A UNIFIED APPROACH TO MULTI-SERVER BULK-ARRIVAL QUEUES USING ROOTS

UNE APPROCHE UNIFIÉE POUR FILES D'ATTENTE D'ARRIVÉES DE MASSE DE MULTISERVEURS À TRAVERS RACINES

A Thesis Submitted

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by

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This thesis is dedicated to all Signalers at the 427 Special Operations Aviation Squadron

"Velox, Versutus, Vigilans"

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ABSTRACT

Kim, James Jaehak, Royal Military College of Canada, 2 April 2018. A unified approach to multi-server bulk-arrival queues using roots. Supervised by Dr. Mohan Chaudhry.

This thesis encompasses new and complete solution procedures that solve multi-server bulk-arrival queues.

In solving $GI^X/M/c$ queues, an elegant and simple solution to determine the distributions of queue-length at different time epochs and the waiting-time for the model are presented. In the past, $GI^X/M/c$ queues have been extensively analyzed using various techniques by many authors. The purpose of this portion of the thesis is to use the roots method to derive the analytic solution for the pre-arrival time epoch probabilities based on the roots of the model's characteristic equation. The solution is then leveraged to compute the waiting-time distributions as well as the case when the inter-batch-arrival times follow heavy-tailed distributions. The method is also extended to solve $GI^X/M/c/N$ queues. Numerical examples are presented.

In solving $GI^X/Geo/c$ queues, a simple solution to determine the distributions of queue-lengths at different observation epochs is presented. In the past, various discrete-time queueing models, particularly multi-server bulk-arrival queues have been solved using complicated methods that lead to incomplete results. The purpose of this portion of the thesis is to show that the roots method can solve $GI^X/Geo/c$ queues. The method works well even for the case when the inter-batch-arrival times follow heavy-tailed distributions. The roots method is also extended to solve $GI^X/Geo/c/N$ queues. Numerical examples are presented.

Finally, the roots method presented in this thesis can be seen as a unified approach for analyzing multi-server bulk-arrival queues that involve continuous and discrete-times, finite and infinite-buffers, and light and heavy-tailed inter-batch-arrival times.

Keywords: Multi-server; bulk-arrival; queues; roots; continuous-time; discrete-time; finite-buffer; infinite-buffer; light-tailed; heavy-tailed

RÉSUMÉ

Kim, James Jaehak, collège militaire royal du Canada, 2 avril 2018. Une approche unifiée pour files d’attente d’arrivées de masse de multiserveurs à travers racines. Dirigé par M. Mohan Chaudhry, Ph.D.

Cette thèse englobe des procédures sur de nouvelles solutions et solutions complètes qui résolvent les files d'attente d'arrivées de masse de multiserveurs.

En résolvant les files d'attente $GI^X/M/c$, une solution simple et élégante pour les distributions de longueurs de files d'attente à différents temps donnés, ainsi que le temps d'attente pour le modèle, sont présentés. Les files d'attente $GI^X/M/c$ furent auparavant largement analysées par diverses techniques utilisées par de nombreux auteurs. L'objectif de cette partie de la thèse est d'utiliser la méthode des racines pour dériver la solution analytique pour les probabilités de pré-arrivées de temps donnés en termes des racines de l'équation caractéristique du modèle. La solution est ensuite utilisée pour calculer les distributions de temps d'attente, ainsi que dans le cas lorsque les temps d'arrivée d'inter-lots suivent les distributions à hautes valeurs. La méthode est également davantage utilisée pour résoudre les files d'attente $GI^X/M/c/N$. Des exemples numériques sont présentés.

En résolvant les files d'attente $GI^X/Geo/c$, une solution simple pour déterminer les distributions de longueurs de files d'attente à différents temps observés est présentée. Divers modèles de files d'attente à temps discret, en particulier les files d'attente d'arrivées de masse de multiserveurs, furent auparavant résolus en utilisant des méthodes complexes conduisant à des résultats incomplets. L'objectif de cette partie de la thèse est de démontrer que la méthode des racines peut résoudre les files d'attente $GI^X/Geo/c$. La méthode fonctionne également dans le cas lorsque les temps d'arrivée d'inter-lots suivent les distributions à hautes valeurs. La méthode des racines est également utilisée pour résoudre les files d'attente $GI^X/Geo/c/N$. Des exemples numériques sont présentés.

Enfin, la méthode des racines présentée dans cette thèse peut être considérée comme une approche unifiée pour files d'attente d'arrivées de masse de multiserveurs qui impliquent des temps continus et discrets, mémoires-tampon finies et infinies, ainsi que temps d'arrivée d'inter-lots à hautes et faibles valeurs.

Mots-clés: Multiserveurs; arrivées de masse; files d'attente; racines; temps continus; temps discrets; mémoires-tampon finies; mémoires-tampon infinies; hautes valeurs; faibles valeurs

LIST OF PUBLICATIONS

This thesis is based on the following submitted and published papers:

Chapters 2 and 3:

1. Kim, J.J. and Chaudhry, M.L.: Simple and Computationally Efficient Method for Internet-Type Queues ($GI/M/1$ and $GI^X/M/1$) with Power-Tailed Inter-Arrival Times. A Tribute to Professor Jyotiprasad Medhi. Gauhati University Press. 18-24 (2017)

2. Chaudhry, M.L. and Kim, J.J.: Analytically Elegant and Computationally Efficient Results in terms of Roots for the $GI^X/M/c$ Queueing System. Queueing Systems. 82(1-2), 237-257 (2016)

3. Chaudhry, M.L. and Kim, J.J.: A Short Note on the $GI^X/M/c$ Queues Involving Heavy-Tailed Inter-Batch-Arrival Distributions. Submitted to OPSEARCH (2017)

4. Chaudhry, M.L. and Kim, J.J.: Novel way of determining the $GI/M/1/N$ queues involving heavy-tailed inter-arrival times. Submitted to Mathematical Problems in Engineering (2017)

5. Kim, J.J. and Chaudhry, M.L.: A Novel Way of Treating the Finite-Buffer Queue $GI/M/c/N$ Using Roots. International Journal of Mathematical Models and Methods in Applied Sciences. 11, 286-289 (2017)

Chapters 4 and 5:

6. Chaudhry, M.L. and Kim, J.J.: Analytically Simple and Computationally Efficient Solution to $GI^X/Geom/1$ Queues Involving Heavy-Tailed Distributions. O.R. Letters. 44(5), 655-657 (2016)

7. Kim, J.J. and Chaudhry, M.L.: Analytically Simple and Computationally Efficient Results for the $GI^X/Geom/c$ Queues. Submitted to Annals of Operations Research (2017)

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(1) $p_j^- = \sum_{i=0}^{\infty} p_i^- P_{i,j}\,, (j \geq 0)$

(2) $1 = B(z^{-1})K(z)$

(3) $p_j^- = \sum_{h=1}^{r} C_h z_h^j\,, (j \geq c)$

(4) $p_j^- = \sum_{i=0}^{c-1} p_i^- P_{i,j} + \sum_{i=c}^{\infty} \sum_{h=1}^{r} C_h z_h^i \, P_{i,j}\,, (j \geq 0)$

(5) $p_j^- = \begin{cases} \text{determined above}, & (0 \leq j \leq c-1) \\ \sum_{h=1}^{r} C_h z_h^j, & (j \geq c) \end{cases}$

(6) $W_F(0) = \sum_{i=0}^{c-1} p_i^-$

(7) $W_F(t) = P(W_F \leq t) = W_F(0) + \sum_{i=c}^{\infty} p_i^- \int_0^t \frac{(c\mu v)^{i-c}}{(i-c)!} (c\mu) e^{-c\mu v} \, dv$

(8) $W_R(0) = \sum_{f=1}^{r} \sum_{i=0}^{c-f} p_i^- \, r_f$

(9) $W_R(t) = P(W_R \leq t) = W_R(0) + \sum_{f=1}^{r} r_f \sum_{i=c-f+1}^{\infty} p_i^- \int_0^t \frac{(c\mu v)^{i-c}}{(i+f-c-1)!} (c\mu)^f v^{f-1} e^{-c\mu v} \, dv$

(10) $L_s = \sum_{i=1}^{\infty} i \, p_i$

(11) $L_q = \sum_{i=c+1}^{\infty} (i-c) \, p_i$

(12) $W_{q_1}^- = \int_0^{\infty} t dW_F(t)$

(13) $W_q^- = \int_0^{\infty} t dW_R(t)$

(14) $\sum_{j=0}^{c-1} (c-j) p_j = c(1-\rho)$

(15) $p_j^- = \sum_{i=0}^{N} p_i^- \, P_{i,j}, (0 \leq j \leq N)$

(16) $0 = \sum_{j=1}^{N} \left(\sum_{i=0}^{N} z^i \, P_{i,j} - z^j \right)$

(17) $p_j^- = \sum_{h=1}^{N} C_h z_h^j\,, (1 \leq j \leq N)$

(18) $0 = \sum_{h=1}^{N} C_h \left(\sum_{i=j-h}^{N} z_h^i \, P_{i,j} - z_h^j \right), (1 \leq j \leq N)$

(19) $$p_j^- = \sum_{h=1}^{N} C_h z_h^j \, , (0 \le j \le N)$$

(20) $$\omega(j|i) = \begin{cases} \binom{i}{j} \mu^j (1-\mu)^{i-j}, (0 \le j \le i < c) \\ \binom{c}{j} \mu^j (1-\mu)^{c-j}, (0 \le j \le c \le i) \end{cases}$$

(21) $$Q_j^- = \sum_{i=0}^{\infty} Q_i^- P_{i,j} \, , (j \ge 0)$$

(22) $$1 = B(z^{-1}) K(z)$$

(23) $$Q_j^- = \sum_{h=1}^{r} C_h z_h^j \, , (j \ge c)$$

(24) $$Q_j^- = \sum_{i=0}^{c-1} Q_i^- P_{i,j} + \sum_{i=c}^{\infty} \sum_{h=1}^{r} C_h z_h^j P_{i,j} \, , (j \ge 0)$$

(25) $$Q_j^- = \begin{cases} \text{determined above}, (0 \le j \le c-1) \\ \sum_{h=1}^{r} C_h z_h^j, \qquad (j \ge c) \end{cases}$$

(26) $$Q_l = \sum_{j=l}^{l+c} Q_j^o \omega(j-l|j) \, , (l \ge 0)$$

(27) $$Q_l = \sum_{j=l}^{c-1} Q_j^o \omega(j-l|j) + \sum_{j=c}^{l+c} \sum_{h=1}^{r} E_h z_h^j \omega(j-l|j), (l \ge 0)$$

(28) $$\sum_{j=0}^{c-1} Q_j^o + \sum_{j=c}^{\infty} \sum_{h=1}^{r} E_h z_h^j = 1$$

(29) $$Q_j^o = \begin{cases} \text{determined above}, (0 \le j < c) \\ \sum_{h=1}^{r} E_h z_h^j, \qquad (j \ge c) \end{cases}$$

(30) $$Q_k^- = \sum_{j=k}^{n+c} P_j^- \omega(j-k|j) \, , (k \ge 0)$$

(31) $$P_j^- = \sum_{h=1}^{r} F_h z_h^j \, , (j \ge c+1)$$

(32) $$Q_k^- = \sum_{j=k}^{c} P_j^- \omega(j-k|j) + \sum_{j=c+1}^{k+c} \sum_{h=1}^{r} F_h z_h^j \omega(j-k|j), (k \ge 0)$$

(33) $$\sum_{j=0}^{c} P_j^- + \sum_{j=c+1}^{\infty} \sum_{h=1}^{r} F_h z_h^j = 1$$

(34) $$P_j^- = \begin{cases} \text{determined above}, (0 \le j \le c) \\ \sum_{h=1}^{r} F_h z_h^j, \qquad (j \ge c+1) \end{cases}$$

(35) $$W_s = \frac{\bar{a} L_s^o}{\bar{b}}$$

(36) $$W_q = \frac{\bar{a} L_q^o}{\bar{b}}$$

(37) $\sum_{j=0}^{c-1}(c-j)Q_j^o = c(1-\rho)$

(38) $\omega(j|i) = \begin{cases} \binom{i}{j}\mu^j(1-\mu)^{i-j}, (0 \le j \le i < c \le N) \\ \binom{c}{j}\mu^j(1-\mu)^{c-j}, (0 \le j \le c \le i \le N) \end{cases}$

(39) $Q_j^- = \sum_{i=0}^{N} Q_i^- P_{i,j}, (0 \le j \le N)$

(40) $Q_j^- = \sum_{h=1}^{N} C_h z_h^j, (1 \le j \le N)$

(41) $0 = \sum_{h=1}^{N} C_h \left(\sum_{i=j-h}^{N} z_h^i P_{i,j} - z_h^j\right), (1 \le j \le N)$

(42) $Q_j^- = \sum_{h=1}^{N} C_h z_h^j, (0 \le j \le N)$

(43) $Q_l = \sum_{j=l}^{\min(l+c,N)} Q_j^o \omega(j-l|j), (0 \le l \le N)$

(44) $Q_l = \sum_{j=l}^{\min(l+c,N)} \sum_{h=1}^{N} E_h z_h^j \omega(j-l|j), (1 \le l \le N)$

(45) $Q_0^o + \sum_{j=1}^{N} \sum_{h=1}^{N} E_h z_h^j = 1$

(46) $Q_j^o = \begin{cases} \text{determined above}, (j=0) \\ \sum_{h=1}^{r} E_h z_h^j, \quad (1 \le j \le N) \end{cases}$

(47) $Q_k^- = \sum_{j=k}^{\min(n+c,N)} P_j^- \omega(j-k|j), (0 \le k \le N)$

(48) $Q_k^- = \sum_{j=k}^{\min(n+c,N)} \sum_{h=1}^{N} F_h z_h^j \omega(j-k|j), (0 \le k \le N)$

(49) $P_0^- + \sum_{j=1}^{N} \sum_{h=1}^{r} F_h z_h^j = 1$

(50) $P_j^- = \begin{cases} \text{determined above}, (j=0) \\ \sum_{h=1}^{r} F_h z_h^j, \quad (1 \le j \le N) \end{cases}$

(51) $0 = \sum_{h=1}^{r} C_h z_h^j \left(\sum_{l=1}^{r} b_l z_h^{-l} \sum_{n=0}^{l-j-1} k_n z_h^n\right), (c \le j \le r-1)$

(52) $\sum_{h=1}^{r} \frac{C_h}{z_h} = \sum_{h=1}^{r} \frac{C_h}{z_h^2} = \cdots = \sum_{h=1}^{r} \frac{C_h}{z_h^{r-c-1}} = \sum_{h=1}^{r} \frac{C_h}{z_h^{r-c}} = 0$

(53) $0 = \sum_{h=1}^{r} C_h z_h^j \left(\sum_{l=1}^{r} b_l z_h^{-l} \sum_{n=0}^{l-j-1} k_n z_h^n\right), (1 \le j \le c-1)$

(54) $\sum_{h=1}^{r} \frac{C_h}{1-z_h} = 1$

(55) $$p_j^- = \sum_{h=1}^{r} C_h z_h^j , (j \geq 0)$$

(56) $$0 = p_0^-\left(P_{0,0} - 1\right) + p_1^- P_{1,0} + p_2^- P_{2,0} + \ldots + p_{c-1}^- P_{c-1,0} + \sum_{i=c}^{\infty}\left(\sum_{h=1}^{r} C_h z_h^i\right) P_{i,0}$$

$$0 = p_0^- P_{0,1} + p_1^-\left(P_{1,1} - 1\right) + p_2^- P_{2,1} + \ldots + p_{c-1}^- P_{c-1,1} + \sum_{i=c}^{\infty}\left(\sum_{h=1}^{r} C_h z_h^i\right) P_{i,1}$$

$$\vdots$$

$$0 = p_0^- P_{0,c+r-2} + p_1^- P_{1,c+r-2} + p_2^- P_{2,c+r-2} + \ldots + p_{c-1}^- P_{c-1,c+r-2} + \sum_{h=1}^{r} C_h \sum_{i=c}^{\infty}\left(z_h^i P_{i,c+r-2} - z_h^{c+r-2}\right)$$

(57) $$1 = p_0^- + p_1^- + p_2^- + \cdots + \sum_{j=c}^{\infty}\left(\sum_{h=1}^{r} C_h z_h^j\right)$$

(58) $$p_j^- = \begin{cases} \text{determined above}, & (0 \leq j < c) \\ \sum_{h=1}^{r} C_h z_h^j, & (r < c \leq j) \end{cases}$$

1 INTRODUCTION

1.1 Problem Description

In the following two subsections, we describe the remaining problems in the multi-server bulk-arrival queues.

1.1.1 Problem description for $GI^X/M/c$ and $GI^X/M/c/N$ queues

A queue forms whenever and wherever demand exceeds supply. It is for this reason that a study of queues naturally emerged as a practical field of study known as queueing theory. Among many different types of queues, multi-server bulk-arrival queues are particularly useful in modeling cases where a group of customers joins a queue in a system that consists of two or more simultaneous servers. Queues of this type are easily seen in our daily lives: A barber shop, supermarket, hospital admissions, seaports, information transmission systems, and many other settings. In the past, extensive studies have been carried out on continuous-time multi-server bulk-arrival queues.

In particular, $GI^X/M/c$ queues have been studied using various techniques. Though different methods can often lead to the same result, it is always good to have simpler and more efficient ways to solve the model at hand analytically. In addition, the results for the $GI^X/M/c$ model involving heavy-tailed inter-batch-arrival times are missing in literature.

Finite-buffer queues are characterized by the imposed limitation to the amount of waiting room such that when the line reaches a certain length, no further customers are allowed to enter until space becomes available. The $GI^X/M/c/N$ queue is the finite-buffer counterpart of $GI^X/M/c$ queue. Traditionally, $GI^X/M/c/N$ queues have been analyzed using laborious techniques that lead to non-explicit results. Nevertheless,

$GI^X/M/c/N$ queues are important since many real-life scenarios that resemble $GI^X/M/c$ queues actually entail some degree of limited occupancy.

A simple and unified approach to treat $GI^X/M/c$ and $GI^X/M/c/N$ queues would be beneficial when designing efficient algorithms with quicker computing time. It would also facilitate further mathematical understanding of continuous-time bulk-arrival multi-server queues.

1.1.2 Problem description for $GI^X/Geo/c$ and $GI^X/Geo/c/N$ queues

The study of discrete-time queues is fairly recent relative to its continuous-time counterpart. However, the field quickly gained value among queueing theorists and researchers due to the digitization of information technology, particularly in the area of signal processing, microcomputers, and computer networks.

The $GI^X/Geo/c$ queues are often the model of choice when measuring the performance of existing technologies, however, available literature on $GI^X/Geo/c$ queues is not as extensive as that of $GI^X/M/c$ queues. In addition, the results of $GI^X/Geo/c$ queues involving heavy-tailed inter-batch-arrival times are missing in literature.

The $GI^X/Geo/c/N$ queues are the finite-buffer counterpart of $GI^X/Geo/c$ queues. In digital communication systems, the rejection policies of $GI^X/Geo/c/N$ queues are critical since both partial and total rejections of packets lead to data loss. Such rejections must be monitored, understood, and managed in order to effectively balance the processor's speed against the fidelity of information. However, the existing work on $GI^X/Geo/c/N$ queues remains incomplete and the analytical work involved can be simplified for wider use.

A simple and unified approach to treat $GI^X/Geo/c$ and $GI^X/Geo/c/N$ queues would be beneficial for applications while advancing the mathematical literature.

1.2 Thesis Objectives

The objective of this thesis is to present a simple and unified approach to solve multi-server bulk-arrival queues that involve

- Continuous and discrete-times
- Infinite and finite-buffers
- Light and heavy-tailed inter-batch-arrival times

2 ANALYTICAL RESULTS IN $GI^X/M/c$ AND $GI^X/M/c/N$ QUEUES

Readers may refer to Appendix A.1 for a brief summary of the probability theory, stochastic processes, and Markov processes, which are all important topics that underlie continuous-time queueing theory. The definitions and properties of a continuous random variable (r.v.) and its moments, Laplace transform (L.T.) and Laplace-Stieltjes transform (L-S.T.) are provided in Appendix A.2. The basic mathematical construct of a queueing system, as well as some common theorems and techniques used in continuous-time queueing theory are explained in Appendix B. The rest of the materials that supplement Chapter 2 are available in Appendix C.1.

2.1 Literature review

In studying the $GI^X/M/c$ queueing system, different methods and techniques have been used by several authors to solve for, and develop relations between various probability distributions and performance measures. We provide a brief history behind some authors' work and explain their solution procedures to solve the multi-server bulk-arrival queueing system with exponential service times.

One of the pioneering works on the topic of multi-server bulk-arrival queues appears in the text by Neuts [39], who presented an algorithmic solution for $GI^X/M/c$ queues that involve vectors and matrices. He first introduced the dual nature of the model's transition probabilities based on the difference between the number of servers (c) and the maximum batch size (r). In solving the $GI^X/M/c$ through the probability generating function (p.g.f.) Zhao [51] also treats the two cases ($c \leq r$ and $c > r$) separately.

