value, is the optimal location of the centre of the annulus. After finding the optimal X, we reverse the rotation and translation process with respect to the pair of demand points giving the optimal solution on the hyperbola or bisector between them. Therefore we obtain X in the original axes. The inner radius of the annulus is unchanged.

3.2 Examples

The following unweighted examples ($w_i = 1, \forall i$) in this chapter are solved by the exact algorithm described earlier, which was coded in MatLab R2012b. Two MatLab functions are used to minimize the objective functions: ”fminsearch” and ”fminbnd.” The first function (fminsearch) uses the simplex algorithm described by Lagarias *et al.* [21]. The other function uses the golden section search whose algorithm is described in Brent [3] and Forsythe *et al.* [15]. Both functions resulted in the same answers most of the time. They resulted in different answers when either one of them stops at a local minimum, which was observed to occur a few times. The fminsearch algorithm gives more accurate answers when the search is in smaller intervals.

Example 3.1. *Seven unweighted demand points were randomly generated in the area* $\mathscr{S} = \{x, y : -20 \leq x, y \leq 20\}$. *The locations of the points are given in Table 3.1. We want to find the best location of an annulus of width* $t = 2$ *that minimizes the sum of distances from demand points to the annulus. Since there are seven demand points, there will be 21 pairs of demand points and up to four objective functions for every pair to be minimized. The centre point of the optimal annulus is* $X^* = (-4.58, -7.68)$ *and the objective value is* $f(A(X^*, r^*)) = 19.72$*, where* $r^* = 14.32$*. The centre point* X^* *was found at the intersection point of three bisectors between the pairs* (D_3, D_4)*,* (D_3, D_5)*, and* (D_4, D_5)*, so in this example there are three demand points on the boundary of the optimal annulus and all of them are on the inner circle (see Figure 3.5). Not only did this optimal solution result in three demand points on the boundary, but also the best annulus resulting from each other pair had three demand points on its boundary. Therefore, two or three pairs could result in the same annulus; for example, the optimal annulus was obtained from the search on three pairs. Also the boundary points on the best annuli are not necessarily on the same circle. For example the second best annulus resulting from the search between pairs has* D_4 *and* D_5 *on its inner circle and* D_3 *on the outer circle with objective* *value* 19.79 [1]. *The third best annulus also has two demand points on the inner circle and one on the outer circle in contrast to the fourth best which has two on the outer circle and one on the inner circle. Figure 3.6 shows the shapes of the four objective functions, given on the cases mentioned earlier in the chapter, for the three pairs* (D_3, D_4), (D_3, D_5), *and* (D_4, D_5). *A summary of the best five solutions resulting from the search between pairs of demand points is given on Table 3.2.*

Table 3.1: Locations of Demand Points Given in Example 3.1.

Demand Point	Location
D_1	(-0.45, 2.71)
D_2	(18.86, -2.94)
D_3	(-15.5, -16.95)
D_4	(9.73, -8.38)
D_5	(5.54, 2.45)
D_6	(3.77, 5.33)
D_7	(-0.06, 17.23)

Table 3.2: A Summary of the Best Five Annuli Coming from the Search between Pairs.

Value	centre Point	Radius	Inner Circle Points	Outer Circle Points
19.72	(-4.58,-7.68)	14.32	D_3 D_4 D_5	ϕ
19.79	(-3.57,-7.31)	13.35	D_4 D_5	D_3
20.01	(-4.29,-8.51)	14.03	D_3 D_4	D_6
20.03	(-3.36,-7.98)	13.10	D_4	D_3 D_6
20.21	(-4.94,-6.61)	14.78	D_3 D_4 D_6	ϕ

The optimal annulus in example 3.1 has three points on its boundary. This raises the question as to whether the optimal annulus always has three demand points on the boundary or not. We know from example 2.1 that the answer is no, but can we use a good heuristic by assuming the optimal annulus has three demand points on the boundary? One hundred examples similar to example 3.1 were generated and solved using the exact algorithm described earlier. The optimal annulus of 97 of them had

[1]Note that this annulus is not the second best in general because moving the centre point of the optimal annulus by an arbitrary small distance will result in a better annulus.