It is further worth noting that including and beyond the model $GI^X/M/c$, various comparisons between the matrix-geometric method used by Neuts [39] and the roots method have been made by several authors: Daigle and Lucantoni [19] state that "whenever the roots method works, it works blindingly fast." Similarly, Janssen and van Leeuwaarden [28] who have successfully used the roots method make a comment, "initially, the potential difficulties of root-finding were considered to be a slur on the unblemished transforms since the determination of the roots can be numerically hazardous and the roots themselves have no probabilistic interpretation. However, Chaudhry et al. [8] have made every effort to dispel the skepticism towards root-finding in queueing theory…." Gouweleeuw [23] states that it is more efficient to use the roots method to get explicit expressions for probabilities from generating functions (g.f.'s). Furthermore, a recent paper by Maity and Gupta [37] compares the spectral theory approach and the roots method. Maity and Gupta [37] indicate several difficulties in getting results using the spectral theory approach, an approach which may be simpler than the matrix-geometric approach as stated in several papers by Chakka (see e.g., Chakka [6]) and others.

When comparing the roots method and the matrix-geometric technique, it is evident that the solution procedure based on the matrix-geometric technique requires a unique algorithmic procedure for each arrival pattern. This is not the case in the roots method since the roots of the characteristic equation form the basis of solution and as such, a simple algorithm can address different arrival patterns. In addition, one can discuss the tail probabilities for both light and high traffic cases using the roots method. In fact, the roots method is simple analytically, notationally, and computationally.

Besides, the matrix-geometric method is sensitive to traffic intensity, whereas the roots method is not.

Historically, when MAPLE and Mathematica could not find a large number of roots (they do now), a software package called QROOT developed by Chaudhry [9] was used by him and his collaborators to find a large number of roots and use them in solving several queueing models. The algorithm for finding such roots is available in some of their papers. In particular, see Chaudhry et al. [8]. It may be remarked here that MAPLE can now not only find roots that are close to each other (a concern expressed by several researchers), but even repeated roots.

While discussing the simple model $M^X/M/c$, Cromie et al. [18] pointed out that both Kabak [29] and Abol'nikov [1] took the incorrect probability mass function (p.m.f.) for the position of the random customer within an incoming batch. This aspect was first noticed and corrected by Burke [5] where, in single-server batch-arrival queue, the distinction between the delays (waiting times until entering service) of the first and randomly positioned customers within an incoming batch was made. Cromie et al. [18] extended this concept to $M^X/M/c$. The two distributions of delay are determined in terms of p.m.f. of the number of customers in the system. Other miscellaneous results which stem from these distributions are also provided. Chaudhry and Kim [13] generalized Burke [5]'s concept to the $GI^X/M/c$ model, where the waiting-time distributions of first and random customers within an incoming batch are both given in terms of the pre-arrival queue-length distribution.

In solving the more general model $E_k^X/M/c$ with Erlang-k inter-batch-arrival time distribution, Holman and Chaudhry [26] introduced the queue-length distribution,

$p_{n,r}$, as a joint probability distribution with n customers in the system $(n \geq 0)$ and the next incoming batch of customers in the r-th phase $(1 \leq r \leq k)$ of the inter-batch-arrival time period. The iterative relations between the $p_{n,r}$ at varying n and r are then expressed in terms of four independent equilibrium equations. The solution to these equations is in terms of a p.g.f., where the denominator of that p.g.f. has $k-1$ roots inside the unit circle which are also the roots for a portion of the same p.g.f.'s numerator. This portion of the numerator, labelled $U(z)$, provides $k-1$ set of equations leading to the solution $p_{n,r}, (n < c)$. The remaining probabilities, $p_{n,r}, (n \geq c)$ are found by combining and then solving the first two of the four equilibrium equations, the $k-1$ set of equations derived from $U(z)$, and the normalizing condition. The challenge with the method introduced by Holman and Chaudhry [26] is in solving a model with large values of c and k. Due to the bi-variate nature of the solution and the transition probabilities, the number of linear equations (ck) becomes large when working with large input parameters.

In deriving various relations in $GI^X/M/c$ queues, Yao et al. [50] first relate the pre-arrival queue-length distribution with its counterpart distributions at random and post-departure time epochs. The procedure to obtain the random time epoch probabilities this way is simpler than what is done by Neuts [39] or Laxmi and Gupta [35]. Yao et al. [50] also present the waiting-time distributions of the first and random customers within an incoming batch. However, these relations are both given in terms of L-S.T.'s. Their inverted forms are independently derived by Chaudhry and Kim [13]. Yao et al. [50] also derive several relations between the performance measures of $GI^X/M/c$ queues. However, some of these relations do not hold when the inter-batch-arrival times follow a heavy-tailed distribution with an infinite mean (see Kim and Chaudhry [33]).

Heavy-tailed distributions constitute a class of probability distributions that are characterized by their slower decay than the exponential distribution. In queueing theory, using heavy-tailed distributions as either an inter-arrival or service time distribution creates models that are, in general, difficult to analyze due to the unique probabilistic properties of heavy-tailed r.v.'s (see Boxma and Cohen [2] or Harris et al. [25]). However, analysis of such models holds significant merit for applications when modeling real life examples; transportation systems, airport security screening, and in digital communication networks (see Leland et al. [36] or Willinger and Paxon [47]).

In solving $GI/M/1$ and $GI/M/c$ queues involving heavy-tailed inter-arrival times, Harris et al. [25] approximate the root of each model's characteristic equation. In extending and refining the method by Harris et al. [25], Kim and Chaudhry [32] solve $GI^X/M/1$ queues involving heavy-tailed inter-batch-arrival times whereas Chaudhry and Kim [15] use the standard root-finding method to compute the solution to $GI^X/M/c$ queues involving heavy-tailed inter-arrival times.

In solving the model $GI^X/M/c/N$ with the finite-buffer N, Laxmi and Gupta [35] present two rejection policies: Partial and total rejections. Given that the size of an incoming batch is larger than the available space, in partial rejection, a portion of an incoming batch is allowed to enter the system. In total rejection, under the same circumstance, an incoming batch is entirely rejected. This results in two different set of transition probabilities for the model $GI^X/M/c/N$. The solution to each rejection policy therefore leads to two different pre-arrival queue-length distributions for $GI^X/M/c/N$ queues. The solutions are then leveraged to determine the queue-length distribution at a random time epoch using the supplementary variable technique. Waiting-time distributions of the first, random, and last customer within an incoming batch are also presented. In discussing the same model, Ferreira and Pacheco [20] apply the uniformization technique to calculate the transient state and steady-state probabilities of the model. (We note that their paper gives a good bibliography of related papers.) Gontijo et al. [21] treat $GI^X/M/c/N$ queues through kernel estimation. In addition, Chaudhry and Kim [16] treat the model $GI/M/1/N$ using roots. By doing so, the solution can be expressed explicitly in terms of the roots of the model's characteristic equation. While their result embarks on the first application of the roots method in $GI/M/1$ type finite-buffer queues, it is also identified that the method remains robust even in the case of heavy-tailed inter-arrival times. Kim and Chaudhry [34] extend this concept in treating $GI/M/c/N$ queues using roots.

In finite-buffer queues, the loss probability can be defined as the probability of a particular queue-length at which rejections start to occur. Gouweleeuw [23] presents an extensive overview of cases where the loss probabilities in finite-buffer queues can be estimated in terms of the queue-length distribution of infinite-buffer queues. Similarly, Kim and Choi [30] express the asymptotic loss probabilities of $GI^X/M/c/N$ queues in terms of the positive root of the characteristic equation of the model $GI^X/M/c$. However, the magnitude of the traffic intensity must always be smaller than 1 in the infinite-buffer queues whereas in the finite-buffer queues it can be smaller, equal to, or greater than 1. Due to this condition, the methods used by Gouweleeuw [23] and Kim and Choi [30] cannot estimate the loss probabilities when the traffic intensity is 1 or greater (see Appendix C.3.4 for more details).

2.2 The $GI^X/M/c$ queues

In this section, we analytically solve $GI^X/M/c$ queues using roots.

2.2.1 Model description

Consider the steady-state aspect of the model $GI^X/M/c$ where the service times, group sizes, and inter-batch-arrival times are mutually independent. There are c parallel servers where each server has the exponential service rate μ. At any time the state of c servers can be categorized into three different cases: Overloaded is the case when all c servers are busy with a queue of at least a single customer, loaded is the case when the system has exactly c customers in the system, and under-loaded is the case when there is at least an idle server. Customers arrive in batches of size X (with maximum size r), which is a r.v. with p.m.f. $b_h = P(X = h)$, $(1 \le h \le r)$, mean $\mu_X = \sum_{h=1}^{r} hb_h$ and p.g.f. $B(z) = \sum_{h=1}^{r} b_h\, z^h, (|z| \le 1)$. Batches of customers arrive at time epochs $T_1, T_2, \dots, T_n, \dots$, and the inter-batch-arrival times $t_{n+1} = T_{n+1} - T_n > 0, (n \ge 0)$ are independently and identically distributed random variables (i.i.d.r.v.'s) T with the probability density function (p.d.f.) $a(t)$, cumulative distribution function (c.d.f.) $A(t) = P(T \le t)$, mean ${}^{1}/_{\lambda}$, and L-S.T. $\bar{a}(s) = \int_0^\infty e^{-st} dA(t)$. Let $N(t)$ be the number of customers in the system at time t. The N_n^- represents the number of customers in the system including the ones, if any, in service just before the arrival instant T_n such that $N_n^- = N(T_n - 0), (n \ge 0)$. The queue-length distribution is $p_j^- = \lim_{n\to\infty} P(N_n^- = j)\,, (j \ge 0)$. Similarly, the queue-length distributions at random and post-departure time epochs are $p_j = \lim_{n\to\infty} P(N_n = j), (j \ge 0)$ and $p_k^+ = \lim_{n\to\infty} P(N_n^+ = k), (k \ge 0)$, respectively. Let D_n be the total number of customers that depart the system over the course of t_n with the steady-state p.m.f. $k_l =$

$\lim_{n\to\infty} \int_0^\infty P(D_n = l \,|t_n = t)dA(t)\,, (l \geq 0)$ and p.g.f. $K(z) = \sum_{l=0}^{\infty} k_l z^l$. The stochastic process $\{N_n^-, n \geq 1\}$ forms a homogenous Markov chain

$$N_{n+1}^- = \begin{cases} N_n^- + X_n - D_n, & (N_n^- + X_n - D_n \geq 0) \\ 0, & (N_n^- + X_n - D_n < 0) \end{cases}$$

Since we deal with the steady-state aspect of the model $GI^X/M/c$, we define the traffic intensity of the system as $\rho = \frac{\lambda\mu_X}{c\mu}, (0 < \rho < 1)$.

2.2.2 The $GI^X/M/c$ queues at a pre-arrival time epoch

To compute the queue-length distribution of $GI^X/M/c$ queues at a pre-arrival time epoch we first define the transition probabilities of the model. Let $P_{i,j}(n) = P[N_{n+1}^- = j \mid N_n^- = i], (i,j \geq 0, n \geq 1)$ be the one-step transition probabilities of $\{N_n^-, n \geq 1\}$. Thus, the steady-state one-step transition probabilities of $GI^X/M/c$ queues are defined as $P_{i,j} \equiv \lim_{n\to\infty} P_{i,j}(n)$ where

$$P_{i,j} = \begin{cases} \sum_{h=1}^{r} b_h k_{i+h-j}\,, (i \geq 0, j \geq 1) \\ 1 - \sum_{k=1}^{\infty} P_{i,k}\,, (i \geq 0, j = 0) \end{cases}$$

with $b_h = 0$ for $h \leq 0$. The p.m.f. of D_n is given by

$$k_{i+h-j}$$

$$= \begin{cases} 0, & (i+h<j) \\ \int_0^\infty \frac{(c\mu t)^{i+h-j}}{(i+h-j)!} e^{-c\mu t} dA(t), & (c \leq j \leq i+h) \\ \int_0^\infty \binom{i+h}{j} (1-e^{-\mu t})^{i+h-j} (e^{-\mu t})^j dA(t), & (1 \leq j \leq i+h \leq c) \\ \int_0^\infty \int_0^t \frac{e^{-c\mu u}(c\mu u)^{i+h-c-1}}{(i+h-c-1)!} c\mu \binom{c}{j} e^{-\mu j(t-u)} \left[1-e^{-\mu(t-u)}\right]^{c-j} du dA(t), & (1 \leq j < c < i+h) \end{cases}$$

For $1 \leq h \leq r,$ the above can be derived by replacing the single arrival notion with the batch-arrival condition $h, (1 \leq h \leq r)$ in the steady-state one-step transition probabilities of $GI/M/c$ queues which are available in Gross and Harris [24] and Breuer and Baum [3]. The following well-known Chapman-Kolmogorov equation (see Appendix A.1 for definition) is used extensively in solving for $\left(p_j^-, j \geq 0\right)$

$$p_j^- = \sum_{i=0}^{\infty} p_i^- P_{i,j}\,, (j \geq 0) \tag{1}$$

Since (1) is a set of j first order linear difference equations with $\left(p_j^-\right)$ being the unknown functions to be determined (see Appendix B.2), we can assume the solution of a general form

$$p_j^- = Cz^j, (j \geq c, C \neq 0)$$

The general solution with a bound $j \geq c$ is purposely chosen so that when it is substituted into (1), the k_{i+h-j} within the $P_{i,j}$ of (1) is fixed at $k_{i+h-j} = \int_0^\infty \frac{(c\mu t)^{i+h-j}}{(i+h-j)!} e^{-c\mu t} dA(t)$. Intuitively this can be understood as all c servers are either overloaded or loaded during T. Such substitution leads to the characteristic equation of $GI^X/M/c$ queues

$$1 = B(z^{-1})K(z) \tag{2}$$

where $K(z) = \sum_{n=0}^{\infty} k_n z^n = \int_0^\infty e^{-c\mu(1-z)t} dA(t) = \overline{a}\big(c\mu(1-z)\big)$. Since (2) has r roots inside the unit circle $|z| = 1$ (see Appendix C.1.2 for details), let these roots be $z_1, z_2, \ldots, z_r$. The general solution becomes r-fold and can be expressed as

$$p_j^- = \sum_{h=1}^{r} C_h z_h^j\,, (j \geq c) \tag{3}$$

where the C_h (yet to be evaluated) for $1 \leq h \leq r$ are the non-zero constants. In completely finding $\left(p_j^-, j \geq 0\right)$ we replace $\left(p_j^-, j \geq c\right)$ in (1) with (3) such that

$$p_j^- = \sum_{i=0}^{c-1} p_i^- P_{i,j} + \sum_{i=c}^{\infty} \sum_{h=1}^{r} C_h z_h^i \, P_{i,j} \, , (j \geq 0) \tag{4}$$

We also consider the normalizing condition

$$\sum_{j=0}^{c-1} p_j^- + \sum_{j=c}^{\infty} \sum_{h=1}^{r} C_h z_h^j = 1$$

By letting $j = 1, 2, \dots, c + r - 1$ in (4) and with the normalizing condition we have the $c + r$ equations that are required to solve for $\left(p_j^-, 0 \leq j \leq c - 1\right)$ and $C_h, (1 \leq h \leq r)$. By solving these equations the $\left(p_j^-, j \geq 0\right)$ are completely found as

$$p_j^- = \begin{cases} \text{determined above}, (0 \leq j \leq c - 1) \\ \displaystyle\sum_{h=1}^{r} C_h z_h^j, \qquad\qquad (j \geq c) \end{cases} \tag{5}$$

As a remark, unlike the way we have determined p_j^-, Neuts [39] and Zhao [51] distinguish the size difference between c and r such that there are two different forms of p_j^- for $c \leq r$ and $c > r$, respectively. Although those two different results can be combined into a single form and expressed as (5), such separation reveals additional mathematical finding (see Appendix C.1.4): The $\left(p_j^-, j \geq 0\right)$ when $c \leq r$ can be expressed entirely as a geometric sum whereas the $\left(p_j^-, j \geq 0\right)$ when $c > r$ can be expressed as a partial geometric sum.

2.2.3 The $GI^X/M/c$ queues at random and post-departure time epochs

The relations of p_j^- with p_j and p_j^+ are later required in determining the performance measures at random and post-departure time epochs. As stated by Yao et al. [50] the relations derived from the standard level-crossing analysis are

$$p_k = \frac{\lambda}{\mu \min(k,c)} \sum_{j=0}^{k-1} p_j^- \left(1 - \sum\nolimits_{h=1}^{k-j-1} b_h\right)$$

$$p_{k-1}^+ = \frac{1}{\mu_X} \sum_{j=0}^{k-1} p_j^- \left(1 - \sum\nolimits_{h=1}^{k-j-1} b_h\right)$$

for $k \geq 1$ and $p_0 = 1 - \sum_{k=1}^{\infty} p_k$. As a remark, the p_k can be found in another way using the random biased sampling (see Appendix B.4).

2.2.4 The waiting-time-in-queue of the first customer within an incoming batch

Let the amount of time spent in queue by the first customer within an incoming batch be $W_F, (W_F \geq 0)$. The c.d.f. of W_F is $W_F(t) = P(W_F \leq t), (t \geq 0)$. The explicit expression of $W_F(t)$ is derived as follows:

In $GI^X/M/c$ queues, the amount of time the first customer within an incoming batch spends in queue should closely resemble the waiting-time-in-queue of an incoming customer in $GI/M/c$ queues (see Gross and Harris [24]). If the first customer within an incoming batch enters service immediately, then the servers must be under-loaded prior to that batch-arrival. In other words,

$$W_F(0) = \sum_{i=0}^{c-1} p_i^- \qquad (6)$$

Complement to (6), if the first customer within an incoming batch is not served immediately, then the servers must be either loaded or overloaded prior to that batch-arrival (i.e. $i \geq c$). Since there are $i - c$ customers waiting in queue prior to a batch-arrival, once a batch arrives, the first customer within that batch must wait until $i - c + 1$ customers in front of him enter service. Hence the waiting time for this customer is $(i - c + 1)$-fold convolution of exponentials with mean $c\mu$, which is Erlang. Removing the condition on i and coupling it with (6), the $W_F(t)$ is found as

$$W_F(t) = P(W_F \leq t) = W_F(0) + \sum_{i=c}^{\infty} p_i^- \int_0^t \frac{(c\mu v)^{i-c}}{(i-c)!} (c\mu) e^{-c\mu v}\, dv \tag{7}$$

2.2.5 The waiting-time-in-queue of the random customer within an incoming batch

Let the amount of time spent in queue by the random customer within an incoming batch be $W_R, (W_R \geq 0)$. The c.d.f. of W_R is $W_R(t) = P(W_R \leq t), (t \geq 0)$. The explicit expression of $W_R(t)$ is derived as follows: If the position of the random customer within an incoming batch is F, then its p.m.f. is $P(F = f) \equiv r_f = \sum_{h=f}^{\infty} \frac{b_h}{\bar{b}}$, $(1 \leq f \leq r)$ (see Appendix B.4 for details). If the random customer within an incoming batch enters service immediately, then $i + f \leq c$ must be true. In other words,

$$W_R(0) = \sum_{f=1}^{r} \sum_{i=0}^{c-f} p_i^- \, r_f \tag{8}$$

Using a similar argument from Subsection 2.2.4 the $W_R(t)$ can be derived as

$$W_R(t) = P(W_R \leq t)$$

$$= W_R(0) + \sum_{f=1}^{r} r_f \sum_{i=c-f+1}^{\infty} p_i^- \int_0^t \frac{(c\mu v)^{i-c}}{(i+f-c-1)!} (c\mu)^f v^{f-1} e^{-c\mu v}\, dv \tag{9}$$

2.2.6 Performance measures

Since the distributions are known, various moments can be calculated. In particular, denote $L_s, L_q, W_{q_1}^-$, and W_q^- as the mean number of customers in the system, the mean number of customers in queue, and mean waiting-time-in-queue of the first and random customers within an incoming batch, respectively. First, the mean number of customers in the system is

$$L_s = \sum_{i=1}^{\infty} i\, p_i \tag{10}$$

and similarly, $L_s^- = \sum_{i=1}^{\infty} i\, p_i^-$ and $L_s^+ = \sum_{i=1}^{\infty} i\, p_i^+$. On the other hand, the mean number of customers in queue is defined as

$$L_q = \sum_{i=c+1}^{\infty} (i-c)\, p_i \tag{11}$$

where $(i-c)$ indicates the number of customers in queue when all servers are busy. The mean number of customers in queue at pre-arrival and post-departure time epochs are $L_q^- = \sum_{i=c+1}^{\infty}(i-c)\, p_i^-$ and $L_q^+ = \sum_{i=c+1}^{\infty}(i-c)\, p_i^+$. Moreover, $W_{q_1}^-$ is

$$W_{q_1}^- = \int_0^{\infty} t dW_F(t) \tag{12}$$

where $W_F(t)$ is given in (7). Similarly, W_q^- is

$$W_q^- = \int_0^{\infty} t dW_R(t) \tag{13}$$

where $W_R(t)$ is given in (9). Further, it can be shown that the average number of idle servers in the model $GI^X/M/c$ can be found using the expression

$$\sum_{j=0}^{c-1}(c-j)p_j = c(1-\rho) \tag{14}$$

The left-hand side of (14) determines the average number of idle servers using the queue-length distribution at a random time epoch. The right-hand side of (14) determines the same number, except that it is independent of the queue-length distribution (hence independent of roots). In the next chapter we use (14) to verify the accuracy of our numerical results, thus demonstrating the robustness of the roots and the roots method (see point 4 in Appendix C.3.2 for more details).

2.2.7 The $GI^X/M/c$ queues involving heavy-tailed inter-batch-arrival times

Assume that inter-batch-arrival times of $GI^X/M/c$ queue under consideration follows a heavy-tail distribution with the p.d.f. $a(t)$, c.d.f. $A(t)$, and L-S.T. $\bar{a}(s) = \int_0^\infty e^{-st} dA(t)$. Due to the probabilistic properties of heavy-tail distributions, there are instances where the mean, variance or higher order moments of the inter-batch-arrival times are infinite. In addition, there are also cases where the L-S.T. of the inter-batch-arrival times is non-closed or non-analytic. To overcome this difficulty we extend the technique by Kim and Chaudhry [32] who solved the $GI^X/M/1$ queue involving the heavy-tailed inter-batch-arrival times using the roots method. In doing so, we replace the $K(z)$ in (2) with $K_\Psi(z)$ such that

$$K_\Psi(z) = \sum_{n=0}^{\Psi} \int_0^\infty \frac{e^{-c\mu t}\,(c\mu t)^n}{n!} dA(t)\, z^n, (0 \le \Psi < \infty)$$

where Ψ is a non-negative integer. The very last term of $K_\Psi(z)$ is $\int_0^\infty \frac{e^{-c\mu t}\,(c\mu t)^\Psi}{\Psi!} dA(t) z^\Psi$ and by assuming that $\int_0^\infty \frac{e^{-c\mu t}\,(c\mu t)^\Psi}{\Psi!} dA(t)$ is a near-zero probability, we can determine the value of Ψ. Based on this notion we have implemented a simple algorithm in MAPLE that determines Ψ (see Appendix C.1.3). This algorithm was used to compute and plot the roots (see Appendices C.3.1) as well as the distributions (see Section 3.2) in $GI^X/M/c$ queues involving heavy-tailed inter-batch-arrival times. For more in-depth numerical analysis on the roots of (2) readers may refer to Appendix C.3.2.

In contrast to our root-finding method, Harris et al. [25] use the transform approximation method (TAM). TAM is based on approximating the L-S.T. of a heavy-tailed inter-arrival time distribution by a geometric sum of Ψ terms such that

$$K_\Psi(z) = \frac{1}{\Psi}\sum_{k=1}^{\Psi} e^{-zX(k)}$$

where $X(k), k = 1,2,\dots,\Psi$, are chosen to cover the outcome space of the original inter-arrival r.v. with $K(X(j)) = (j - 0.5)/\Psi$. Ultimately, Ψ is manually picked so that it satisfies $\lim_{\Psi\to\infty} K_\Psi(z) = K(z)$ where $K(z)$ is the L-S.T. of a heavy-tailed inter-arrival time distribution such that $K_\Psi(z) = \bar{a}_\Psi\big(\mu(1-z)\big)$ for $GI/M/1$ queues and $K_\Psi(z) = \bar{a}_\Psi\big(c\mu(1-z)\big)$ for $GI/M/c$ queues. Therefore, once Ψ is known, solving the equation

$$z = K_\Psi(z)$$

results in the root that is needed to solve the single-arrival model. In explaining TAM, Harris et al. [25] place a caveat on the fact that TAM results in Ψ as large as 10^6. Moreover, Harris et al. [25] lay their analytical foundation of TAM based on single-arrival models such as $GI/M/1$ and $GI/M/c$ queues. Though their method could be extended to bulk-arrival models (i.e. multiple roots), such extension may not only lead to a laborious analytical foundation, but will require even larger Ψ to approximate the L-S.T. of a heavy-tailed inter-batch-arrival time distribution.

In contrast to this, the proposed root-finding method employs Ψ no larger than 100 (see Appendix C.3.1). Since finding the roots at a smaller Ψ equates to a higher efficiency and shorter computing time, it is concluded that the proposed root-finding method is more advantageous than TAM (see Chaudhry and Kim [15] for more details).

2.3 The $GI^X/M/c/N$ queues

In this section, we analytically solve $GI^X/M/c/N$ queues using roots. Some notations from Section 2.2 are redefined in the context of $GI^X/M/c/N$ queues.

2.3.1 Model description

Consider the steady-state aspect of $GI^X/M/c/N$ queueing system where the service times, group sizes, and inter-batch-arrival times are mutually independent. There are c parallel servers where each server has the exponential service rate μ. Customers arrive in batches of size X (with maximum size r) with the p.m.f. $b_h = P(X = h), (1 \le h \le r)$, mean $\mu_X = \sum_{h=1}^{r} h b_h$, and p.g.f. $B(z) = \sum_{h=1}^{r} b_h z^h, (|z| \le 1)$. Batches of customers arrive at time epochs $T_1, T_2, \dots, T_n, \dots$, and the inter-batch-arrival times $t_{n+1} = T_{n+1} - T_n > 0, (n \ge 0)$ are i.i.d.r.v.'s with the p.d.f. $a(t)$, c.d.f. $A(t)$, mean $1/\lambda$, and L-S.T. $\bar{a}(s) = \int_0^\infty e^{-st} dA(t)$. Let $M(t)$ be the number of customers in the system at time t such that $M_n^- = M(T_n - 0), (n \ge 0)$ is the number of customers in the system including the ones, if any, in service just before the arrival instant T_n. The steady-state p.m.f. of M_n^- is $p_j^- = \lim_{n\to\infty} P(M_n^- = j), (j \ge 0)$. Similarly, the queue-length distribution at a random time epoch is $p_j = \lim_{n\to\infty} P(M_n = j), (j \ge 0)$. Let D_n be the total number of customers that depart the system over the course of t_n with the steady-state p.m.f. $k_l = \lim_{n\to\infty} \int_0^\infty P(D_n = l \,|t_n = t) dA(t), (l \ge 0)$ and p.g.f. $K(z) = \sum_{l=0}^{\infty} k_l z^l$. The stochastic process $\{M_n^-, n \ge 1\}$ forms a homogenous Markov chain:

$$M_{n+1}^{-} = \begin{cases} \min(M_n^{-} + X_n - D_n, N), & (M_n^{-} + X_n - D_n \geq 0) \\ 0, & (M_n^{-} + X_n - D_n < 0) \end{cases}$$

The traffic intensity of the system is defined as $\rho = \frac{\lambda \mu_X}{c\mu} > 0$. The model $GI^X/M/c/N$ has the finite-buffer $N, (N \geq c)$ such that an incoming batch is either partially or totally rejected if a batch size $h, (1 \leq h \leq r)$ is larger than the available space $(N - i)$.

2.3.2 The $GI^X/M/c/N$ queues at a pre-arrival time epoch

To compute the queue-length distribution of $GI^X/M/c/N$ queues at a pre-arrival time epoch we first define the transition probabilities of the model. Let $P_{i,j}(n) = P[M_{n+1}^{-} = j \mid M_n^{-} = i], (i, j \geq 0, n \geq 1)$ be the one-step transition probabilities of $\{M_n^{-}, n \geq 1\}$. Thus, the steady-state one-step transition probabilities of $GI^X/M/c/N$ queues are defined as $P_{i,j} \equiv \lim_{n\to\infty} P_{i,j}(n)$. Given that a batch can be either partially or totally rejected, there are two different set of transition probabilities that correspond to each rejection policy: In the case of partial rejection, the $P_{i,j}$ are defined as

$$P_{i,j} = \begin{cases} \displaystyle\sum_{h=j-i}^{N-i} k_{i+h-j} b_h + k_{N-j} \sum_{h=N-i+1}^{r} b_h\,, & (j > i \geq 0, j \geq c) \\ \displaystyle\sum_{h=1}^{N-i} k_{i+h-j} b_h + k_{N-j} \sum_{h=N-i+1}^{r} b_h, & (c \leq j \leq i) \\ \displaystyle\sum_{h=\max(1,j-i)}^{N-i} V_{i+h,j} b_h + V_{N,j} \sum_{h=N-i+1}^{r} b_h\,, & (i \geq 0, 1 \leq j \leq c-1) \end{cases}$$

In the case of total rejection, the $P_{i,j}$ are defined as

$$P_{i,j}=\begin{cases}\displaystyle\sum_{h=j-i}^{N-i}k_{i+h-j}b_h, & (j>i\geq 0, j\geq c)\\ \displaystyle\sum_{h=1}^{N-i}k_{i+h-j}b_h+k_{i-j}\sum_{h=N-i+1}^{r}b_h, & (c\leq j\leq i)\\ \displaystyle\sum_{h=\max(1,j-i)}^{N-i}V_{i+h,j}b_h+V_{i,j}\sum_{h=N-i+1}^{r}b_h, & (i\geq 0, 1\leq j\leq c-1)\end{cases}$$

where $P_{i,0}=1-\sum_{j=1}^{N}P_{i,j}$, $(0\leq i\leq N)$ and $k_n=\int_0^\infty \frac{(c\mu t)^n}{n!}e^{-c\mu t}dA(t)$, $(n\geq 0$ and $k_n=0$ for $n<0)$. The $V_{i+h,j}$ are defined as

$$V_{i+h,j}$$

$$=\begin{cases}0, & (i+h<j)\\ \displaystyle\int_0^\infty\binom{i+h}{j}(1-e^{-\mu t})^{i+h-j}(e^{-\mu t})^j dA(t), & (1\leq j\leq i+h\leq c)\\ \displaystyle\int_0^\infty\int_0^t\frac{e^{-c\mu u}(c\mu u)^{i+h-c-1}}{(i+h-c-1)!}c\mu\binom{c}{j}e^{-\mu j(t-u)}\left[1-e^{-\mu(t-u)}\right]^{c-j}du\,dA(t), & (1\leq j<c<i+h)\end{cases}$$

As a remark, the $P_{i,j}$ presented above match with those given by Laxmi and Gupta [35].

In presenting a novel way of treating $GI^X/M/c/N$ queues using roots we extend the result by Chaudhry and Kim [16] as well as Kim and Chaudhry [34] who solve $GI/M/1/N$ and $GI/M/c/N$ queues, respectively, using roots. In doing so, we define the Chapman-Kolmogorov equation of $GI^X/M/c/N$ queues as

$$p_j^-=\sum_{i=0}^{N}p_i^-P_{i,j}, (0\leq j\leq N) \tag{15}$$

which is a set of j first order linear difference equations. As a remark, $N=0$ indicates that no customers are allowed in the system (i.e. $p_0^-=1$) hence this case can be ignored. Whether the $P_{i,j}$ follow partial or total rejection policy, both cases can be solved altogether by assuming the solution of a general form $p_j^- = Cz^j, (1 \le j \le N, C \ne 0)$. By substituting the general solution into (15), we have

$$Cz^j = \sum_{i=0}^{N} Cz^i \, P_{i,j}, (1 \le j \le N)$$

$$0 = \sum_{i=0}^{N} z^i \, P_{i,j} - z^j$$

By summing both sides of the above over $1 \le j \le N$ we have the characteristic equation of $GI^X/M/c/N$ queues as

$$0 = \sum_{j=1}^{N} \left(\sum_{i=0}^{N} z^i \, P_{i,j} - z^j \right) \tag{16}$$

Since (16) is an N-th degree polynomial, solving it gives N roots. Let these roots be $z_1, z_2, \dots, z_N$ such that the solution becomes

$$p_j^- = \sum_{h=1}^{N} C_h z_h^j \, , (1 \le j \le N) \tag{17}$$

where the $C_h, (1 \le h \le N)$ are the unknown non-zero constant coefficients. To determine these unknowns, we substitute (17) into (15) such that it leads to

$$\sum_{h=1}^{N} C_h z_h^j = \sum_{i=0}^{N} \sum_{h=1}^{N} C_h z_h^i \, P_{i,j}, (1 \le j \le N)$$

The expression above can be rearranged to

$$0 = \sum_{h=1}^{N} C_h \left(\sum_{i=j-h}^{N} z_h^i \, P_{i,j} - z_h^j \right), (1 \le j \le N) \tag{18}$$

To make (17) also true for the case when $j = 0$, we establish the normalizing condition as

$$1 = \sum_{j=0}^{N} \sum_{h=1}^{N} C_h z_h^j$$

The normalizing condition, in conjunction with letting $j = 1,2, \dots, N-1$ in (18), gives N equations. Solving these equations give the solution to $GI^X/M/c/N$ queues in terms of roots as

$$p_j^- = \sum_{h=1}^{N} C_h z_h^j, (0 \le j \le N) \tag{19}$$

2.3.3 The $GI^X/M/c/N$ queues at a random time epoch

Using the random biased sampling (see Appendix B.4) the queue-length distribution of $GI^X/M/c/N$ queues at a random time epoch (say $p_j, 0 \le j \le N$) can be explicitly expressed as $p_j = \sum_{i=0}^{N} p_i^- P_{i,j}^*, (1 \le j \le N)$ where the $(p_i^-, 0 \le i \le N)$ are available in (19), $p_0 = 1 - \sum_{j=1}^{N} p_j$, and $P_{i,j}^*$ are $P_{i,j}$ (for both partial and total rejections) except $A(t)$ is replaced with $A_R(t)$ where $A_R(t) = \lambda \int_0^t [1 - A(w)]dw, (0 < w \le t)$. This way of computing p_j is analytically simpler and computationally more efficient than the previous method (see e.g., Laxmi and Gupta [35] who relates p_j with p_i^- through the use of a supplementary variable). As a remark, when $\lambda \to 0$ (see next Subsection) $p_0 \to 1$ since $\sum_{j=1}^{N} \sum_{i=0}^{N} p_i^- P_{i,j}^* = 0$ (such phenomena are numerically demonstrated in Section 3.4). The waiting-time distributions, blocking probabilities, and performance measures of $GI^X/M/c/N$ queues can all be found in terms of (19) using the relations derived by Laxmi and Gupta [35].

2.3.4 The $GI^X/M/c/N$ queues involving heavy-tailed inter-batch-arrival times

The analytical results in Section 2.3 remain robust even if the inter-batch-arrival times follow heavy-tailed distributions (both non-closed and non-analytic form of L-

S.T.'s). Unlike in $GI^X/M/c$ queues involving heavy-tailed inter-batch-arrival times, no manipulation of the characteristic equation is required when solving $GI^X/M/c/N$ queues involving heavy-tailed inter-batch-arrival times. Numerical examples are provided in Section 3.4.

2.4 Conclusions

In Section 2.1 we introduced some earlier work done by others on the continuous-time multi-server bulk-arrival queues.

In Section 2.2 we applied the roots method to solve $GI^X/M/c$ queues. By interpreting the Chapman-Kolmogorov equation as a set of linear difference equations we can express the solution in terms of roots. We also derived the explicit expressions of the waiting-time-in-queue distributions, as well, treated $GI^X/M/c$ queues involving heavy-tailed inter-batch-arrival times.

In Section 2.3 we applied the roots method to solve $GI^X/M/c/N$ queues. While this embarks on the first application of the roots method to treat continuous-time finite-buffer multi-server bulk-arrival queues, the method remains robust even in the case of heavy-tailed inter-batch-arrival times.

3 NUMERICAL RESULTS IN $GI^X/M/c$ AND $GI^X/M/c/N$ QUEUES

This chapter provides all the numerical results that complement the analytical work provided in Chapter 2. It is organized in the following manner: The $GI^X/M/c$ queues involving light-tailed inter-batch-arrival times are discussed in Section 3.1 and $GI^X/M/c$ queues involving heavy-tailed inter-batch-arrival times are discussed in Section 3.2. In addition, whereas Section 3.3 deals with $GI^X/M/c/N$ queues involving light-tailed inter-batch-arrival times, Section 3.4 deals with $GI^X/M/c/N$ queues involving heavy-tailed inter-batch-arrival times. All computations were performed on MAPLE, calibrated at the ninth decimal place. In presenting our numerical results, all results were rounded to four decimal places.

3.1 The $GI^X/M/c$ queues involving light-tailed inter-batch-arrival times

In computing the queue-length distributions of $GI^X/M/c$ queues at a pre-arrival, random, and post-departure time epochs, we consider the inter-batch-arrival patterns to be exponential and deterministic.

3.1.1 Exponential inter-batch-arrival times

The inter-batch-arrival pattern is exponential (M) with $a(t) = \lambda e^{-\lambda t}, (\lambda, t > 0)$. The parameters taken are $\rho = 0.5, b_1 = 0.4, b_2 = 0.6, \mu = 1,$ and $c = 3$. This gives $\mu_X = 1.6$ and $\lambda = 0.9375$.

Table 1: Various distributions in $M^X/M/3$ queue

j	p_j^-	p_j	p_j^+	t	$W_F(t)$	$W_R(t)$
0	0.2680	0.2680	0.1675	0.0000	0.7125	0.6401
1	0.2513	0.2513	0.2575	0.0114	0.7162	0.6448
2	0.1931	0.1931	0.2149	0.1544	0.7584	0.6991
3	0.1074	0.1074	0.1396	0.4566	0.8287	0.7879
4	0.0698	0.0698	0.0839	0.6940	0.8694	0.8387
5	0.0419	0.0419	0.0524	0.7740	0.8809	0.8529
6	0.0262	0.0262	0.0321	0.8990	0.8956	0.8711
7	0.0160	0.0160	0.0198	0.9974	0.9078	0.8862
8	0.0099	0.0099	0.0122	1.0123	0.9094	0.8881
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	1.0000	1.0000	1.0000	9.9999	0.9999	0.9999

$$\sum_{j=0}^{c-1}(c-j)p_j = 1.5000 \qquad c(1-\rho) = 1.5000$$

As expected, $p_j^- = p_j, (j \geq 0)$ due to the Poisson Arrivals See Time Averages (P.A.S.T.A.) property (see Wolff [49]).

3.1.2 Deterministic inter-batch-arrival times

The inter-batch-arrival pattern is Deterministic (D) with $\lambda = 1$. The parameters taken are $\rho = 0.4888, b_1 = 0.4, b_2 = 0.2, b_3 = 0.2, b_4 = 0.2, \mu = 1.5,$ and $c = 3$. This gives $\mu_X = 2.2$.

Table 2: Various distributions in $D^X/M/3$ queue

j	p_j^-	p_j	p_j^+	t	$W_F(t)$	$W_R(t)$
0	0.4866	0.2246	0.2211	0.0000	0.9435	0.7431
1	0.3413	0.3243	0.2878	0.0114	0.9452	0.7508
2	0.1157	0.2110	0.2341	0.1544	0.9626	0.8309
3	0.0332	0.1144	0.1529	0.4566	0.9836	0.9269
4	0.0145	0.0747	0.0677	0.6940	0.9915	0.9625
5	0.0054	0.0331	0.0230	0.7740	0.9932	0.9701
6	0.0019	0.0112	0.0080	0.8990	0.9952	0.9784
7	0.0007	0.0039	0.0031	0.9974	0.9963	0.9841

8	0.0002	0.0015	0.0011	1.0123	0.9965	0.9848
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Sum	0.9999	0.9999	0.9999	9.9999	0.9999	0.9999

$$\sum_{j=0}^{c-1}(c-j)p_j = 1.3020 \qquad c(1-\rho) = 1.3020$$

3.2 The $GI^X/M/c$ queues involving heavy-tailed inter-batch-arrival times

In computing the queue-length distributions of $GI^X/M/c$ queues at a pre-arrival, random, and post-departure time epochs, we consider the inter-batch-arrival patterns to be inverse-Gaussian, Pareto, standard log-normal, and Burr.

3.2.1 Inverse-Gaussian inter-batch-arrival times

The inter-batch-arrival pattern is inverse-Gaussian ($IG[\alpha, k]$) with $a(t) = \sqrt{\frac{k}{2\pi t^3}}e^{-\frac{k(t-\alpha)^2}{2\alpha^2 t}}, (t > 0)$. The parameters taken are $\alpha = 2, k = 4, \rho = 0.3268, b_1 = 0.325, b_2 = 0.1, b_3 = 0.075, b_4 = 0.05, b_5 = 0.125, b_6 = 0.025, b_7 = 0.025, b_8 = 0.05$, $b_9 = 0.125, b_{10} = 0.05, b_{11} = 0.025, b_{15} = 0.025, \mu = 1,$ and $c = 7$ This gives $\mu_X = 4.575$ and $\lambda = \alpha$.