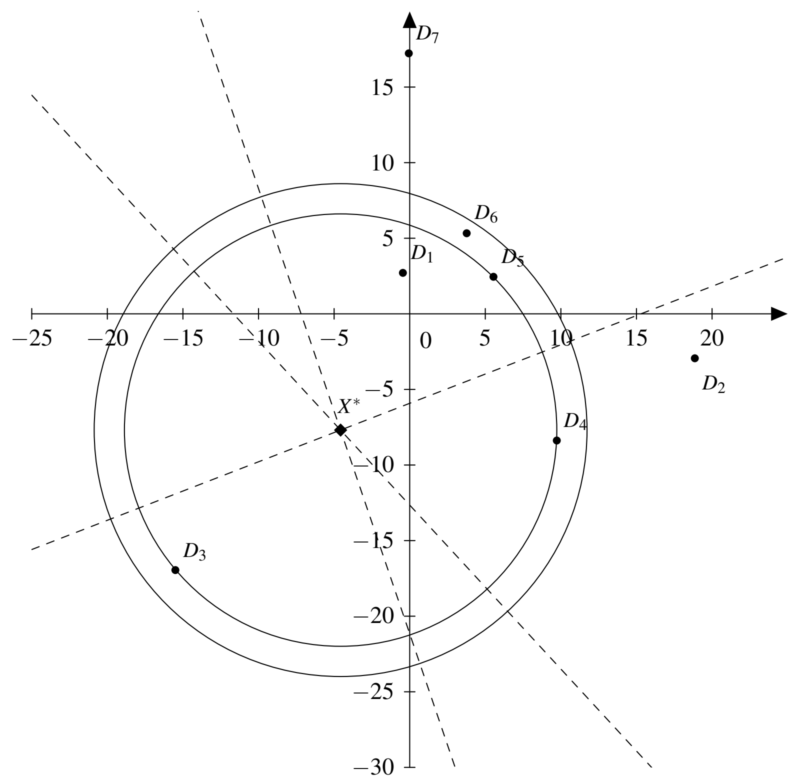
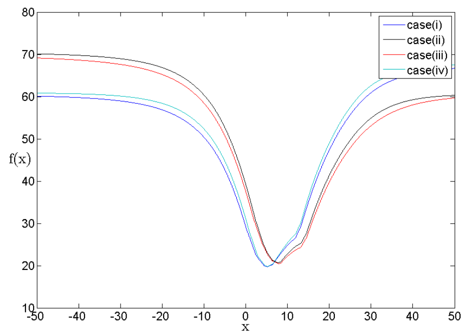
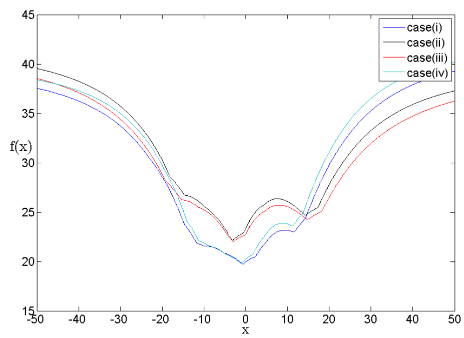


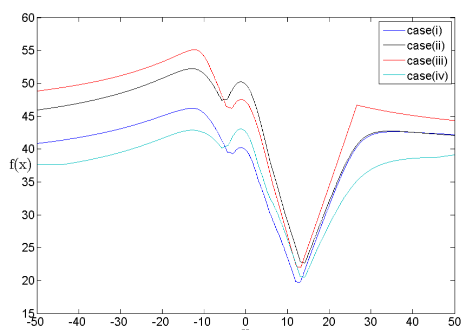
Figure 3.5: The optimal annulus in example 3.1.



(a) The search between the pair (D_3, D_4).



(b) The search between the pair (D_3, D_5).



(c) The search between the pair (D_4, D_5).