Table 3: Various distributions in $IG^X[2,4]/M/7$ queue

j	p_j^-	p_j	p_j^+	t	$W_F(t)$	$W_R(t)$
0	0.4200	0.3071	0.0918	0.0000	0.9470	0.6984
1	0.2285	0.2100	0.1119	0.0114	0.9481	0.7041
2	0.1240	0.1280	0.1136	0.1544	0.9601	0.7696
3	0.0751	0.0866	0.1093	0.4566	0.9774	0.8685
4	0.0478	0.0625	0.1034	0.6940	0.9857	0.9161
5	0.0311	0.0473	0.0891	0.7740	0.9877	0.9280
6	0.0206	0.0340	0.0793	0.8990	0.9904	0.9433
7	0.0137	0.0259	0.0716	0.9974	0.9921	0.9532

8	0.0106	0.0234	0.0629	1.0123	0.9923	0.9545
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	1.0000	1.0000	1.0000	4.9999	0.9999	0.9999

$$\sum_{j=0}^{c-1}(c-j)p_j = 4.7125 \qquad c(1-\rho) = 4.7125$$

3.2.2 Pareto inter-batch-arrival times

The inter-batch-arrival pattern is Pareto ($Pareto[\alpha, k]$) with $a(t) = \frac{k^\alpha \alpha}{t^{\alpha+1}}, (0 < k \leq t)$. The parameters taken are $\alpha = 1.5, k = 2, \rho = 0.3631, b_1 = 0.325, b_2 = 0.1, b_3 = 0.075, b_4 = 0.05, b_5 = 0.125, b_6 = 0.025, b_7 = 0.025, b_8 = 0.05, b_9 = 0.125, b_{10} = 0.05, b_{11} = 0.025, b_{15} = 0.025, \mu = 0.3,$ and $c = 7$. This gives $\mu_X = 4.575$ and $\lambda = 0.1667$.

Table 4: Various distributions in $Pareto^X[1.5, 2]/M/7$ queue

j	p_j^-	p_j	p_j^+	t	$W_F(t)$	$W_R(t)$
0	0.2288	0.3735	0.0500	0.0000	0.8419	0.5676
1	0.1877	0.1271	0.0748	0.0114	0.8426	0.5694
2	0.1421	0.0950	0.0875	0.1544	0.8514	0.5916
3	0.1062	0.0741	0.0928	0.4566	0.8685	0.6361
4	0.0783	0.0589	0.0937	0.6940	0.8807	0.6684
5	0.0572	0.0476	0.0876	0.7740	0.8845	0.6788
6	0.0417	0.0371	0.0813	0.8990	0.8903	0.6945
7	0.0301	0.0295	0.0751	0.9974	0.8947	0.7064
8	0.0251	0.0273	0.0685	1.0123	0.8953	0.7081
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	1.0000	1.0000	1.0000	21.9999	0.9999	0.9999

$$\sum_{j=0}^{c-1}(c-j)p_j = 4.4583 \qquad c(1-\rho) = 4.4583$$

3.2.3 Standard log-normal inter-batch-arrival times

The inter-batch-arrival pattern is standard log-normal ($SLN[\sigma^2]$) with $a(t) = \frac{1}{t\sqrt{2\pi\sigma^2}} e^{-\frac{\ln^2(t)}{2\sigma^2}}, (t, \sigma > 0)$. The parameters taken are $\sigma^2 = 0.25, \rho = 0.4437, b_1 = 0.325, b_2 = 0.1, b_3 = 0.075, b_4 = 0.05, b_5 = 0.125, b_6 = 0.025, b_7 = 0.025, b_8 = 0.05, b_9 = 0.125,\ b_{10} = 0.05, b_{11} = 0.025, b_{15} = 0.025, \mu = 1.3$, and $c = 7$. This gives $\mu_X = 4.575$ and $\lambda = e^{-\frac{\sigma^2}{2}} = 0.8825$.

Table 5: Various distributions in $SLN[0.25]^X/M/7$ queue

j	p_j^-	p_j	p_j^+	t	$W_F(t)$	$W_R(t)$
0	0.2823	0.1668	0.0617	0.0000	0.9120	0.6410
1	0.2379	0.1916	0.0936	0.0114	0.9141	0.6487
2	0.1532	0.1454	0.1041	0.1544	0.9367	0.7365
3	0.0994	0.1077	0.1051	0.4566	0.9676	0.8628
4	0.0656	0.0816	0.1020	0.6940	0.9811	0.9190
5	0.0439	0.0634	0.0920	0.7740	0.9842	0.9324
6	0.0298	0.0476	0.0826	0.8990	0.9882	0.9491
7	0.0203	0.0366	0.0748	0.9974	0.9906	0.9594
8	0.0162	0.0332	0.0668	1.0123	0.9909	0.9607
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	1.0000	1.0000	1.0000	3.9999	0.9999	0.9999

$$\sum_{j=0}^{c-1}(c-j)p_j = 3.8943 \qquad c(1-\rho) = 3.8943$$

3.2.4 Burr inter-batch-arrival times

The inter-batch-arrival pattern is Burr (Burr$[k, u]$) with $a(t) = uk\frac{t^{u-1}}{(1+t^u)^{k+1}}, (k, u > 0, t \geq u)$. The parameters taken are $k = 2, u = 3, \rho = 0.5405, b_1 = 0.325, b_2 = 0.1, b_3 = 0.075, b_4 = 0.05, b_5 = 0.125, b_6 = 0.025, b_7 = 0.025, b_8 =$

$0.05, b_9 = 0.125, b_{10} = 0.05, b_{11} = 0.025, b_{15} = 0.025, \mu = 1.5,$ and $c = 7$. This gives $\mu_X = 4.575$ and $\lambda = \frac{1}{kB(k-1/u, 1+1/u)} = 1.2405$.

Table 6: Various distributions in Burr$[2,3]^X/M/7$ queue

j	p_j^-	p_j	p_j^+	t	$W_F(t)$	$W_R(t)$
0	0.1839	0.1011	0.0402	0.0000	0.8397	0.5608
1	0.2073	0.1520	0.0724	0.0114	0.8432	0.5697
2	0.1578	0.1370	0.0882	0.1544	0.8823	0.6727
3	0.1128	0.1112	0.0941	0.4566	0.9372	0.8240
4	0.0801	0.0889	0.0947	0.6940	0.9621	0.8933
5	0.0570	0.0717	0.0891	0.7740	0.9680	0.9099
6	0.0409	0.0562	0.0820	0.8990	0.9756	0.9310
7	0.0299	0.0443	0.0754	0.9974	0.9802	0.9442
8	0.0253	0.0408	0.0691	1.0123	0.9808	0.9459
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	1.0000	1.0000	1.0000	3.9999	0.9999	0.9999

$$\sum_{j=0}^{c-1}(c-j)p_j = 3.2165 \qquad c(1-\rho) = 3.2165$$

As a remark, $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$ where $Re(a), Re(b) > 0$.

3.3 The $GI^X/M/c/N$ queues involving light-tailed inter-batch-arrival times

In computing the queue-length distributions of $GI^X/M/c/N$ queues at pre-arrival and random time epochs, we consider the inter-batch-arrival patterns to be exponential and Erlang. We present three different cases of ρ in each table where $\rho < 1, \rho = 1$, and $\rho > 1$.

3.3.1 Exponential inter-batch-arrival times

The inter-batch-arrival pattern is exponential (M) with $a(t) = \lambda e^{-\lambda t}, (\lambda, t > 0)$. The parameters taken are $b_1 = 0.4, b_2 = 0.6, \mu = 1, c = 3, N = 5, \rho = 0.5, 1,$ and 2. This gives $\mu_X = 1.6, \mu = 1, \lambda = 0.9375, 1.875,$ and 3.75.

Table 7: Various distributions in $M^X/M/3/5$ queue

	p_j^- (partial rejection)			p_j^- (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 2$	$\rho = 0.5$	$\rho = 1$	$\rho = 2$
0	0.2877	0.0851	0.0116	0.2918	0.0916	0.0142
1	0.2697	0.1595	0.0436	0.2735	0.1716	0.0543
2	0.2073	0.1974	0.0948	0.2103	0.2124	0.1178
3	0.1154	0.1832	0.1512	0.1170	0.1971	0.1879
4	0.0749	0.1885	0.2601	0.0760	0.2028	0.3232
5	0.0450	0.1865	0.4386	0.0314	0.1246	0.3026
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	p_j (partial rejection)			p_j (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 2$	$\rho = 0.5$	$\rho = 1$	$\rho = 2$
0	0.2877	0.0851	0.0116	0.2918	0.0916	0.0142
1	0.2697	0.1595	0.0436	0.2735	0.1716	0.0543
2	0.2073	0.1974	0.0948	0.2103	0.2124	0.1178
3	0.1154	0.1832	0.1512	0.1170	0.1971	0.1879
4	0.0749	0.1885	0.2601	0.0760	0.2028	0.3232
5	0.0450	0.1865	0.4386	0.0314	0.1246	0.3026
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

As expected, $p_j^- = p_j, (0 \le j \le N)$ due to the P.A.S.T.A. property.

3.3.2 Erlang inter-batch-arrival times

The inter-batch-arrival pattern is Erlang (E_m) with $a(t) = \frac{(m\lambda)^m t^{m-1} e^{-m\lambda t}}{(m-1)!}, (m, \lambda, t > 0)$. The parameters taken are $m = 2, b_1 = 0.4, b_2 = 0.6, \mu =$

$1, c = 3, N = 5, \rho = 0.5, 1, \text{and } 2$. This gives $\mu_X = 1.6, \mu = 1, \lambda = {}^{\rho c \mu m}/_{\mu_X} =$ $1.875, 3.75, \text{and } 7.5.$

Table 8: Various distributions in $E_2^X/M/3/5$ queue

	p_j^- (partial rejection)			p_j^- (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 2$	$\rho = 0.5$	$\rho = 1$	$\rho = 2$
0	0.0731	0.0063	0.0003	0.0800	0.0086	0.0005
1	0.1738	0.0333	0.0034	0.1896	0.0453	0.0061
2	0.2245	0.0902	0.0193	0.2428	0.1207	0.0337
3	0.2009	0.1641	0.0709	0.2122	0.2109	0.1175
4	0.1894	0.2957	0.2473	0.1855	0.3407	0.3572
5	0.1383	0.4105	0.6588	0.0900	0.2738	0.4849
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	p_j (partial rejection)			p_j (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 2$	$\rho = 0.5$	$\rho = 1$	$\rho = 2$
0	0.0499	0.0040	0.0002	0.0548	0.0056	0.0003
1	0.1371	0.0235	0.0022	0.1501	0.0322	0.0041
2	0.2041	0.0694	0.0135	0.2227	0.0947	0.0242
3	0.2055	0.1377	0.0533	0.2229	0.1849	0.0935
4	0.2098	0.2727	0.2062	0.2237	0.3541	0.3444
5	0.1937	0.4926	0.7247	0.1259	0.3285	0.5334
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

3.4 The $GI^X/M/c/N$ queues involving heavy-tailed inter-batch-arrival times

In computing the queue-length distributions of $GI^X/M/c/N$ queues at pre-arrival and random time epochs, we consider the inter-batch-arrival patterns to be Cauchy and Lévy. We present three different cases of μ in each table where $\mu < 1, \mu = 1$, and $\mu > 1$.

3.4.1 Standard Cauchy inter-batch-arrival times

The inter-batch-arrival pattern is standard Cauchy ($SCauchy$) with $a(t) = \frac{2}{\pi(1+t^2)}, (t > 0)$. The parameters taken are $\lambda = \infty, b_1 = 0.4, b_2 = 0.6, \mu = 1$, $c = 3, N = 5, \mu = 0.5, 1$, and 1.5. This gives $\mu_X = 1.6$ and $\rho = 0$.

Table 9: Various distributions in $SCauchy^X/M/3/5$ queue

	p_j^- (partial rejection)			p_j^- (total rejection)		
j	$\mu = 0.5$	$\mu = 1$	$\mu = 1.5$	$\mu = 0.5$	$\mu = 1$	$\mu = 1.5$
0	0.2405	0.4354	0.5591	0.2445	0.4372	0.5600
1	0.2153	0.2581	0.2418	0.2207	0.2597	0.2423
2	0.2054	0.1724	0.1317	0.2119	0.1736	0.1320
3	0.1382	0.0741	0.0423	0.1442	0.0748	0.0424
4	0.1128	0.0405	0.0185	0.1186	0.0409	0.0185
5	0.0878	0.0194	0.0067	0.0601	0.0140	0.0050
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	p_j (partial rejection)			p_j (total rejection)		
j	$\mu = 0.5$	$\mu = 1$	$\mu = 1.5$	$\mu = 0.5$	$\mu = 1$	$\mu = 1.5$
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

3.4.2 Lévy inter-batch-arrival times

The inter-batch-arrival pattern is Lévy (Lévy$[k, u]$) with $a(t) = \sqrt{\frac{k}{2\pi}} \frac{e^{-\frac{k}{2(t-u)}}}{(t-u)^{3/2}}, (k, u > 0, t > u)$. The parameters taken are $\lambda = \infty, k = 0.7, u = 0, b_1 = 0.4, b_2 = 0.6$, $c = 3, N = 5, \mu = 0.5, 1$, and 1.5. This gives $\mu_X = 1.6$ and $\rho = 0$.

Table 10: Various distributions in Lévy$^{X}[0.7, 0]/M/3/5$ queue

	p_j^- (partial rejection)			p_j^- (total rejection)		
j	$\mu = 0.5$	$\mu = 1$	$\mu = 1.5$	$\mu = 0.5$	$\mu = 1$	$\mu = 1.5$
0	0.4187	0.5737	0.6680	0.4203	0.5743	0.6682
1	0.2218	0.2258	0.2052	0.2236	0.2263	0.2054
2	0.1798	0.1327	0.0955	0.1822	0.1331	0.0956
3	0.0875	0.0422	0.0219	0.0901	0.0425	0.0219
4	0.0582	0.0192	0.0077	0.0599	0.0190	0.0075
5	0.0340	0.0066	0.0018	0.0239	0.0049	0.0014
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	p_j (partial rejection)			p_j (total rejection)		
j	$\mu = 0.5$	$\mu = 1$	$\mu = 1.5$	$\mu = 0.5$	$\mu = 1$	$\mu = 1.5$
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

3.5 Conclusions

The numerical results based on Chapter 2 are presented in Chapter 3. In doing so, all different cases are considered (the $GI^X/M/c$ and $GI^X/M/c/N$ queues involving light and heavy-tailed inter-batch-arrival times).

Since the queue-length distributions are in terms of roots, the characteristic equation plays a pivotal role in treating $GI^X/M/c$ and $GI^X/M/c/N$ queues. When the inter-batch-arrival times of the $GI^X/M/c$ queues follow heavy-tailed distributions, the characteristic equation needs to be modified to allow the computation of roots. This aspect of $GI^X/M/c$ queues is isolated and studied further in Appendix C.3.1 and C.3.2. On the other hand, no manipulation of the characteristic equation is needed when treating $GI^X/M/c/N$ queues using roots.

4 ANALYTICAL RESULTS IN $GI^X/Geo/c$ AND $GI^X/Geo/c/N$ QUEUES

Readers may refer to Appendix A.1 for a brief summary of probability theory, stochastic processes, and Markov processes, which are all important topics that lead to discrete-time queueing theory. The definitions and properties of a discrete r.v. and its moments, g.f., and p.g.f. are provided in Appendix A.3. The basic mathematical construct of a queueing system, as well as some common theorems and techniques used in discrete-time queueing theory are explained in Appendix B. The rest of the materials that supplement Chapter 4 are available in Appendix C.2.

4.1 Literature review

The study of discrete-time queues is relatively recent compared to its continuous-time counterpart. Over the last few decades, the field quickly gained value among queueing theorists and researchers due to the digitization of information technology, particularly in the area of signal processing devices, microcomputers, and computer networks. In analyzing discrete-time queues, many researchers have recognized that the continuous-time models are no longer suitable to accurately measure the performance of systems in which the basic operational units are digital, such as machine cycle time, bits, and packets (see Miyazawa and Takagi [38] for details). For further details on the discrete-time models and telecommunications, one may see Bruneel and Kim [4]. In this connection, see also Takagi [45].

In discrete-time queueing theory, various techniques have been introduced by many researchers to analyze the $GI/Geo/1$ queues. Some of their techniques include the imbedded Markov chain, supplementary variable, semi-Markov process, birth and death process, matrix-geometric, combinatorial, and numerical methods. What is pervasive across these techniques is in their final product: The $GI/Geo/1$ queues have three distinct queue-length distributions at three different time epochs. Though each queue-length distribution has its own importance, they all have other purposes as well. For instance, the queue-length distribution at a pre-arrival time epoch is used to compute the actual waiting-time distribution, and in the case of queues with finite-buffer it is used to evaluate the blocking probability (see Chaudhry and Gupta [10]). The queue-length distribution at a random time epoch is needed to compute the virtual waiting-time, whereas the queue-length distribution at an outside observer's time epoch is used to obtain various performance measures such as the mean waiting-time-in-queue using Little's law. Moreover, relations between the queue-length distributions at different time epochs involve interesting mathematical derivations.

In comparing $GI/Geo/1$ queues with its continuous-time counterpart, the $GI/M/1$ queues, the main difference between the two models is in the measurement of time. While a continuous-time model has a time parameter that is a real number, a discrete-time model has a time parameter that is an integer. In $GI/Geo/1$ queues, the time axis is divided into individual time slots where the duration of 1 time slot is a single unit of time. In each individual time slot, two events may occur: arrival and departure. When an arrival occurs before a departure it is called $GI/Geo/1$ queues with Early Arrival System (EAS) and when a departure occurs before an arrival, it is called the $GI/Geo/1$ queue with Late Arrival System (LAS). Further, in $GI/Geo/1$ queues with LAS, if a server is idle and a customer arrives, then either his service can start immediately or in the next time slot. In the former case it is known as an immediate access (IA), whereas in the latter case it is known as a delayed access (DA). When discussing $GI/Geo/1$ type queues, there exist six queue-length distributions: Pre-arrival, random, and outside observer's time epochs (three for EAS and three for LAS-DA).

The queueing model $GI^X/Geo/1$ considers the model $GI/Geo/1$ with a batch-arrival. The earliest work on $GI^X/Geo/1$ queues appears to be that of Vinck and Bruneel [46] who use the complex contour integration technique. Though they provide an analytical solution to $GI^X/Geo/1$ queues with EAS at different time epochs, they do not provide the corresponding numerical results. Furthermore, in their work, the analysis of $GI^X/Geo/1$ queues with LAS-DA is missing, yet it is deemed to be an important aspect of $GI^X/Geo/1$ queues when considering applications in telecommunications (see Takagi [45]). In response to this, Chaudhry and Gupta [11] provide the first complete solution to $GI^X/Geo/1$ queues using the supplementary variable technique. One of the main contributions of their work is that they do not stop after finding the p.g.f. which was a common way to conclude the analysis of a queueing model at that time, but perform the inversion of the p.g.f.. Doing so enables the finding of queue-length distributions in terms of the roots of the model's characteristic equation. This technique is referred to as the roots method.

Historically, the roots method was dismissed by some queueing theorists due to perceived difficulties in computing the roots of a model's characteristic equation. Neuts states (see Stidham [44]) "in discussing matrix-analytic solutions, I had pointed out that when the Rouché roots coincide or are close together, the method of roots could be numerically inaccurate. When I finally got copies of Crommelin's papers, I was elated to read that, for the same reasons as I, he was concerned about the clustering of roots. In

1932, Crommelin knew; in the 1980, many of my colleagues did not…." Readers can refer to Neuts [39] for his other comments on the roots method. In the 1980s, commercial computing software such as MATLAB and MAPLE were not able to find the roots (they do now). To address the issue of root-finding in queueing theory, the QROOT software was developed by M.L. Chaudhry in the early 1990s and demonstrated that such roots can be found (see Chaudhry [9]). The roots method was then successfully adopted to solve a wide variety of queueing problems as noted by Janssen and van Leeuwaarden [28] who write "initially, the potential difficulties of root-finding were considered to be a slur on the unblemished transforms since the determination of the roots can be numerically hazardous and the roots themselves have no probabilistic interpretation. However, Chaudhry et al. [8] have made every effort to dispel the skepticism towards root-finding in queueing theory…."

While the roots method is simply another way of solving queueing problems, there are added advantages to it as well. Gouweleeuw [23] states "it is more efficient to use the roots method to get explicit expressions for probabilities from g.f.'s." Furthermore, a recent paper by Maity and Gupta [37] compares the spectral theory approach against the roots method. Maity and Gupta [37] identify several difficulties in getting results using the spectral theory approach, an approach which may be simpler than the matrix-geometric approach as stated in several papers by Chakka (see e.g., Chakka [6]). As well, Daigle and Lucantoni [19] state "whenever the roots method works, it works blindingly fast." The roots method, when compared to other methods, is deemed analytically elegant and computationally efficient.

However, while the roots method historically only dealt with queues involving light-tailed distributions, more recent research by Harris et al. [25] conclude that the roots method cannot solve the queues that involve heavy-tailed inter-batch-arrival times. Heavy-tailed distributions constitute a class of probability distributions that are characterized by their slower decay than the exponential or geometric distribution. When considering heavy-tailed distributions as an inter-arrival time distribution, the consensus among some researchers is that the roots method cannot be applied due to the unique probabilistic properties of heavy-tailed r.v.'s. In sharing this view, Harris et al. [25] state that "the standard root-finding problem gets complicated particularly when the inter-arrival time distribution possesses a complicated non-closed form or non-analytic L-S.T.." The same difficulty persists in discrete-time queues since discrete heavy-tailed probability distributions such as Weibull and Log-Normal distributions do not have a closed form p.g.f.. In addition, the discrete Pareto distribution, for certain values of its parameter (see later in Table 11), has an infinite mean just like the continuous Pareto distribution. Nevertheless, the heavy-tailed distributions are useful tools in modeling real life examples such as in finance, signal processing, and physical or biological systems (see Willinger and Paxon [47], Leland et al. [36], Park et al. [40, 41], and Pitkow [43]). In particular, heavy-tailed distributions (synonymously referred to as the power, long or fat-tailed distribution) are particularly useful when modeling the inter-arrival times of network packets and connection sizes under heavy traffic congestion (see Harris et al. [25]).

Despite the benefits of the roots method (i.e. analytically simple and numerically efficient) the solution to discrete-time multi-server bulk-arrival queues using roots is missing in the literature. There exists some work on $GI/Geo/c$ and $GI^X/Geo/c$ queues using different methods: In solving $GI/Geo/c$ queues, Chan and Maa [7] use the imbedded Markov chain technique to derive the queue-length distribution for EAS at a pre-arrival time epoch only. Similarly, Wittevrongel et al. [48] solve the $GI^X/Geo/c$ queues using complex analysis and contour integration while only considering the EAS with no numerical results. Lastly, Chaudhry et al. [12] solve $GI^X/Geo/c$ queues using the supplementary variable technique while only considering light-tailed inter-batch-arrival times. The results by Chaudhry et al. [12] are in terms of iterative relations between different queue-length distributions which can be analytically laborious and numerically inefficient.

The $GI^X/Geo/c/N$ queues are the finite-buffer counterpart of $GI^X/Geo/c$ queues. In real life applications, many contexts that resemble $GI^X/Geo/c$ queues entail some degree of finite-buffer (e.g., maximum data buffer, packet size, processing speed, etc). In solving the model $GI^X/Geo/c/N$, Goswami and Samanta [22] use the supplementary variable technique and follow a similar solution procedure used by Chaudhry et al. [12] and Laxmi and Gupta [35]. However, like the others' methods that they have followed, the results by Goswami and Samanta [22] are in a non-explicit form that involve lengthy expressions. The solution to $GI^X/Geo/c/N$ queues with LAS-DA as well as the case of heavy-tailed inter-batch-arrival times are missing in the literature on queues.

4.2 The $GI^X/Geo/c$ queues

In this section, we analytically solve $GI^X/Geo/c$ queues using roots. Some notations from Section 2.2 are redefined in the context of $GI^X/Geo/c$ queues.

4.2.1 Model description

Consider the model $GI^X/Geo/c$ in which customers arrive in batches of size X with a maximum size $r, (0 < r < \infty)$. The p.m.f. of X is $b_h = P(X = h),\ \ (1 \leq h \leq r)$

with mean $\bar{b} = \sum_{h=1}^{r} hb_h$ and p.g.f. $B(z) = \sum_{h=1}^{r} b_h z^h, (|z| \le 1)$. The n-th inter-batch-arrival time, say $T_n, (n \ge 1)$ is a discrete-time period that is measured from the moment just before the n-th batch-arrival (say at time t) to the moment just before the $(n+1)$-th batch-arrival (say at time $t+m$). Inter-batch-arrival times are i.i.d.r.v.'s distributed as T which is divided into m time slots. The T has a p.m.f. $a_m = P(T = m), (a_0 = 0, m \ge 1)$, mean $\bar{a} = \sum_{m=1}^{\infty} ma_m = {}^{1}\!/_{\lambda}$, and p.g.f. $A(z) = \sum_{m=1}^{\infty} a_m z^m, (|z| \le 1)$. The batch sizes are independent of the inter-batch-arrival times. There are c parallel servers in the model where each has a service time Y that is independent to one another and geometrically distributed as

$$P(Y = y) = (1-\mu)^{y-1}\mu, (0 < \mu < 1, y \ge 1)$$

At any time the state of servers can be categorized into three different cases: Overloaded is the case when all c servers are busy with a queue of at least a single customer, loaded is the case when the system has exactly c customers in the system, and under-loaded is the case when there is at least an idle server. The mean service time of a server is $E[Y] = {}^{1}\!/_{\mu}$ and traffic intensity of the model is $\rho = {}^{\bar{b}}\!/_{\bar{a}c\mu}, (0 < \rho < 1)$. In addition, let $\omega(j|i)$ be the probability of an event that j customers are served out of i customers within a single time slot. The conditional probability $\omega(j|i)$ is a binomial distribution

$$\omega(j|i) = \begin{cases} \binom{i}{j} \mu^j (1-\mu)^{i-j}, (0 \le j \le i < c) \\ \binom{c}{j} \mu^j (1-\mu)^{c-j}, (0 \le j \le c \le i) \end{cases} \tag{20}$$

where $\omega(0|0) = 1$ and $\binom{a}{b} = 0$ for $a < b$ or $a < 0$. Lastly, given that the system $GI^X/Geo/c$ is a discrete-time queueing model it has two different but related aspects

(EAS and LAS-DA). Since the model is solved under the steady-state condition, the queue-length distribution of the LAS-IA is identical to that of the EAS.

4.2.2 The $GI^X/Geo/c$ queues with EAS at a pre-arrival time epoch

In $GI^X/Geo/c$ queues with EAS, the n-th batch arrives before an event of customer departures in the n-th inter-batch-arrival time. This is illustrated in Figure 1 below.

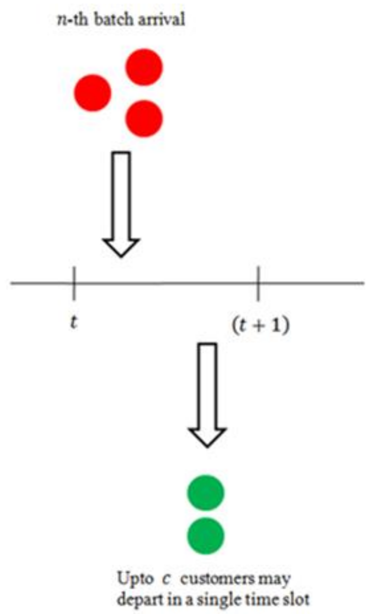

Figure 1: The $GI^X/Geo/c$ queues with EAS

Let N_n^- be the number of customers in the system, including those, if any, receiving service just before the n-th batch-arrival (i.e. at the n-th pre-arrival time epoch). The stochastic process $\{N_n^-, n \geq 1\}$ forms a homogenous Markov chain:

$$N_{n+1}^- = \begin{cases} N_n^- + X_n - D_n, & (N_n^- + X_n - D_n \geq 0) \\ 0, & (N_n^- + X_n - D_n < 0) \end{cases}$$

where D_n is the total number of customers that depart the system during T_n and X_n is the size of the n-th batch-arrival. In considering the steady-state aspect of $GI^X/Geo/c$ queues with EAS, N_n^- becomes N^- as $n \to \infty$ and has the steady-state p.m.f. $Q_j^- = \lim_{n\to\infty} P(N_n^- = j), (j \geq 0)$. Let $P_{i,j}(n) = P[N_{n+1}^- = j \mid N_n^- = i], (i, j \geq 0, n \geq 1)$ be the one-step transition probabilities of $\{N_n^-, n \geq 1\}$. Thus, the steady-state one-step transition probabilities of $GI^X/Geo/c$ queues with EAS are defined as $P_{i,j} \equiv \lim_{n\to\infty} P_{i,j}(n)$ where

$$P_{i,j} = \begin{cases} \sum_{h=1}^{r} b_h k_{i+h-j}, (i \geq 0, j \geq 1) \\ 1 - \sum_{k=1}^{\infty} P_{i,k}, (i \geq 0, j = 0) \end{cases}$$

where $b_h = 0$ for $h \leq 0$. As $n \to \infty$, D_n becomes D with the steady-state p.m.f. $k_j = \lim_{n\to\infty} P(D_n = j), (j \geq 0)$, which is defined as

$$k_{i+h-j} = \begin{cases} 0, & (i+h<j) \\ \sum_{m=1}^{\infty} a_m \binom{cm}{i+h-j} (1-\mu)^{cm-(i+h-j)} (\mu)^{i+h-j}, & (c \leq j \leq i+h) \\ \sum_{m=1}^{\infty} a_m \binom{i+h}{j} (1-\mu)^{mj} (1-(1-\mu)^m)^{i+h-j}, & (1 \leq j \leq i+h \leq c) \\ \sum_{m=1}^{\infty} \left[\sum_{w=1}^{m} \sum_{s=j}^{c} \sum_{l=c-s+1}^{c} \binom{c(w-1)}{i+h-l-s} \mu^{i+h-l-s} (1-\mu)^{c(w-1)-(i+h-l-s)} \times \right. & \\ \left. \binom{c}{l} \mu^l (1-\mu)^{c-l} \binom{S}{j} (1-\mu)^{(m-w)j} (1-(1-\mu)^{m-w})^{s-j} \right] a_m, & (1 \leq j < c < i+h) \end{cases}$$

for $1 \leq h \leq r$. The $P_{i,j}$ can be derived by replacing the single-arrival notion with a batch-arrival condition $h, (1 \leq h \leq r)$ in the steady-state one-step transition probabilities of $GI/Geo/c$ queues which are available in Chan and Maa [7]. Furthermore, the same results are also available in Chaudhry et al. [12]. Lastly, in expressing $\left(Q_j^-, j \geq 0\right)$ in terms of roots, the global balance equation (see Appendix A.1 for definition)

$$Q_j^- = \sum_{i=0}^{\infty} Q_i^- P_{i,j}, (j \geq 0) \tag{21}$$

is used extensively. Since (21) is a set of j first order linear difference equations with $\left(Q_j^-\right)$ being the unknown functions to be determined, we can assume the solution of a general form

$$Q_j^- = Cz^j, (j \geq c, C \neq 0)$$

The general solution with a bound $j \geq c$ is purposely chosen so that when it is substituted into (21), the k_{i+h-j} within the $P_{i,j}$ of (21) is fixed at $k_{i+h-j} = \sum_{m=1}^{\infty} a_m \binom{cm}{i+h-j} (1-\mu)^{cm-(i+h-j)} (\mu)^{i+h-j}$ (intuitively this can be understood as all c servers are either overloaded or loaded during T). This substitution leads to the characteristic equation of $GI^X/Geo/c$ queues

$$1 = B(z^{-1})K(z) \tag{22}$$

where $K(z) = \sum_{j=0}^{\infty} k_j z^j = \sum_{m=1}^{\infty} a_m (\mu z + 1 - \mu)^{cm} = A\{(1 - \mu + \mu z)^c\}$. As a remark, while (22) is superficially identical to (2), these two characteristic equations are fundamentally different since (2) takes place in continuous-time whereas (22) takes place in discrete-time. The two characteristic equations can be related as shown in Appendix C.2.3. Since (22) has r roots inside the unit circle $|z| = 1$ (see the proof in Appendix C.2.1), let these roots be $z_1, z_2, \dots, z_r$. Hence the Q_j^- for $j \geq c$ becomes r-fold and can be expressed as

$$Q_j^- = \sum_{h=1}^{r} C_h z_h^j, (j \geq c) \tag{23}$$

where the C_h (yet to be evaluated) for $1 \leq h \leq r$ are the non-zero constants. In completely finding $\left(Q_j^-, j \geq 0\right)$ we replace $\left(Q_j^-, j \geq c\right)$ in (21) with (23) such that

$$Q_j^- = \sum_{i=0}^{c-1} Q_i^- P_{i,j} + \sum_{i=c}^{\infty} \sum_{h=1}^{r} C_h z_h^j P_{i,j}, (j \geq 0) \tag{24}$$

We also consider the normalizing condition

$$\sum_{j=0}^{c-1} Q_j^- + \sum_{j=c}^{\infty}\sum_{h=1}^{r} C_h z_h^j = 1$$

By letting $j = 1, 2, \ldots, c + r - 1$ in (24) and with the normalizing condition we have the $c + r$ equations that are required to solve for $\left(Q_j^-, 0 \le j \le c - 1\right)$ and $C_h, (1 \le h \le r)$. By solving these equations the $\left(Q_j^-, j \ge 0\right)$ are completely found as

$$Q_j^- = \begin{cases} \text{determined above}, (0 \le j \le c - 1) \\ \displaystyle\sum_{h=1}^{r} C_h z_h^j, \qquad (j \ge c) \end{cases} \tag{25}$$

4.2.3 The $GI^X/Geo/c$ queues with EAS at a random time epoch

In the steady-state aspect of $GI^X/Geo/c$ queues with EAS, a pre-arrival time epoch falls immediately before a batch-arrival whereas a random time epoch falls at any instant between two consecutive batch-arrivals. To illustrate this concept, let R_n be the discrete-time period measured from t to $t + i, (1 \le i \le m)$. In addition, R_n are i.i.d.r.v.'s distributed as R with the p.m.f. $a_i^* = P(R = i), (a_0^* = 0, i \ge 1)$. The visual illustration of R_n and T_n for $GI^X/Geo/c$ queues with EAS are provided in Figure 2 where R_n consist of i time slots which are the duration of time measured from the n-th pre-arrival time epoch to the $(n + 1)$-th random time epoch. On the other hand, T_n consists of m time slots which are the duration of time measured from the n-th pre-arrival time epoch to the $(n + 1)$-th pre-arrival time epoch.

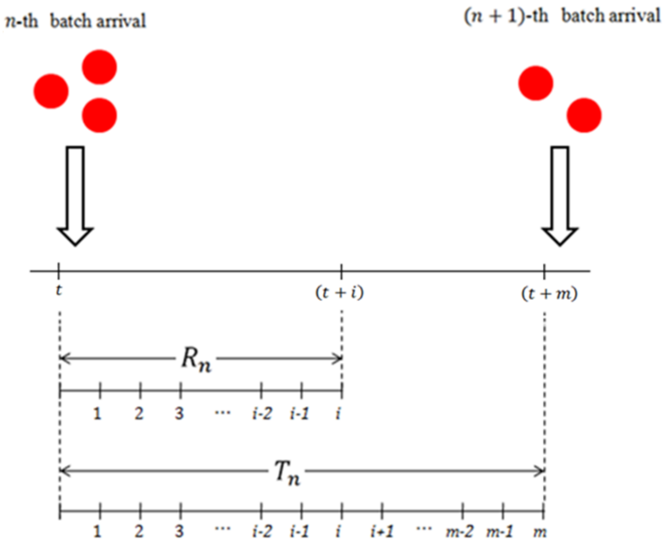

Figure 2: The $GI^X/Geo/c$ queues with EAS at pre-arrival and random time epochs

Based on the discrete renewal theory and random biased sampling (see Appendix B.4), if R consists of i time slots then its p.m.f. $(a_i^*, 1 \le i \le m)$ can be expressed in terms of $(a_m, m \ge 1)$ such that

$$a_i^* = \frac{1}{\bar{a}}(a_i + a_{i+1} + .. + a_{m-2} + a_{m-1} + a_m), (1 \le i \le m)$$

As $m \to \infty$, the above expression becomes

$$a_i^* = \frac{1}{\bar{a}} \lim_{m\to\infty} \sum_{l=i}^{m} a_l \,, (i \ge 1)$$

In addition, let $\left(Q_j, j \ge 0\right)$ be the steady-state p.m.f. of the number of customers in the system at a random time epoch for $GI^X/Geo/c$ queues with EAS. Using the above relation, we can determine $\left(Q_j, j \ge 0\right)$ with the following global balance equation:

$$Q_j = \sum_{i=0}^{\infty} Q_i^- P_{i,j}^* \,, (j \ge 1)$$

where $(Q_i^-, i \geq 0)$ are available in (25) and the $P_{i,j}^*$ are the transition probabilities from Subsection 4.2.2 except $(a_m, m \geq 1)$ are replaced with $(a_m^*, m \geq 1)$. In addition, $Q_0 = 1 - \sum_{j=1}^{\infty}\sum_{i=0}^{\infty} Q_i^- P_{i,j}^*$ holds. The Geometric Arrivals See Time Averages (G.A.S.T.A.) property holds if $a_l = \lambda(1-\lambda)^{l-1}, (l \geq 1, 0 < \lambda < 1)$ are substituted into the relation between a_m and a_i^* (see Takagi [45] for more details). Doing so leads to $a_i^* = a_i, (i \geq 1)$, hence due to this property, $Q_j = Q_j^-, (j \geq 0)$ in $Geo^X/Geo/c$ queues (see later in Subsection 5.1.1).

4.2.4 The $GI^X/Geo/c$ queues with EAS at an outside observer's time epoch

By definition, an outside observer's time epoch in $GI^X/Geo/c$ queues with EAS falls anywhere just after a batch-arrival and immediately before an event of customer departures (see Figure 3 below). Let $(Q_j^o, j \geq 0)$ be the queue-length distribution of $GI^X/Geo/c$ queues with EAS at an outside observer's time epoch.

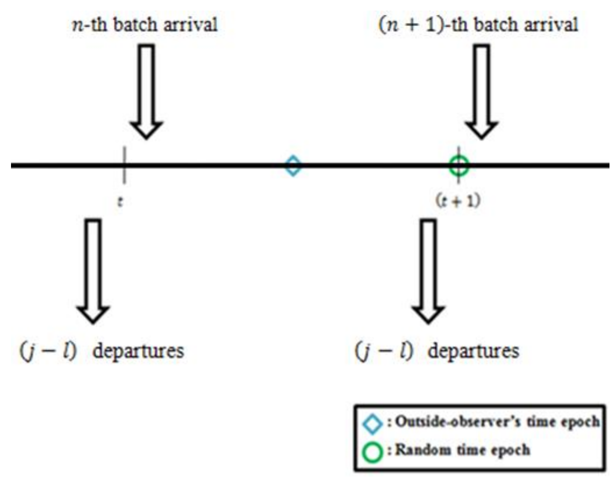

Figure 3: Outside observer's and random time epochs in $GI^X/Geo/c$ queues with EAS

In the above figure, the n-th inter-batch-arrival time period is a single time slot $(\mathrm{i.e.}\, m = 1)$ hence the $(n+1)$-th random and $(n+1)$-th pre-arrival time epochs coincide at $t+1$ (this can be verified by letting $m = 1$ in Figure 2).

In Figure 3, if there are j customers in the system at an outside observer's time epoch (with probability Q_j^o) and there are l customers in the system at a random time epoch (with probability Q_l) then there must be an event of $j-l$ customer departures between an outside observer's and random time epochs with the probability $\omega(j-l|j)$. Based on this notion, we can form the relation

$$Q_l = \sum_{j=l}^{l+c} Q_j^o \omega(j-l|j)\,, (l \geq 0) \tag{26}$$

Since (26) is a set of l first order linear difference equations with $\left(Q_j^o\right)$ being the unknown functions to be determined, we assume that $\left(Q_j^o, j \geq c\right)$ is a geometric sum that consists of the roots of (22) but with different constant coefficients (hence they are unknown). Let these unknown constant terms be $E_h, (1 \leq h \leq r)$, then the $\left(Q_j^o, j \geq 0\right)$ can be expressed as

$$Q_j^o = \sum_{h=1}^{r} E_h z_h^j\,, (j \geq c)$$

By replacing $\left(Q_j^o, j \geq c\right)$ in (26) with the above geometric sum we have

$$Q_l = \sum_{j=l}^{c-1} Q_j^o \omega(j-l|j) + \sum_{j=c}^{l+c} \sum_{h=1}^{r} E_h z_h^j\, \omega(j-l|j), (l \geq 0) \tag{27}$$

We also consider the normalizing condition

$$\sum_{j=0}^{c-1} Q_j^o + \sum_{j=c}^{\infty} \sum_{h=1}^{r} E_h z_h^j = 1 \tag{28}$$

We let $l = 1, 2, \ldots, c + r - 1$ in (27) and with (28) it leads to $c + r$ equations that are required to solve for $\left(Q_j^o, 0 \le j \le c - 1\right)$ and $E_h, (1 \le h \le r)$. By solving these equations the $\left(Q_j^o, j \ge 0\right)$ are completely found as

$$Q_j^o = \begin{cases} \text{determined above}, & (0 \le j \le c - 1) \\ \displaystyle\sum_{h=1}^{r} E_h z_h^j, & (j \ge c) \end{cases} \tag{29}$$

4.2.5 The $GI^X/Geo/c$ queues with LAS-DA at a pre-arrival time epoch

In $GI^X/Geo/c$ queues with LAS-DA, the n-th batch arrives after an event of customer departures in the n-th inter-batch-arrival time. This is illustrated in Figure 4 below.