Figure 3.6: The four objective functions of the pairs (D_3, D_4), (D_3, D_5), and (D_4, D_5) in example 3.1.

three demand points on the boundary distributed as follows: three points on the outer circle in 8 examples, two on the outer and one on the inner circle in 32 examples, one on the outer and two on the inner in 45 examples, and in 12 examples three demand points are on the inner circle. Only three examples resulted in optimal annuli with only two demand points on the boundary. In 4 examples, the radius length of the optimal solution was found to be greater than 40, which means the optimal solution is more a strip than an annulus in these examples, (the number might increase if $t > 2$). The average CPU time for solving one of the instances with 7 demand points is 0.495 seconds. Therefore using the heuristic in a small problem might not be very effective.

To see if using the heuristic, finding the optimal annulus among annuli that have three demand points on the boundary, to solve the problem is accurate and effective or not, we repeated the last experiment with larger problems. We generated 100 examples with 10 demand points randomly located in the same area as in example 3.1, and used the exact algorithm to solve the problems. By repeating the process for five demand point increments up to 50 demand points, we have a total of 900 examples. Out of 900 examples, the optimal annulus has two demand points on its boundary in five examples only, which means the heuristic is accurate, especially for problems with 30 demand points and more.

Table 3.3 shows the average CPU times (in seconds) per example, and the number of solutions having 2, 3, or 4 demand points on the boundary. It also shows the number of solutions with a long radius, which means the solution resembles a strip. We notice that as the number of demand points increases, the chance of having a strip as the optimal answer fades (this information could change if the given width is different).

The results also show that when the number of demand points increases, the location of the optimal annulus centre point approaches the centre point of the area. Figure 3.7 illustrates the relation between the number of demand points and the average distance to the centre point of the area from the optimal annulus centre point.

The average CPU time of solving one problem with 10 demand points is 1.135 seconds, while it is 50.563 seconds for a problem with 50 demand points. Thus the heuristic should be very effective for larger problems since the search along the hyperbolas and bisectors is not required by the heuristic. The number of combinations

Table 3.3: A Summary of the Results of 900 examples.

n	Avg CPU Times	Boundary Points 2	3	4	Long Radius
10	1.135	3	97	0	2
15	2.811	1	99	0	0
20	5.546	0	100	0	1
25	9.369	1	97	2	0
30	14.394	0	97	3	0
35	21.055	0	95	5	0
40	29.031	0	100	0	0
45	39.186	0	100	0	0
50	50.563	0	100	0	0

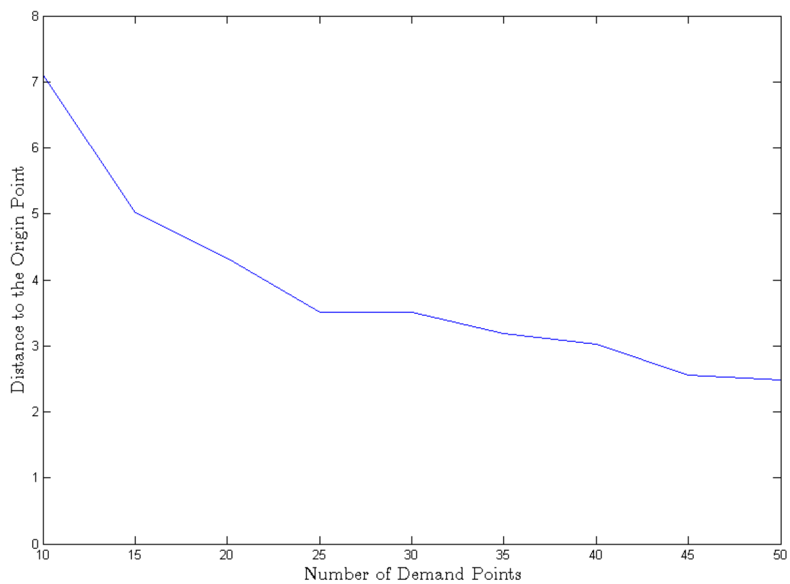


Figure 3.7: The relation between the number of demand points and the average distance from the optimal centre point to the origin.

of any three demand points being on an annulus boundary is 8. The centre point of the annulus with three demand points on the boundary should be the intersection point of the three bisectors when the three demand points are in one circle, either the inner or the outer (two cases), or the intersection point of the bisector between two demand points and the hyperbola between one of the two points and the third demand point (six cases). For example if the points D_i and D_j are on one circle and D_k is on the other of an annulus, the candidate centre points of the annulus are the intersection points of the bisector between D_i and D_j and the hyperbola between D_k and one of D_i and D_j.(Because there are two branches of the hyperbola, there will be two intersection points: at one of them, D_k will be on the inner circle, and the opposite at the other intersection point.) As a result, to solve a problem with n demand points using the heuristic, we need to find the best annulus of up to $8\binom{n}{3}$ annuli. This should happen in much faster time than solving the problem using the exact algorithm, which requires $4\binom{n}{2}$ numerical searches along bisectors and hyperbolas.

Chapter 4

Finding an Annulus with Fixed Radius

4.1 Objective Function

In Chapter 3, in which the annulus to be located has a variable radius r, it was proved that the optimal annulus must have at least two demand points on its boundary. However, when the radius is fixed ($r = r_0$), the function will transform from a function of three variables $f(A(X,r))$ to a function of two variables $f(X)$, or equivalently $f(x,y)$, and the incidence property is no longer satisfied. Although the function may have a nice shape after fixing the radius, the function is still neither convex nor concave. For example, a set of 20 unweighted demand points is given in the area $\mathscr{S} = \{(x,y) : -10 \leq x,y \leq 10\}$, as in Figure 4.1. The objective function is the sum of the distances from these points to an annulus of given width and given radius. Suppose the width is $t = 2$ and the radius is $r = 4$. The three dimensional shape of the function is illustrated in Figure 4.2. The objective function will keep the same form as the form in the variable radius case, except it will be a function of only two variables:

$$\min f(X) = \sum_{i\in J^-} w_i(r_0 - d_i(X)) + \sum_{i\in J^+} w_i(d_i(X) - (r_0 + t)). \qquad (4.1)$$

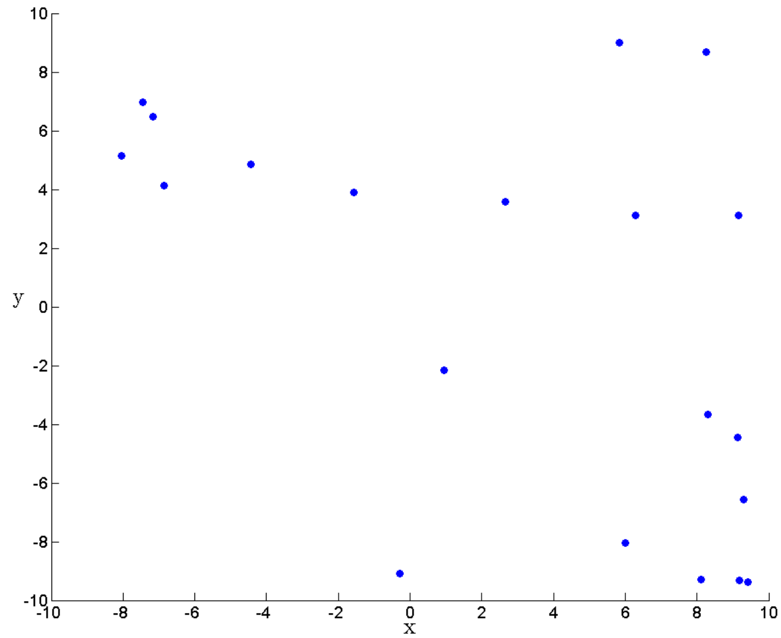


Figure 4.1: A set of 20 unweighted demand points.

4.2 Applications

The partial coverage distance annulus with fixed radius may be very useful in locating an undesirable facility, also known as a noxious or obnoxious facility. The concept of the location of undesirable facilities was introduced early in the 1970s. Later, because some travel between demand points and undesirable facilities is necessary, the terms semi-desirable and semi-obnoxious were introduced (see Eiselt and Laporte [13] and Melachrinoudis [24]). Since it still remains undesirable to locate these facilities close to demand points, we will use the term *undesirable facilities* to refer to semi-desirable and undesirable facilities.