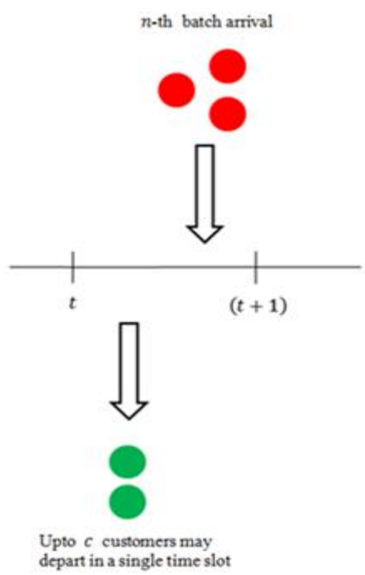

Figure 4: The $GI^X/Geo/c$ queues with LAS-DA

Let M_n^- be the number of customers in the system, including the ones, if any, in service just before the n-th batch-arrival. The M_n^- becomes M^- as $n \to \infty$ and has a steady-state p.m.f. $P_j^- = \lim_{n\to\infty} P(M_n^- = j), (j \ge 0)$. Before proceeding to find P_j^-, it is worth mentioning that P_j^- can be found independently of $Q_k^-, (0 \le j \le k)$ (as done by

Chaudhry and Kim [14] in solving the model $GI^X/Geo/1$). However, we leverage Q_k^- from Subsection 4.2.2 to determine P_j^- for the purpose of demonstrating continuity in our solution procedure. Let $P_j^-(s)$ be the p.m.f. of there being j customers in the system $GI^X/Geo/c$ with LAS-DA at time s and let $Q_k^-(t)$ be the p.m.f. of there being $k, (0 \leq k \leq j)$ customers in the system $GI^X/Geo/c$ with EAS at time t. Figure 5 illustrates that in EAS, the n-th batch arrives at the beginning of the time slot $(t, t+1)$, whereas in LAS-DA, the same n-th batch would arrive at the end of the time slot $(s, s+1)$.

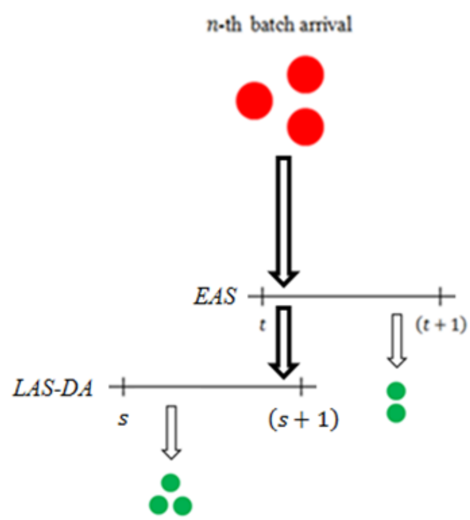

Figure 5: A comparison between EAS and LAS-DA at the n-th batch-arrival

Since there can be an event of up to c customer departures during (s, t) we can form the relation between $Q_k^-(t)$ and $P_j^-(s)$ as

$$Q_k^-(t) = \sum_{j=k}^{n+c} P_j^-(s)\omega(j-k|j), (k \geq 0)$$

As a remark, $0 \leq k \leq j$ indicates that if no customers depart during (s, t) then $k = j$ with probability $\omega(0|j)$, whereas any customer departure during (s, t) results in $k < j$ with probability $\omega(j-k|j)$. By letting $Q_n^- = \lim_{t\to\infty} Q_n^-(t), (n \geq 0)$ and $P_j^- = \lim_{s\to\infty} P_j^-(s)$, $(j \geq 0)$, so that we have the queue-length distributions, we have

$$Q_k^- = \sum_{j=k}^{n+c} P_j^- \omega(j-k|j), (k \geq 0) \tag{30}$$

Since (30) is a set of k first order linear difference equations with $\left(P_j^-\right)$ being the unknown functions to be determined, we can make a similar assumption as the one made in Subsection 4.2.4 and let

$$P_j^- = \sum_{h=1}^{r} F_h z_h^j, (j \geq c+1) \tag{31}$$

where the $F_h, (1 \leq h \leq r)$ in (31) are the unknown non-zero constants. As a remark, (31) differs from (23) since we have $j \geq c+1$ in (31) but $j \geq c$ in (23). This difference is due to the property of $GI^X/Geo/c$ queues with LAS-DA: Assume that there are initially no customers in the system and then a batch of customers arrives after some inter-batch-arrival time. If the next inter-batch-arrival time is 1 time slot (with probability a_1), then no customers access any of the c servers even if there are idle servers, whereas if the inter-batch-arrival time is at least two time slots (with probability $a_m, m \geq 2$), then customers start accessing the idle servers at the beginning of the second time slot. To solve for the $\left(P_j^-, 0 \leq j \leq c\right)$ and $F_h, (1 \leq h \leq r)$, (30) is alternatively expressed as,

$$Q_k^- = \sum_{j=k}^{c} P_j^- \omega(j-k|j) + \sum_{j=c+1}^{k+c} \sum_{h=1}^{r} F_h z_h^j \omega(j-k|j), (k \geq 0) \tag{32}$$

We also consider the normalizing condition

$$\sum_{j=0}^{c} P_j^- + \sum_{j=c+1}^{\infty} \sum_{h=1}^{r} F_h z_h^j = 1 \tag{33}$$

We let $k = 1,2, \dots, c + r$ in (32) and with (33) it leads to $c + r + 1$ equations that are required to solve for the $\left(P_j^-, 0 \le j \le c\right)$ and $F_h, (1 \le h \le r)$. By solving these equations the $\left(P_j^-, j \ge 0\right)$ are completely found as

$$P_j^- = \begin{cases} \text{determined above}, (0 \le j \le c) \\ \sum_{h=1}^{r} F_h z_h^j, \qquad\qquad (j \ge c+1) \end{cases} \tag{34}$$

4.2.6 The $GI^X/Geo/c$ queues with LAS-DA at other time epochs

In addition, let the $\left(P_j, j \ge 0\right)$ and $\left(P_j^o, j \ge 0\right)$ be the steady-state p.m.f.'s of the number of customers in $GI^X/Geo/c$ queues with LAS-DA at random and outside observer's time epochs, respectively. In LAS-DA, an outside observer's time epoch falls in a time interval that begins just after a departure and immediately before a batch-arrival. Based on this notion $P_j^o = P_j$ for $j \ge 0$. In addition, we have

$$Q_j^o = P_j^o = P_j, (j \ge 0)$$

where the $\left(Q_j^o, j \ge 0\right)$ are available in (29).

4.2.7 P.m.f. of the waiting-time-in-queue and performance measure

Let T_q be the amount of time the random customer within an incoming batch spends in queue upon batch-arrival and until entering service. As mentioned by Chaudhry and Templeton [17] if the position of the random customer within an incoming batch is F, then its p.m.f. is $P(F = f) \equiv r_f = \sum_{h=f}^{\infty} \frac{b_h}{\bar{b}}$, $(1 \le f \le r)$. Since $GI^X/Geo/c$ queues are infinite-buffer queues, the waiting time distribution remains the same in both LAS-DA and EAS (see Hunter [27] and Chaudhry and Gupta [11]). Let the p.m.f. of T_q be $w_k = P\left(T_q = k\right), (k \ge 0)$. As stated by Chaudhry et al. [12], the $(w_k, k \ge 0)$ are defined as

$$w_k = \begin{cases} \sum_{i=1}^{c} \hat{Q}_i, & (k=0) \\ \sum_{d=1}^{ck} \sum_{j=\max(0,d-c)}^{d-1} \binom{c(k-1)}{j} \mu^j (1-\mu)^{c(k-1)-j} \sum_{l=d-j}^{c} \binom{c}{l} \mu^l (1-\mu)^{c-l} \hat{Q}_{c+d}, & (k \geq 1) \end{cases}$$

where $\hat{Q}_i = \sum_{l=0}^{i-1} Q_l^- \, r_{i-l}, (i \geq 1)$. As a remark, letting $r_1 = 1$ in the definition of w_k leads to the p.m.f. of the waiting-time-in-queue of the first customer within an incoming batch.

The model $GI^X/Geo/c$ has various performance measures. Since the distributions are known, various moments can be calculated. In particular, denote L_s^o as the mean number of customers in the system and L_q^o as the mean number of customers in queue, both at an outside observer's time epoch. The L_s^o and L_q^o can be found in terms of the $(Q_j^o, j \geq 0)$ such that

$$L_s^o = \sum_{n=1}^{\infty} n Q_n^o$$

$$L_q^o = \sum_{n=c+1}^{\infty} (n-c) Q_n^o$$

The mean waiting-time in the system (say W_s) and the mean waiting-time-in-queue (say W_q) can be found using Little's law:

$$W_s = \frac{\bar{a} L_s^o}{\bar{b}} \tag{35}$$

$$W_q = \frac{\bar{a} L_q^o}{\bar{b}} \tag{36}$$

In addition, W_q can be found independently from $W_q = \sum_{k=1}^{\infty} k w_k$. The average number of idle servers in $GI^X/Geo/c$ queues is defined as

$$\sum_{j=0}^{c-1}(c-j)Q_j^o = c(1-\rho) \tag{37}$$

The left-hand side of (37) determines the average number of idle servers in terms of the queue-length distribution at an outside observer's time epoch. The right-hand side of (37) determines the same number, except that it is independent of the $\left(Q_j^o, j \geq 0\right)$ (hence independent of roots). In the next chapter we use (37) to verify the accuracy of our numerical results, thus demonstrating the robustness of the roots and the roots method (see point 4 in Appendix C.3.2 for more details).

4.2.8 The $GI^X/Geo/c$ queues involving heavy-tailed inter-batch-arrival times

Heavy-tailed distributions distinguish themselves from light-tailed distributions by having a significantly slower rate of decay. In $GI^X/Geo/c$ queues, a slow decaying arrival time distribution renders a very lengthy (or an infinite) mean inter-batch-arrival time that equates to a very small (or a zero) arrival rate (λ). The λ is directly proportional to ρ and as $\rho \rightarrow 0$ the roots of the characteristic equation of $GI^X/Geo/c$ queues converge toward the origin (see later in Table 23) which was a concern to some researchers.

Heavy-tailed distributions are believed to impose another challenge on the roots method due to its non-closed form or non-existent p.g.f.'s. While light-tailed distributions have a closed form of $A(z)$ (see e.g., the first two rows in Table 11), this is not the case for Weibull, standard log-normal (SLN), and Pareto distributions. Moreover, directly solving (22) with an a_m that follows the Pareto distribution with $M \leq 1$ will consume a very lengthy computing time (or not compute at all) due to the infinite series of a p.m.f. that decays at an extremely slow rate.

Table 11: Discrete probability distributions

GI	$a_m, (m \geq 1)$	$A(z), (\lvert z \rvert \leq 1)$	Parameter(s)	Mean
$Geometric[\lambda]$	$\lambda(1-\lambda)^{m-1}$	$\frac{\lambda z}{1-(1-\lambda)z}$	$\lambda > 0$	${}^{1}/_{\lambda}$
$Poisson[\lambda]$	$\frac{\lambda^{m-1}}{(m-1)!}e^{-\lambda}$	$ze^{-\lambda(1-z)}$	$\lambda > 0$	$1+\lambda$
$Weibull[\beta]$	$\frac{\beta^{-\sqrt{m}}}{\beta' - 1}$	$\frac{1}{\beta'-1}\sum_{m=1}^{\infty}\beta^{-\sqrt{m}}z$	$\beta > 0, \beta' = \sum_{j=0}^{\infty}\beta^{-\sqrt{j}}$	$\sum_{m=1}^{\infty}\frac{m\beta^{-\sqrt{m}}}{\beta'-1}$
SLN	$\varphi e^{-\frac{ln^2(m)}{2}}$	$\varphi\sum_{m=1}^{\infty}e^{-\frac{ln^2(m)}{2}}z^m$	$\varphi = \sum_{j=1}^{\infty}e^{-\frac{ln^2(j)}{2}}$	$\varphi\sum_{m=1}^{\infty}me^{-\frac{ln^2(m}{2}}$
$Pareto[M]$	$\frac{\delta}{m^{M+1}}$	Does not exist	$M > 0$, $\delta = \varphi(M+1)^{-1}$	$\begin{cases}\infty, & (M \leq k \\ {}^{\varphi(M)}/_{\varphi(M+1)}, & (M > k\end{cases}$

As a remark, in $Pareto[M]$ the function $\varphi(\cdot)$ is the Riemann zeta function. The k-th moment of $Pareto[M]$ exists as $E[T^k] = \varphi(M-k+1)/\varphi(M+1)$ if $1 \leq k < M$ is true. On the contrary, the $E[T^k] \to \infty$ if $k \geq M$.

Solving $GI^X/Geo/c$ queues involving heavy-tailed inter-batch-arrival times requires a different approach than the one traditionally used to deal with $GI^X/Geo/c$ queues involving light-tailed inter-batch-arrival times. To accomplish this, we extend the technique by Chaudhry and Kim [14] who solve $GI^X/Geo/1$ queues involving heavy-tailed inter-batch-arrival times using the roots method. In doing so, we replace the $K(z)$ in (22) with $K_\Psi(z)$ where

$$K_\Psi(z) = \sum_{m=1}^{\Psi} a_m\,(\mu z + 1 - \mu)^{cm}, (1 \leq \Psi < \infty)$$

where Ψ is a non-zero integer. Modern computing software such as MAPLE can easily determine the roots of (22) when using the above expression of $K(z)$. However, one must choose an adequately sized Ψ given that the value of Ψ is indirectly proportional to the rate of decay of a_m (larger Ψ is required for the a_m with a slower decay). To offset this balance, we have implemented a simple algorithm in MAPLE that determines Ψ (see Appendix C.2.2). This algorithm was used to compute several results in $GI^X/Geo/c$ queues involving heavy-tailed inter-batch-arrival times (see later in Section 5.2).

4.3 The $GI^X/Geo/c/N$ queues

In this section, we analytically solve $GI^X/Geo/c/N$ queues using roots. Some notations from Section 2.3 are redefined in the context of $GI^X/Geo/c/N$ queues.

4.3.1 Model description

Consider the steady-state aspect of the model $GI^X/Geo/c/N$ in which customers arrive in batches of size X with a maximum size $r, (0 < r < \infty)$. The p.m.f. of X is $(b_h, 1 \le h \le r)$ with mean $\bar{b} = \sum_{h=1}^{r} hb_h$ and p.g.f. $B(z) = \sum_{h=1}^{r} b_h z^h, (|z| \le 1)$. The n-th inter-batch-arrival time, say $T_n, (n \ge 1)$ is a discrete-time period that is measured from the moment just before the n-th batch-arrival (say at time t) to the moment just before the $(n+1)$-th batch-arrival (say at time $t+m$). Inter-batch-arrival times are i.i.d.r.v.'s distributed as T which is divided into m time slots. The T has a p.m.f. $a_m = P(T = m), (a_0 = 0, m \ge 1)$, mean $\bar{a} = \sum_{m=1}^{\infty} ma_m = {}^{1}/_{\lambda}$, and p.g.f. $A(z) = \sum_{m=1}^{\infty} a_m z^m, (|z| \le 1)$. There are c parallel servers in the model where each has a service time Y that is independent to one another and geometrically distributed as

$$P(Y = y) = (1-\mu)^{y-1}\mu, (0 < \mu < 1, y \ge 1)$$

The mean service time of a server is $E[Y] = {}^{1}/_{\mu}$ and traffic intensity of the model is $\rho = {}^{\bar{b}}/_{\bar{a}c\mu} > 0$. The batch sizes, inter-batch-arrival times, and services times are independent of one another. In addition, let $\omega(j|i)$ be the probability of an event that j customers are served out of i customers within a single time slot. The conditional probability $\omega(j|i)$ then follows a binomial distribution

$$\omega(j|i) = \begin{cases} \binom{i}{j}\mu^j(1-\mu)^{i-j}, (0 \le j \le i < c \le N) \\ \binom{c}{j}\mu^j(1-\mu)^{c-j}, (0 \le j \le c \le i \le N) \end{cases} \tag{38}$$

with $\omega(0|0) = 1$ and $\binom{a}{b} = 0$ for $a < b$ or $a < 0$. Since the $GI^X/Geo/c/N$ is a discrete-time queueing model, it has two different but related aspects (EAS and LAS-DA). Since the model is solved under the steady-state condition, the queue-length distribution of LAS-IA is identical to that of the EAS. Lastly, the model $GI^X/Geo/c/N$ has the finite-buffer $N, (N \ge c)$ such that an incoming batch is either partially or totally rejected if a batch size $h, (1 \le h \le r)$ is larger than the available space $(N - c)$.

4.3.2 The $GI^X/Geo/c/N$ queues with EAS at a pre-arrival time epoch

Let M_n^- be the number of customers in the system, including those, if any, receiving service just before the n-th batch-arrival (i.e. at the n-th pre-arrival time epoch). The stochastic process $\{M_n^-, n \ge 1\}$ forms a homogenous Markov chain:

$$M_{n+1}^- = \begin{cases} \min(M_n^- + X_n - D_n, N), & (M_n^- + X_n - D_n \ge 0) \\ 0, & (M_n^- + X_n - D_n < 0) \end{cases}$$

where D_n is the total number of customers that depart the system during T_n, X_n is the size of the n-th batch-arrival and N is the finite-buffer. In considering the steady-state aspect of $GI^X/Geo/c/N$ queues with EAS, M_n^- becomes M^- as $n \to \infty$ and has the steady-state p.m.f. $Q_j^- = \lim_{n\to\infty} P(M_n^- = j), (j \ge 0)$. Let $P_{i,j}(n) = P[M_{n+1}^- = j \mid M_n^- = i], (i, j \ge 0, n \ge 1)$ be the one-step transition probabilities of $\{M_n^-, n \ge 1\}$. Thus, the steady-state one-step transition probabilities of $GI^X/Geo/c/N$ queues are defined as $P_{i,j} \equiv \lim_{n\to\infty} P_{i,j}(n)$. Given that a batch can either be partially or totally rejected, there are two different set of transition probabilities that correspond to each rejection policy: In the case of partial rejection, the $P_{i,j}$ are defined as

$$P_{i,j} = \begin{cases} \sum_{h=j-i}^{N-i} k_{i+h-j} b_h + k_{N-j} \sum_{h=N-i+1}^{r} b_h, & (j > i \geq 0, j \geq c) \\ \sum_{h=1}^{N-i} k_{i+h-j} b_h + k_{N-j} \sum_{h=N-i+1}^{r} b_h, & (c \leq j \leq i) \\ \sum_{h=\max(1,j-i)}^{N-i} V_{i+h,j} b_h + V_{N,j} \sum_{h=N-i+1}^{r} b_h, & (i \geq 0, 1 \leq j \leq c-1) \end{cases}$$

In the case of total rejection the $P_{i,j}$ are

$$P_{i,j} = \begin{cases} \sum_{h=j-i}^{N-i} k_{i+h-j} b_h, & (j > i \geq 0, j \geq c) \\ \sum_{h=1}^{N-i} k_{i+h-j} b_h + k_{i-j} \sum_{h=N-i+1}^{r} b_h, & (c \leq j \leq i) \\ \sum_{h=\max(1,j-i)}^{N-i} V_{i+h,j} b_h + V_{i,j} \sum_{h=N-i+1}^{r} b_h, & (i \geq 0, 1 \leq j \leq c-1) \end{cases}$$

where $P_{i,0} = 1 - \sum_{j=1}^{N} P_{i,j}, (0 \leq i \leq N)$ and $k_n = \sum_{m=1}^{\infty} a_m \binom{cm}{i+h-j} (1 - \mu)^{cm-(i+h-j)} (\mu)^{i+h-j}, (n \geq 0, k_n = 0 \text{ for } n < 0)$. The $V_{i+h,j}$ are defined as

$$V_{i+h,j} = \begin{cases} 0, & (i+h < j) \\ \sum_{m=1}^{\infty} a_m \binom{i+h}{j} (1-\mu)^{mj} (1-(1-\mu)^m)^{i+h-j}, & (1 \leq j \leq i+h \leq c) \\ \sum_{m=1}^{\infty} \left[\sum_{w=1}^{m} \sum_{s=j}^{c} \sum_{l=c-s+1}^{c} \binom{c(w-1)}{i+h-l-s} \mu^{i+h-l-s} (1-\mu)^{c(w-1)-(i+h-l-s)} \times \right. & \\ \left. \binom{c}{l} \mu^l (1-\mu)^{c-l} \binom{S}{j} (1-\mu)^{(m-w)j} (1-(1-\mu)^{m-w})^{s-j} \right] a_m, & (1 \leq j < c < i+h) \end{cases}$$

As a remark, the $P_{i,j}$ presented above match with those by Goswami and Samanta [22].