For the past four decades, push models have been introduced and developed to solve undesirable location problems. As mentioned in Chapter 1, Church and Garfinkel [9] introduced the maxisum model to locate undesirable facilities. The objective of this model is to maximize the sum of distances from demand points to the new facility. Another common model to solve undesirable facility location problems is the maximin model. The aim of the maximin model is to locate the undesirable

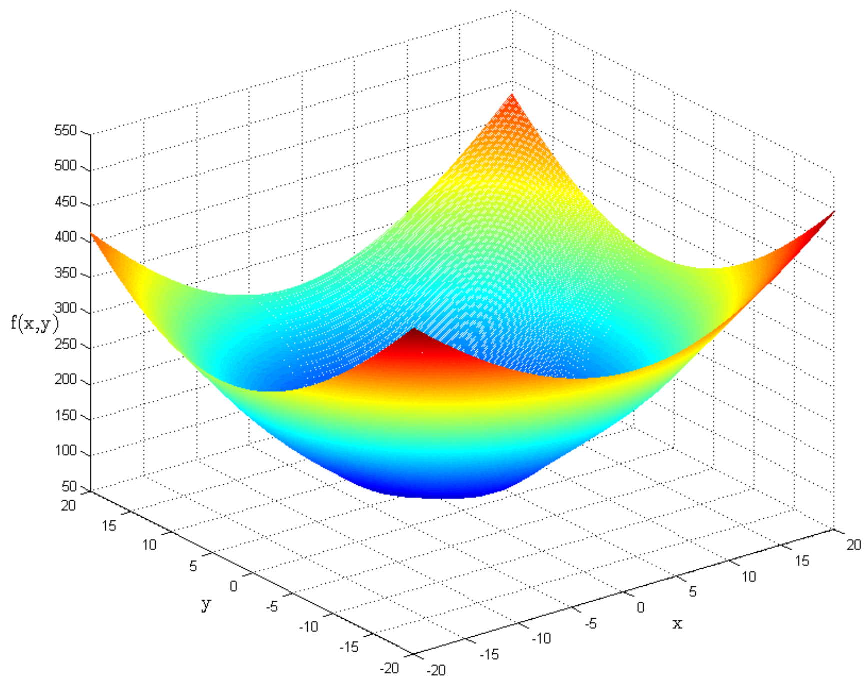


Figure 4.2: The shape of the objective function for fixed radius $r_0 = 4$.

facility such that the distance to the facility from the closest demand point is maximized, i.e., the objective is maximizing the minimum distance between the demand points and the new facility. This is equivalent to finding the largest empty disc. The problem of finding the largest empty disc was introduced and solved by Shamos and Hoey [28], but it was first used in location problems by Dasarathy and White [10] and Drezner and Wesolowsky [11]. *Empty covering* is another term used to refer to these models. Minimal Covering models are related to empty covering models, but the difference is that the radius of the disc is given in the minimal covering models and the objective is to locate the disc such as the number of covered demand points is minimized. Drezner and Wesolowsky [12] introduced and studied minimal covering models. More details about undesirable facilities, their types, and different models employed to solve the problem can be found in Melachrinoudis [24] and Hosseini and Esfahani [19].

The objective of push models mentioned above suggests that the optimal solution is to locate the undesirable facility at infinity. Such results are not applicable in real life problems. In fact, the location of the undesirable facility should not be close to demand points, but it must not be out of reach. There is no point in locating an airport, for example, at three hours travel from the closest demand point because that will be a good solution. Therefore, in push models, there must be some constraints to avoid similar situations. In our model, we do not need to add constraints to the problem since we are minimizing the sum of distances to the annulus, in which the undesirable facility is the centre of the annulus. Thus the facility will not be very distant because the objective value will increase as the facility moves away from demand points.

Most push models are not designed for different kinds of undesirable facilities problems. One of the advantages of using our model to locate an undesirable facility is that the model can be used for all kinds of undesirable facilities. For example, if the new facility is undesirable because of its hazardous nature (e.g., nuclear plant, explosive storage), then the maxisum model is not the best choice to locate the facility because the optimal location might result in one demand point or more being very close to the facility. The empty covering (maximin) model would be a better option than the maxisum model to locate a hazardous facility. However, the radius of the largest empty disc might be much greater than the safe distance from the facility, which will cause unnecessary traveling costs, or shorter than the safe distance, leading to demand points being in the danger zone. Furthermore, the minimal covering model will minimize the number of demand points in the danger zone, but it does not guarantee that no points will be close to the hazardous facility.

Therefore our model can be used effectively to find the optimal locations for these facilities. Even if the new facility is not dangerous, but it is undesirable because of some unpleasant effects to the neighborhood such as noise and traffic (e.g., airports, retail stores, sports stadiums), our model can be used. To control the degree of undesirability, a parameter λ is added to equation 4.1 and the objective of our model to find a minisum annulus with fixed radius becomes

$$\min f(X) = \lambda \sum_{i \in J^-} w_i(r_0 - d_i(X)) + (1-\lambda) \sum_{i \in J^+} w_i(d_i(X) - (r_0 + t)), \tag{4.2}$$

where $0 < \lambda < 1$. As the degree of undesirability increases, the value of λ should increase. If the new facility is extremely dangerous to the surrounding areas then increasing λ to a value close to one will insure that no demand points will be inside the inner circle (surrounding area of the facility). On the other hand, if there is no harm to demand points near the facility but it is undesirable because of the noise, for example, then being far from the facility might be as unpleasant as being close. In this case, λ can be set to equal 0.5, or even less than 0.5 if demand points prefer to be close more than being very far. Setting λ to be 0.5 does not mean the undesirable facility will be located near demand points since the goal of our model is to minimize the sum of distances to the annulus not to the centre (the facility).

For example, consider locating a nuclear power plant, where demand points are the nearby cities. According to *Ready*, a national public service advertising campaign designed for emergencies preparation in the U.S., radiation exposure might harm people within 10 miles of a nuclear power plant. Therefore we use the minisum model to locate an annulus with fixed radius, where the plant is the centre of the annulus and the radius is fixed to be $r = 10$. Now for a worker in the plant, assume that traveling more than 25 miles is uncomfortable, so the width of the annulus is $t = 15$. That means it is dangerous for the worker to be within 10 miles from the facility and unpleasant for him to be more than 25 miles away from the facility. We can call the zone between 10 to 25 miles from the plant as the comfort zone (see

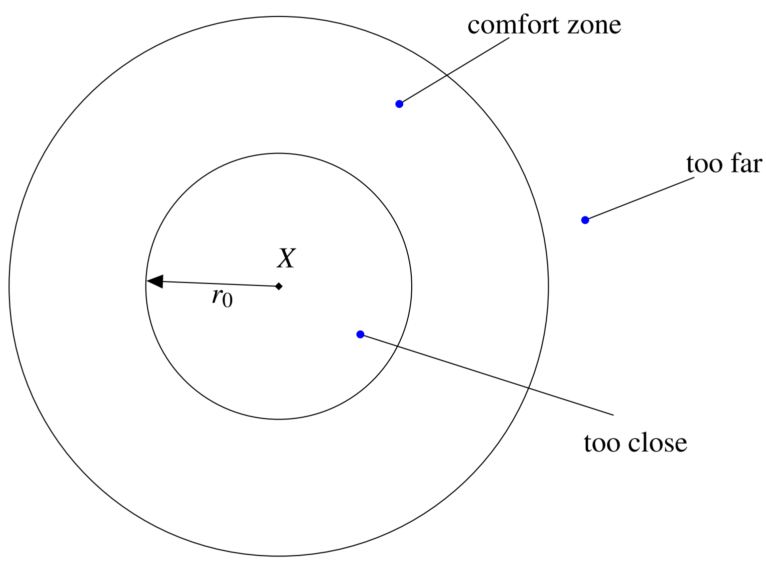


Figure 4.3: Locating an undesirable facility using a minisum annulus model.

Figure 4.2). The weight of each city can be proportional to its population. Setting $\lambda = 0.8$ guarantees that no demand point will be far inside the inner circle (i.e., no city will be much less than 10 miles from the plant), unless it will be extremely costly otherwise. To find the optimal location of the plant, we minimize the function

$$f(X) = 0.8 \sum_{i \in J^-} w_i(10 - d_i(X)) + 0.2 \sum_{i \in J^+} w_i(d_i(X) - (25)).$$

Locating the plant using this model will not only keep cities out of the danger zone, but also minimize transportation costs between the plant and the cities.