The global balance equation of $GI^X/Geo/c/N$ queues with EAS is defined as

$$Q_j^- = \sum_{i=0}^{N} Q_i^- P_{i,j}, (0 \leq j \leq N) \tag{39}$$

which is a set of j first order linear difference equations. As a remark, $N = 0$ indicates that no customers are allowed in the system (i.e. $Q_0^- = 1$) hence this case can be ignored. Whether the $P_{i,j}$ follow partial or total rejection policy, both cases can be solved altogether by assuming the solution of a general form $Q_j^- = Cz^j, (1 \leq j \leq N, C \neq 0)$. By substituting the general solution into (39), we have

$$Cz^j = \sum_{i=0}^{N} Cz^i P_{i,j}, (1 \leq j \leq N)$$

$$0 = \sum_{i=0}^{N} z^i P_{i,j} - z^j$$

By summing both sides of the above over $1 \leq j \leq N$ we have the characteristic equation of $GI^X/Geo/c/N$ queues with EAS:

$$0 = \sum_{j=1}^{N} \left(\sum_{i=0}^{N} z^i P_{i,j} - z^j \right)$$

Since the above is an N-th degree polynomial, solving it gives N roots. Let these roots be $z_1, z_2, \dots, z_N$ such that the solution becomes

$$Q_j^- = \sum_{h=1}^{N} C_h z_h^j, (1 \leq j \leq N) \tag{40}$$

where the $C_h, (1 \leq h \leq N)$ are the unknown constant coefficients. To compute these unknowns, we substitute (40) into (39) such that it leads to

$$\sum_{h=1}^{N} C_h z_h^j = \sum_{i=0}^{N} \sum_{h=1}^{N} C_h z_h^i \, P_{i,j}, (1 \le j \le N)$$

The above expression can be rearranged to

$$0 = \sum_{h=1}^{N} C_h \left(\sum_{i=j-h}^{N} z_h^i \, P_{i,j} - z_h^j \right), (1 \le j \le N) \tag{41}$$

To make (40) also true for the case when $j = 0$, we establish the normalizing condition as

$$1 = \sum_{j=0}^{N} \sum_{h=1}^{N} C_h z_h^j$$

The normalizing condition, in conjunction with letting $j = 1,2, \dots, N - 1$ in (41), gives N equations. Solving these equations gives the solution to $GI^X/Geo/c/N$ queues with EAS in terms of roots as

$$Q_j^- = \sum_{h=1}^{N} C_h z_h^j, (0 \le j \le N) \tag{42}$$

4.3.3 The $GI^X/Geo/c/N$ queues with EAS at a random time epoch

The queue-length distributions of $GI^X/Geo/c/N$ queues with EAS at a random time epoch (say $Q_l, 0 \le l \le N$) and (42) can be related in the same manner as we have done in $GI^X/Geo/c$ queues with EAS (see Subsection 4.2.3).

4.3.4 The $GI^X/Geo/c/N$ queues with EAS at an outside observer's time epoch

In relating the queue-length distributions of $GI^X/Geo/c/N$ queues with EAS at random and outside observer's time epochs we adopt the relation (26) from Subsection 4.2.4 and then modify it to comply with the finite-buffer N: Unlike in $GI^X/Geo/c$ queues with EAS where Q_j^o has no upper-bound (i.e. $j \ge 0$), in $GI^X/Geo/c/N$ queues with EAS the Q_j^o has an upper-bound (i.e. $0 \leq j \leq N$) due to the finite-buffer N. Based on this notion, (26) is modified to

$$Q_l = \sum_{j=l}^{\min(l+c,N)} Q_j^o \omega(j-l|j), (0 \leq l \leq N) \tag{43}$$

Since (43) is a set of l first order linear difference equations, we assume that the $\left(Q_j^o, 1 \leq j \leq N\right)$ is a geometric sum that consists of the same roots as those in (40) but with different constant coefficients (hence they are unknown). Let these non-zero unknown constant terms be $E_h, (1 \leq h \leq N)$, then we have

$$Q_j^o = \sum_{h=1}^{N} E_h z_h^j, (1 \leq j \leq N)$$

By substituting the above geometric sum in (43) we have

$$Q_l = \sum_{j=l}^{\min(l+c,N)} \sum_{h=1}^{N} E_h z_h^j \omega(j-l|j), (1 \leq l \leq N) \tag{44}$$

and the normalizing condition

$$Q_0^o + \sum_{j=1}^{N} \sum_{h=1}^{N} E_h z_h^j = 1 \tag{45}$$

By letting $l = 1,2, \ldots, N$ in (44) we have N equations that can be solved with (45) such that the $\left(Q_j^o, 0 \leq j \leq N\right)$ are completely found as

$$Q_j^o = \begin{cases} \text{determined above}, (j = 0) \\ \sum_{h=1}^{r} E_h z_h^j, \qquad (1 \leq j \leq N) \end{cases} \tag{46}$$

4.3.5 The $GI^X/Geo/c/N$ queues with LAS-DA at a pre-arrival time epoch

In developing a relation between the pre-arrival queue-length distributions of $GI^X/Geo/c/N$ queues with EAS and LAS-DA we adopt (30) from Subsection 4.2.5 and then modify it to comply with the finite-buffer N. Doing so leads to

$$Q_k^- = \sum_{j=k}^{\min(n+c,N)} P_j^- \omega(j-k|j)\,, (0 \le k \le N) \tag{47}$$

Since (47) is a set of l first order linear difference equations, we assume that the $\left(P_j^-, 1 \le j \le N\right)$ is a geometric sum that consists of the same roots as those in (40) but with different constant coefficients (hence they are unknown). Let these unknown constant terms be $F_h, (1 \le h \le N)$, then we have

$$P_j^- = \sum_{h=1}^{N} F_h z_h^j\,, (1 \le j \le N)$$

By substituting the above geometric sum in (47) we have

$$Q_k^- = \sum_{j=k}^{\min(n+c,N)} \sum_{h=1}^{N} F_h z_h^j \,\omega(j-k|j)\,, (0 \le k \le N) \tag{48}$$

and the normalizing condition

$$P_0^- + \sum_{j=1}^{N} \sum_{h=1}^{r} F_h z_h^j = 1 \tag{49}$$

By letting $l = 1,2, \ldots, N$ in (48) we have N equations that can be solved with (49) such that the $\left(P_j^-, 0 \le j \le N\right)$ are completely found as

$$P_j^- = \begin{cases} \text{determined above}, (j = 0) \\ \displaystyle\sum_{h=1}^{r} F_h z_h^j, \qquad (1 \le j \le N) \end{cases} \tag{50}$$

4.3.6 The $GI^X/Geo/c/N$ queues with LAS-DA at other time epochs

The relation $Q_j^o = P_j^o = P_j, (j \geq 0)$ in $GI^X/Geo/c$ queues from Subsection 4.2.6 also holds in $GI^X/Geo/c/N$ queues for $0 \leq j \leq N$.

4.3.7 The $GI^X/Geo/c/N$ queues involving heavy-tailed inter-batch-arrival times

The analytical results in Section 4.3 remain robust even if the inter-batch-arrival times follow heavy-tailed distributions (both non-closed and non-existent form of p.g.f's). Unlike in $GI^X/Geo/c$ queues involving heavy-tailed inter-batch-arrival times, no manipulation of the characteristic equation is needed when solving $GI^X/Geo/c/N$ queues involving heavy-tailed inter-batch-arrival times. Numerical examples are provided in Section 5.4.

4.4 Conclusions

In Section 4.1 we introduced some earlier work done by others in discrete-time multi-server bulk-arrival queues.

In Section 4.2 we applied the roots method to solve $GI^X/Geo/c$ queues. By interpreting the global balance equation as a set of linear difference equations we can express the solution in terms of roots. We also treated $GI^X/Geo/c$ queues involving heavy-tailed inter-batch-arrival times.

In Section 4.3 we applied the roots method to solve $GI^X/Geo/c/N$ queues. While this embarks on the first application of the roots method to treat discrete-time finite-buffer multi-server bulk-arrival queues, the method remains robust even in the case of heavy-tailed inter-batch-arrival times.

5 NUMERICAL RESULTS IN $GI^X/Geo/c$ AND $GI^X/Geo/c/N$ QUEUES

This chapter provides all the numerical results that complement the analytical work provided in Chapter 4. It is organized in the following manner: The $GI^X/Geo/c$ queues involving light-tailed inter-batch-arrival times in Section 5.1 and $GI^X/Geo/c$ queues involving heavy-tailed inter-batch-arrival times in Section 5.2. In addition, $GI^X/Geo/c/N$ queues involving light-tailed inter-batch-arrival times in Section 5.3 and $GI^X/Geo/c/N$ queues involving heavy-tailed inter-batch-arrival times in Section 5.4. All computations were performed on MAPLE, calibrated at the fifteenth decimal place. In presenting our numerical results, all results were rounded to four decimal places except in the case when the first four decimal places are all zero.

5.1 The $GI^X/Geo/c$ queues involving light-tailed inter-batch-arrival times

In computing the queue-length distributions of $GI^X/Geo/c$ queues at a pre-arrival, random, and outside observer's time epochs, we consider the inter-batch-arrival patterns to be geometric and deterministic.

5.1.1 Geometric inter-batch-arrival times

The inter-batch-arrival pattern is geometric (Geo) with $a_m = \lambda(1-\lambda)^{m-1}$, $(0 < \lambda < 1, m \geq 1)$ and $\lambda = 0.25$. The parameters taken are $b_1 = 0.6, b_5 = 0.4, \mu = 0.3,$ and $c = 4$. This gives $\bar{b} = 2.6, \bar{a} = 4,$ and $\rho = 0.5417$.

Table 12: Various distributions in $Geo^X/Geo/4$ queue

j	Q_j^-	Q_j	Q_j^o	P_j^-	P_j	P_j^o	m	w_k
0	0.3002	0.3002	0.2251	0.2251	0.2251	0.2251	0	0.3998
1	0.2142	0.2142	0.2057	0.2057	0.2057	0.2057	1	0.1425
2	0.1162	0.1162	0.1193	0.1193	0.1193	0.1193	2	0.0984
3	0.0798	0.0798	0.0773	0.0773	0.0773	0.0773	3	0.0699
4	0.0690	0.0690	0.0638	0.0638	0.0638	0.0638	4	0.0510
5	0.0579	0.0579	0.0838	0.0838	0.0838	0.0838	5	0.0376
6	0.0405	0.0405	0.0605	0.0605	0.0605	0.0605	6	0.0277
7	0.0293	0.0293	0.0396	0.0396	0.0396	0.0396	7	0.0204
8	0.0226	0.0226	0.0293	0.0293	0.0293	0.0293	8	0.0150
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	Sum	0.9999

$\sum_{j=0}^{c-1} Q_j^o\,(c-n) = 1.8333$	$c(1-\rho) = 1.8333$

As expected, $Q_j^- = Q_j$ and $Q_j^o = P_j^- = P_j = P_j^o$ for $j \geq 0$ due to the G.A.S.T.A. property.

5.1.2 Deterministic inter-batch-arrival times

The inter-batch-arrival pattern is deterministic (D) with $a_4 = 1$. The parameters taken are $b_1 = 0.6, b_5 = 0.4, \mu = 0.35,$ and $c = 5$. This gives $\bar{b} = 2.6, \bar{a} = 4,$ and $\rho = 0.3714$.

Table 13: Various distributions in $D^X/Geo/5$ queue

j	Q_j^-	Q_j	Q_j^o	P_j^-	P_j	P_j^o	m	w_k
0	0.5684	0.3582	0.2162	0.4316	0.2162	0.2162	0	0.9559
1	0.2994	0.3236	0.3340	0.3308	0.3340	0.3340	1	0.0719
2	0.0973	0.1434	0.1640	0.1470	0.1640	0.1640	2	0.0171
3	0.0272	0.0820	0.0898	0.0628	0.0898	0.0898	3	0.0039
4	0.0059	0.0520	0.0546	0.0202	0.0546	0.0546	4	0.0009
5	0.0013	0.0272	0.0846	0.0055	0.0846	0.0846	5	0.0002
6	0.0003	0.0099	0.0399	0.0016	0.0399	0.0399	6	$3.9399\text{x}10^{-5}$
7	0.0001	0.0028	0.0126	0.0004	0.0126	0.0126	7	$8.4349\text{x}10^{-6}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	Sum	0.9999

$\sum_{j=0}^{c-1} Q_j^o \,(c-n) = 3.1429$	$c(1-\rho)$ $= 3.1429$

5.2 The $GI^X/Geo/c$ queues involving heavy-tailed inter-batch-arrival times

In computing the queue-length distributions of $GI^X/Geo/c$ queues at a pre-arrival, random, and outside observer's time epochs, we consider the inter-batch-arrival patterns to be Weibull and standard log-normal.

5.2.1 Weibull inter-batch-arrival times

The inter-batch-arrival pattern is Weibull ($Weibull[M]$) with $a_m = M^{-\sqrt{m}}/(V-1)$, $\left(m \geq 1, V = \sum_{j=0}^{\infty} M^{-\sqrt{j}} > 1\right)$. The parameters taken are $M = 2, b_1 = 0.425, b_2 = 0.325, b_3 = 0.075, b_4 = 0.05, b_5 = 0.125, \mu = 0.11,$ and $c = 5$. This gives $\bar{b} = 2.125$, $\bar{a} = 13.7054$, and $\rho = 0.2819$.

Table 14: Various distributions in $Weibull^X[2]/Geo/5$ queue

j	Q_j^-	Q_j	Q_j^o	P_j^-	P_j	P_j^o	m	w_k
0	0.3300	0.4172	0.4338	0.3041	0.4338	0.4338	0	0.1996
1	0.2279	0.2097	0.2033	0.2161	0.2033	0.2033	1	0.0382
2	0.1606	0.1291	0.1323	0.1623	0.1323	0.1323	2	0.0314
3	0.1002	0.0756	0.0805	0.1053	0.0805	0.0805	3	0.0258
4	0.0635	0.0458	0.0505	0.0690	0.0505	0.0505	4	0.0212
5	0.0411	0.0286	0.0347	0.0497	0.0347	0.0347	5	0.0174
6	0.0270	0.0188	0.0228	0.0327	0.0228	0.0228	6	0.0143
7	0.0176	0.0122	0.0149	0.0215	0.0149	0.0149	7	0.0117
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	Sum	0.9999

$\sum_{j=0}^{c-1} Q_j^o \,(c-n) = 3.5905$	$c(1-\rho)$ $= 3.5905$

5.2.2 Standard log-normal inter-batch-arrival times

The inter-batch-arrival pattern is standard log-normal (SLN) with $a_m = Ve^{\frac{-[\ln(m)]^2}{2}}, \left(m \geq 1, V = 1/\sum_{j=1}^{\infty} e^{\frac{-[\ln(j)]^2}{2}}\right)$. The parameters taken are $b_1 = 0.425, b_2 = 0.325, b_3 = 0.075, b_4 = 0.05, b_5 = 0.125, \mu = 0.2,$ and $c = 5$. This gives $\bar{b} = 2.125, \bar{a} = 4.6519,$ and $\rho = 0.4568$.

Table 15: Various distributions in $SLN^X/Geo/5$ queue

j	Q_j^-	Q_j	Q_j^o	P_j^-	P_j	P_j^o	m	w_k
0	0.2080	0.2765	0.2339	0.1630	0.2339	0.2339	0	0.4175
1	0.2188	0.2141	0.1860	0.1844	0.1860	0.1860	1	0.0805
2	0.1811	0.1646	0.1602	0.1737	0.1602	0.1602	2	0.0591
3	0.1253	0.1107	0.1190	0.1335	0.1190	0.1190	3	0.0432
4	0.0819	0.0715	0.0837	0.0955	0.0837	0.0837	4	0.0315
5	0.0549	0.0477	0.0630	0.0725	0.0630	0.0630	5	0.0229
6	0.0396	0.0344	0.0464	0.0535	0.0464	0.0464	6	0.0167
7	0.0278	0.0241	0.0331	0.0382	0.0331	0.0331	7	0.0122
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	Sum	0.9999

$\sum_{j=0}^{c-1} Q_j^o\,(c-n) = 2.7160$	$c(1-\rho) = 2.7160$

5.3 The $GI^X/Geo/c/N$ queues involving light-tailed inter-batch-arrival times

In computing the queue-length distributions of the $GI^X/Geo/c/N$ queues at a pre-arrival, random, and outside observer's time epochs, we consider the inter-batch-arrival patterns to be geometric and deterministic. We present three different cases of ρ in each table where $\rho < 1, \rho = 1,$ and $\rho > 1$.

5.3.1 Geometric inter-batch-arrival times

The inter-batch-arrival pattern is geometric (Geo) with $a_m = \lambda(1-\lambda)^{m-1}$, $(0 < \lambda < 1, m \geq 1)$. The parameters taken are $b_1 = 0.25, b_2 = 0.25, b_3 = 0.25$, $b_4 = 0.25, \bar{a} = 5, c = 3, N = 5, \rho = 0.5, 1,$ and 2 . This gives $\bar{b} = 2.5$, $\mu = 0.3333, 0.1667,$ and 0.0833.

Table 16: Various distributions in $Geo^X/Geo/3/5$ queue

	Q_j^- (partial rejection)			Q_j^- (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 2$	$\rho = 0.5$	$\rho = 1$	$\rho = 2$
0	0.4518	0.1621	0.0303	0.4771	0.2013	0.0536
1	0.2338	0.1932	0.0768	0.2463	0.2395	0.1355
2	0.1398	0.1832	0.1248	0.1424	0.2199	0.2134
3	0.0924	0.1682	0.1738	0.0814	0.1664	0.2324
4	0.0606	0.1728	0.2707	0.0420	0.1216	0.2293
5	0.0217	0.1205	0.3236	0.0108	0.0513	0.1359
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	Q_j (partial rejection)			Q_j (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 2$	$\rho = 0.5$	$\rho = 1$	$\rho = 2$
0	0.4518	0.1621	0.0303	0.4771	0.2013	0.0536
1	0.2338	0.1932	0.0768	0.2463	0.2395	0.1355
2	0.1398	0.1832	0.1248	0.1424	0.2199	0.2134
3	0.0924	0.1682	0.1738	0.0814	0.1664	0.2324
4	0.0606	0.1728	0.2707	0.0420	0.1216	0.2293
5	0.0217	0.1205	0.3236	0.0108	0.0513	0.1359
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	Q_j^o (partial rejection)			Q_j^o (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 2$	$\rho = 0.5$	$\rho = 1$	$\rho = 2$
0	0.3615	0.1297	0.0243	0.3816	0.1611	0.0429
1	0.2096	0.1627	0.0629	0.2209	0.2017	0.1111
2	0.1461	0.1643	0.1052	0.1572	0.2089	0.1908
3	0.1152	0.1615	0.1506	0.1166	0.1828	0.2293
4	0.0943	0.1735	0.2368	0.0873	0.1568	0.2496
5	0.0734	0.2083	0.4201	0.0364	0.0887	0.1764
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	P_j^- (partial rejection)			P_j^- (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 2$	$\rho = 0.5$	$\rho = 1$	$\rho = 2$
0	0.3615	0.1297	0.0243	0.3816	0.1611	0.0429
1	0.2096	0.1627	0.0629	0.2209	0.2017	0.1111
2	0.1461	0.1643	0.1052	0.1572	0.2089	0.1908
3	0.1152	0.1615	0.1506	0.1166	0.1828	0.2293
4	0.0943	0.1735	0.2368	0.0873	0.1568	0.2496
5	0.0734	0.2083	0.4201	0.0364	0.0887	0.1764
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	P_j (partial rejection)			P_j (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 2$	$\rho = 0.5$	$\rho = 1$	$\rho = 2$
0	0.3615	0.1297	0.0243	0.3816	0.1611	0.0429
1	0.2096	0.1627	0.0629	0.2209	0.2017	0.1111
2	0.1461	0.1643	0.1052	0.1572	0.2089	0.1908
3	0.1152	0.1615	0.1506	0.1166	0.1828	0.2293
4	0.0943	0.1735	0.2368	0.0873	0.1568	0.2496
5	0.0734	0.2083	0.4201	0.0364	0.0887	0.1764
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	P_j^o (partial rejection)			P_j^o (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 2$	$\rho = 0.5$	$\rho = 1$	$\rho = 2$
0	0.3615	0.1297	0.0243	0.3816	0.1611	0.0429
1	0.2096	0.1627	0.0629	0.2209	0.2017	0.1111
2	0.1461	0.1643	0.1052	0.1572	0.2089	0.1908
3	0.1152	0.1615	0.1506	0.1166	0.1828	0.2293
4	0.0943	0.1735	0.2368	0.0873	0.1568	0.2496
5	0.0734	0.2083	0.4201	0.0364	0.0887	0.1764
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

As expected, $Q_j^- = Q_j$ and $Q_j^o = P_j^- = P_j = P_j^o$ for $0 \leq j \leq N$ due to the G.A.S.T.A. property.