An example of an undesirable facility that is not dangerous is an airport. The objective function is similar to the previous example. Only some or all of the parameters are changed. For example, λ might be set to $\lambda = 0.5$. The parameter values, for the same kind of undesirable facility, are subject to be changed based on different factors (e.g., demand point disposition, travel costs). Thus, the values might be chosen based on opinion surveys among demand points. Our model aims to locate the undesirable facility such that as many demand points as possible are in the comfort zone, and control the number of demand points inside and outside the comfort zone.

Note. *When $t = 0$, the problem transforms to the circle problem, and similarly to the annulus problem, the parameter λ might be included in the objective function of the circle problem. Thus finding a circle with fixed radius using a minisum model can also be used to locate undesirable facilities.*

In conclusion, finding a partial coverage distance annulus with a fixed radius using a minisum model has many advantages for undesirable facility location problems. The model does not have to add constraints to the problem. It might be used efficiently to locate any kind of undesirable facility by changing the parameters. Moreover, the objective of the model is to choose the location that considers the cost of transportation that keeps the demand points at a safe distance from the facility.

Chapter 5

Conclusions

5.1 Summary

In this thesis, we study the problem of finding a minisum annulus with given width using the partial coverage distance model. The radius of the inner circle of the annulus is either variable or given, and both cases are presented. The objective is to locate an annulus such that demand points covered by the annulus are served at no cost, and uncovered demand points are served at additional cost. This problem is motivated by two problems. The first problem is locating a disc using partial coverage distance models introduced by Brimberg *et al.* [4]. The second problem that motivated the subject of this thesis is the minisum circle problem. In fact, the minisum annulus is the general case of the minisum circle.

The properties and theories that are presented by Brimberg *et al.* [6] and used to find the optimal minisum circle are generalized to find the optimal minisum annulus. The main property that was generalized from the circle problem to the annulus problem is the incidence property, which is that two demand points must be on the boundary of the optimal annulus.

Unlike the circle problem in which the optimal centre must be on a bisector of two demand points, the incidence property for the annulus indicates that the centre of the optimal annulus may otherwise be on a defined hyperbola between two demand points. Using this property, an exact algorithm to find the optimal minisum annulus was introduced. After solving numerous examples by the exact algorithm, we observed that almost all optimal annuli resulted in three demand points on the boundary. This result suggests the use of a fast heuristic that requires three demand points to be on the boundary of the annulus. Note that an analogous heuristic was used to solve the circle problem in Brimberg *et al.* [6].

The incidence property and the heuristic help solve the problem of finding an annulus in which the radius of the inner circle is unknown. In the case that the inner radius is fixed, the optimal annulus does not necessarily have demand points on its boundary. Therefore, the problem cannot be solved by the exact algorithm or the heuristic used to solve the problem with unknown inner radius.

One of the applications of the problem of locating an annulus with fixed radius is to locate an undesirable facility, in which the undesirable facility is the centre of the annulus. Different push models are introduced in the literature and used to locate different kinds of undesirable facilities; however, the model we introduce to locate the optimal minisum annulus with fixed radius can be a good alternative to those models. It aims at keeping demand points away from the facility, and at the same time, minimizing the costs of travel. Also the model is flexible, so it can be adjusted to locate undesirable facilities with different degrees of undesirability.

5.2 Future Work

Since the partial coverage distance model was introduced recently, many directions can be studied for annular facilities or for other facility problems. In this thesis we covered the problem of locating a minisum annulus. We introduced the model and derived an exact algorithm to solve the case when the inner radius is variable. In the fixed radius case, only the model was introduced. Thus, methods to solve the problem of finding an annulus with fixed radius can be further investigated. Furthermore, we might introduce the problem of finding an annulus with fixed inner radius and variable width, or fixed outer radius and variable width.

Another direction to be discussed is expanding the problem to use other norms to calculate the distances. We used the Euclidean norm to measure distances. Rectangular distance and Chebyshev distance are other common distance functions. Also problems in which the distance is measured by a general norm might be considered.

The problem we introduced is finding the location of one new facility. We may study locating more than one minisum annulus using the partial coverage distance model. also locating a maxisum annulus using the partial coverage distance model can be investigated.

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