5.3.2 Deterministic inter-batch-arrival times

The inter-batch-arrival pattern is deterministic (D) with $a_7 = 0.5, a_{10} = 0.2,$ and $a_{15} = 0.3$. The parameters taken are $b_1 = 0.2, b_5 = 0.3, b_{10} = 0.5,\ c = 4, N = 7, \rho = 0.86, 1,$ and 2. This gives $\bar{a} = 10, \bar{b} = 6.7$, $\mu = 0.1948, 0.1675,$ and 0.0838.

Table 17: Various distributions in $D^X/Geo/4/7$ queue

	Q_j^- (partial rejection)			Q_j^- (total rejection)		
j	$\rho = 0.86$	$\rho = 1$	$\rho = 2$	$\rho = 0.86$	$\rho = 1$	$\rho = 2$
0	0.4226	0.3228	0.0477	0.7793	0.7197	0.4103
1	0.2729	0.2693	0.1160	0.1490	0.1746	0.2536
2	0.1595	0.1837	0.1616	0.0502	0.0674	0.1466
3	0.0840	0.1151	0.1785	0.0161	0.0267	0.0917
4	0.0384	0.0630	0.1747	0.0041	0.0084	0.0532
5	0.0171	0.0329	0.1678	0.0011	0.0026	0.0307
6	0.0050	0.0114	0.1139	0.0002	0.0005	0.0115
7	0.0007	0.0019	0.0397	$1.8084\text{x}10^{-5}$	0.0001	0.0025
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	Q_j (partial rejection)			Q_j (total rejection)		
j	$\rho = 0.86$	$\rho = 1$	$\rho = 2$	$\rho = 0.86$	$\rho = 1$	$\rho = 2$
0	0.2212	0.1620	0.0205	0.6392	0.5861	0.3269
1	0.2045	0.1864	0.0600	0.1759	0.1935	0.2368
2	0.1462	0.1504	0.0952	0.0710	0.0826	0.1439
3	0.1141	0.1245	0.1193	0.0471	0.0551	0.1035
4	0.0990	0.1117	0.1396	0.0363	0.0423	0.0772
5	0.0945	0.1112	0.1798	0.0244	0.0307	0.0679
6	0.0775	0.0962	0.2085	0.0051	0.0078	0.0328
7	0.0429	0.0576	0.1771	0.0011	0.0019	0.0111
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	Q_j^o (partial rejection)			Q_j^o (total rejection)		
j	$\rho = 0.86$	$\rho = 1$	$\rho = 2$	$\rho = 0.86$	$\rho = 1$	$\rho = 2$
0	0.1790	0.1297	0.0158	0.6002	0.5501	0.3064
1	0.1857	0.1659	0.0494	0.1841	0.1992	0.2323
2	0.1357	0.1374	0.0813	0.0714	0.0828	0.1417
3	0.1089	0.1167	0.1047	0.0478	0.0559	0.1046
4	0.0968	0.1077	0.1257	0.0365	0.0427	0.0780

5	0.1063	0.1189	0.1679	0.0478	0.0524	0.0807
6	0.0855	0.1038	0.2039	0.0096	0.0131	0.0408
7	0.1021	0.1199	0.2513	0.0026	0.0039	0.0157
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	P_j^- (partial rejection)			P_j^- (total rejection)		
j	$\rho = 0.86$	$\rho = 1$	$\rho = 2$	$\rho = 0.86$	$\rho = 1$	$\rho = 2$
0	0.3649	0.2752	0.0384	0.7460	0.6871	0.3882
1	0.2590	0.2493	0.0987	0.1582	0.1808	0.2499
2	0.1693	0.1834	0.1432	0.0614	0.0766	0.1476
3	0.1063	0.1316	0.1651	0.0245	0.0363	0.0970
4	0.0585	0.0849	0.1735	0.0075	0.0136	0.0608
5	0.0301	0.0511	0.1837	0.0022	0.0047	0.0378
6	0.0102	0.0204	0.1410	0.0004	0.0010	0.0151
7	0.0017	0.0040	0.0564	4.3013×10^{-5}	0.0001	0.0035
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	P_j (partial rejection)			P_j (total rejection)		
j	$\rho = 0.86$	$\rho = 1$	$\rho = 2$	$\rho = 0.86$	$\rho = 1$	$\rho = 2$
0	0.1790	0.1297	0.0158	0.6002	0.5501	0.3064
1	0.1857	0.1659	0.0494	0.1841	0.1992	0.2323
2	0.1357	0.1374	0.0813	0.0714	0.0828	0.1417
3	0.1089	0.1167	0.1047	0.0478	0.0559	0.1046
4	0.0968	0.1077	0.1257	0.0365	0.0427	0.0780
5	0.1063	0.1189	0.1679	0.0478	0.0524	0.0807
6	0.0855	0.1038	0.2039	0.0096	0.0131	0.0408
7	0.1021	0.1199	0.2513	0.0026	0.0039	0.0157
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	P_j^o (partial rejection)			P_j^o (total rejection)		
j	$\rho = 0.86$	$\rho = 1$	$\rho = 2$	$\rho = 0.86$	$\rho = 1$	$\rho = 2$
0	0.1790	0.1297	0.0158	0.6002	0.5501	0.3064
1	0.1857	0.1659	0.0494	0.1841	0.1992	0.2323
2	0.1357	0.1374	0.0813	0.0714	0.0828	0.1417
3	0.1089	0.1167	0.1047	0.0478	0.0559	0.1046
4	0.0968	0.1077	0.1257	0.0365	0.0427	0.0780
5	0.1063	0.1189	0.1679	0.0478	0.0524	0.0807
6	0.0855	0.1038	0.2039	0.0096	0.0131	0.0408
7	0.1021	0.1199	0.2513	0.0026	0.0039	0.0157

Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

5.4 The $GI^X/Geo/c/N$ queues involving heavy-tailed inter-batch-arrival times

In computing the queue-length distributions of $GI^X/Geo/c/N$ queues at a pre-arrival, random, and outside observer's time epochs, we consider the inter-batch-arrival patterns to be Weibull and standard log-normal. We present three different cases of ρ in each table where $\rho < 1, \rho = 1$, and $\rho > 1$.

5.4.1 Weibull inter-batch-arrival times

The inter-batch-arrival pattern is Weibull ($Weibull[M]$) with $a_m = M^{-\sqrt{m}}/(V-1), \left(m \geq 1, V = \sum_{j=0}^{\infty} M^{-\sqrt{j}} > 1\right)$. The parameters taken are $M = 2, N = 5, b_1 = 0.425, b_2 = 0.325, b_3 = 0.075, b_4 = 0.05, b_5 = 0.125, c = 4, \rho = 0.5, 1,$ and 3. This gives $\bar{b} = 2.125$, $\bar{a} = 13.7054$, and $\mu = 0.0775, 0.0388,$ and 0.0129.

Table 18: Various distributions in $Weibull^X[2]/Geo/4/5$ queue

	Q_j^- (partial rejection)			Q_j^- (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 3$	$\rho = 0.5$	$\rho = 1$	$\rho = 3$
0	0.2428	0.0917	0.0138	0.2714	0.1138	0.0174
1	0.2223	0.1385	0.0261	0.2587	0.1851	0.0446
2	0.1898	0.1714	0.0617	0.2081	0.2187	0.1055
3	0.1433	0.1834	0.1223	0.1383	0.2084	0.1944
4	0.1111	0.1915	0.2339	0.0820	0.1676	0.3001
5	0.0908	0.2235	0.5423	0.0416	0.1064	0.3379
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	Q_j (partial rejection)			Q_j (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 3$	$\rho = 0.5$	$\rho = 1$	$\rho = 3$
0	0.3737	0.1911	0.0910	0.4037	0.2177	0.0955
1	0.2163	0.1574	0.0310	0.2419	0.1988	0.0517
2	0.1608	0.1721	0.0736	0.1687	0.2050	0.1166
3	0.1114	0.1663	0.1377	0.1025	0.1756	0.1968

4	0.0791	0.1552	0.2313	0.0564	0.1279	0.2681
5	0.0588	0.1578	0.4353	0.0270	0.0752	0.2713
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	Q_j^o (partial rejection)			Q_j^o (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 3$	$\rho = 0.5$	$\rho = 1$	$\rho = 3$
0	0.3566	0.1851	0.0906	0.3845	0.2100	0.0949
1	0.2075	0.1502	0.0295	0.2337	0.1904	0.0494
2	0.1595	0.1660	0.0703	0.1704	0.2002	0.1121
3	0.1133	0.1620	0.1314	0.1089	0.1758	0.1905
4	0.0819	0.1520	0.2197	0.0653	0.1356	0.2674
5	0.0812	0.1848	0.4586	0.0372	0.0880	0.2858
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	P_j^- (partial rejection)			P_j^- (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 3$	$\rho = 0.5$	$\rho = 1$	$\rho = 3$
0	0.2253	0.0864	0.0134	0.2511	0.1067	0.0168
1	0.2094	0.1306	0.0248	0.2455	0.1752	0.0425
2	0.1855	0.1633	0.0586	0.2073	0.2111	0.1008
3	0.1431	0.1758	0.1154	0.1448	0.2061	0.1864
4	0.1113	0.1821	0.2164	0.0939	0.1762	0.2975
5	0.1254	0.2618	0.5713	0.0575	0.1247	0.3560
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	P_j (partial rejection)			P_j (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 3$	$\rho = 0.5$	$\rho = 1$	$\rho = 3$
0	0.3566	0.1851	0.0906	0.3845	0.2100	0.0949
1	0.2075	0.1502	0.0295	0.2337	0.1904	0.0494
2	0.1595	0.1660	0.0703	0.1704	0.2002	0.1121
3	0.1133	0.1620	0.1314	0.1089	0.1758	0.1905
4	0.0819	0.1520	0.2197	0.0653	0.1356	0.2674
5	0.0812	0.1848	0.4586	0.0372	0.0880	0.2858
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	P_j^o (partial rejection)			P_j^o (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 3$	$\rho = 0.5$	$\rho = 1$	$\rho = 3$
0	0.3566	0.1851	0.0906	0.3845	0.2100	0.0949
1	0.2075	0.1502	0.0295	0.2337	0.1904	0.0494
2	0.1595	0.1660	0.0703	0.1704	0.2002	0.1121

3	0.1133	0.1620	0.1314	0.1089	0.1758	0.1905
4	0.0819	0.1520	0.2197	0.0653	0.1356	0.2674
5	0.0812	0.1848	0.4586	0.0372	0.0880	0.2858
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

5.4.2 Standard log-Normal inter-batch-arrival times

The inter-batch-arrival pattern is standard log-normal (SLN) with $a_m = Ve^{\frac{-[\ln(m)]^2}{2}}, \left(m \geq 1, V = 1/\sum_{j=1}^{\infty} e^{\frac{-[\ln(j)]^2}{2}}\right)$. The parameters taken are $M = 2, N = 5, b_1 = 0.425, b_2 = 0.325, b_3 = 0.075, b_4 = 0.05, b_5 = 0.125, c = 4, \rho = 0.5, 1, \text{and } 3.$ This gives $\bar{b} = 2.125$, $\bar{a} = 4.6519$ and $\mu = 0.2284, 0.1142, \text{and } 0.0381.$

Table 19: Various distributions in $SLN^X/Geo/4/5$ queue

	Q_j^- (partial rejection)			Q_j^- (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 3$	$\rho = 0.5$	$\rho = 1$	$\rho = 3$
0	0.2825	0.0883	0.0064	0.3201	0.1168	0.0103
1	0.2644	0.1517	0.0223	0.3006	0.2111	0.0425
2	0.1994	0.1938	0.0578	0.2033	0.2450	0.1093
3	0.1299	0.2045	0.1285	0.1081	0.2127	0.2126
4	0.0836	0.1997	0.2682	0.0503	0.1445	0.3171
5	0.0403	0.1621	0.5168	0.0176	0.0700	0.3082
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	Q_j (partial rejection)			Q_j (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 3$	$\rho = 0.5$	$\rho = 1$	$\rho = 3$
0	0.3588	0.1407	0.0185	0.3950	0.1730	0.0246
1	0.2488	0.1687	0.0367	0.2777	0.2213	0.0596
2	0.1757	0.1901	0.0765	0.1773	0.2306	0.1264
3	0.1114	0.1871	0.1440	0.0924	0.1899	0.2169
4	0.0710	0.1746	0.2620	0.0427	0.1253	0.2968
5	0.0342	0.1388	0.4623	0.0150	0.0599	0.2757
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	Q_j^o (partial rejection)			Q_j^o (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 3$	$\rho = 0.5$	$\rho = 1$	$\rho = 3$
0	0.2981	0.1217	0.0171	0.3262	0.1479	0.0224

1	0.2178	0.1442	0.0325	0.2504	0.1923	0.0526
2	0.1767	0.1685	0.0666	0.1910	0.2146	0.1116
3	0.1248	0.1729	0.1233	0.1197	0.1946	0.1957
4	0.0861	0.1673	0.2206	0.0705	0.1533	0.2957
5	0.0965	0.2255	0.5399	0.0422	0.0973	0.3220
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	P_j^- (partial rejection)			P_j^- (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 3$	$\rho = 0.5$	$\rho = 1$	$\rho = 3$
0	0.2188	0.0716	0.0056	0.2464	0.0932	0.0088
1	0.2247	0.1247	0.0190	0.2646	0.1777	0.0362
2	0.1975	0.1671	0.0484	0.2168	0.2238	0.0939
3	0.1440	0.1848	0.1057	0.1395	0.2156	0.1879
4	0.1014	0.1886	0.2177	0.0830	0.1761	0.3134
5	0.1136	0.2633	0.6036	0.0497	0.1136	0.3600
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	P_j (partial rejection)			P_j (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 3$	$\rho = 0.5$	$\rho = 1$	$\rho = 3$
0	0.2981	0.1217	0.0171	0.3262	0.1479	0.0224
1	0.2178	0.1442	0.0325	0.2504	0.1923	0.0526
2	0.1767	0.1685	0.0666	0.1910	0.2146	0.1116
3	0.1248	0.1729	0.1233	0.1197	0.1946	0.1957
4	0.0861	0.1673	0.2206	0.0705	0.1533	0.2957
5	0.0965	0.2255	0.5399	0.0422	0.0973	0.3220
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	P_j^o (partial rejection)			P_j^o (total rejection)		
j	$\rho = 0.5$	$\rho = 1$	$\rho = 3$	$\rho = 0.5$	$\rho = 1$	$\rho = 3$
0	0.2981	0.1217	0.0171	0.3262	0.1479	0.0224
1	0.2178	0.1442	0.0325	0.2504	0.1923	0.0526
2	0.1767	0.1685	0.0666	0.1910	0.2146	0.1116
3	0.1248	0.1729	0.1233	0.1197	0.1946	0.1957
4	0.0861	0.1673	0.2206	0.0705	0.1533	0.2957
5	0.0965	0.2255	0.5399	0.0422	0.0973	0.3220
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

5.5 Conclusions

The numerical results based on Chapter 4 are presented in Chapter 5. In doing so, all different cases are considered (the $GI^X/Geo/c$ and $GI^X/Geo/c/N$ queues involving light and heavy-tailed inter-batch-arrival times).

Since the queue-length distributions are in terms of roots, the characteristic equation plays a pivotal role in treating $GI^X/Geo/c$ and $GI^X/Geo/c/N$ queues. Particularly, when the inter-batch-arrival times of $GI^X/Geo/c$ queues follow heavy-tailed distributions, the characteristic equation needs to be modified to allow the computation of roots. This aspect of $GI^X/Geo/c$ queues is isolated and studied even further in Appendix C.3.1, C.3.2, and C.3.3. On the other hand, no manipulation of the characteristic equation is needed when treating $GI^X/Geo/c/N$ queues using roots.

6 CONCLUSIONS

6.1 Thesis findings

6.1.1 Findings for $GI^X/M/c$ and $GI^X/M/c/N$ queues

- Derived the queue-length distribution at various time epochs in terms of the roots of the characteristic equation.
- Provided computational results in the $GI^X/M/c$ and $GI^X/M/c/N$ queues involving both light and heavy-tailed inter-batch-arrival times.

6.1.2 Findings for $GI^X/Geo/c$ and $GI^X/Geo/c/N$ queues

- Derived the queue-length distribution at various time epochs for both the EAS and LAS-DA in terms of the roots of the characteristic equation.
- Provided computational results in the $GI^X/Geo/c$ and $GI^X/Geo/c/N$ queues involving both light and heavy-tailed inter-batch-arrival times.

6.2 Summary

Multi-server bulk-arrival queues are advanced queues that are widely studied and applied across various fields of study. The purpose of this thesis is to introduce a unified approach, called the roots method, in treating multi-server bulk-arrival queues that involve continuous and discrete-times, infinite and finite-buffers, and light and heavy-tailed inter-batch-arrival times.

In essence, it is our intention to solve an advanced set of queueing models in the most simple and pragmatic way using the roots method. The roots method consists of two steps: Derive the characteristic equation of the model followed by expressing the solution in terms of the roots of the model's characteristic equation.

In Chapter 2 we applied the roots method to find the queue-length distributions of $GI^X/M/c$ and $GI^X/M/c/N$ queues. By interpreting each model's Chapman-Kolmogorov equation as a set of linear difference equations, the solutions can be expressed in terms of roots. In dealing with heavy-tailed inter-batch-arrival times, the characteristic equation of $GI^X/M/c$ queues is modified to comply with the properties of continuous heavy-tailed probability distributions. However in $GI^X/M/c/N$ queues, such modification is not needed. Based on the analytical findings in Chapter 2, various numerical results were computed in Chapter 3.

In Chapter 4 we again applied the roots method to find the queue-length distributions of $GI^X/Geo/c$ and $GI^X/Geo/c/N$ queues. By interpreting each model's global balance equation as a set of linear difference equations, the solutions can be expressed in terms of roots. Interestingly, in both models, the queue-length distribution at an outside observer's time epoch in the EAS and the distributions in the LAS-DA share the same roots. In dealing with heavy-tailed inter-batch-arrival times, the characteristic equation of $GI^X/Geo/c$ queues is modified to comply with the properties of discrete heavy-tailed probability distributions. However in $GI^X/Geo/c/N$ queues, such modification is not needed. Based on the analytical findings in Chapter 4, various numerical results were computed and presented in Chapter 5.

Because the characteristic equation and its roots play a pivotal role in the roots method, we analyze this aspect of the thesis even further and deduce some facts in Appendix C.3. The root finding programs and the analytical proof of the existence of roots are also provided in Appendices C.1 and C.2. Lastly, the characteristic equation of

$GI^X/Geo/c$ can be transformed into that of $GI^X/M/c$ queues as demonstrated in Appendix C.2.3.

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APPENDIX A

This appendix provides all preliminary knowledge that is required to understand this thesis. This appendix begins with a brief summary of the probability theory, stochastic processes, and Markov processes in a progressive manner. The summary then leads to the following:

Under the continuous probability theory (Appendix A.2) the definition of the p.d.f., L.T., and L-S.T. are discussed. Similarly, under the discrete probability theory (Appendix A.3) the definition of the p.m.f., g.f., and p.g.f. are discussed.

A.1 Brief summary of the probability theory, stochastic processes, and Markov processes

The probability theory can be explained by the example of a coin toss. When a coin is tossed, it could lead to two possible outcomes (heads or tails), and each outcome has a probability of 0.5. A r.v. represents a group of outcomes (in this case, heads or tails) and the distribution function (d.f.) allocates probability to each outcome (in this case, 0.5 chance of getting heads and the same for getting tails).

In general, the outcomes of a r.v. can be non-negative real numbers or non-negative integers. In the case of the former, the d.f. of a r.v. becomes a p.d.f. and in the case of the latter, the d.f. of a r.v. becomes a p.m.f.. The c.d.f. is a sum of either the p.d.f.'s or p.m.f.'s from the smallest valued outcome up to a particular outcome of interest. In a holistic sense, r.v.'s can be added, subtracted, multiplied, divided, or collected to describe a system.

A r.v. could also be time sensitive such that a probability of an outcome changes over time. Building on the previous example of a coin toss, the probability of getting tails on the first coin toss (0.5) would be different from the probability of getting five tails in a row $(0.5^5 = 0.03125)$. As explained, it is evident that the probability of an outcome in the future depends on the probabilities of all previous outcomes. In view of this, a collection of time dependent r.v.'s form the stochastic processes, which Parzen [42] describes as the "dynamic part of the probability theory." The concept of the stochastic processes is familiar and extensively applied across various fields including statistical physics (Brownian motion, fluctuations, and thermal noise), communication and control (automatic tracking of moving objects, reproduction of sound and images), and inventory control (minimizing time-of-delivery lag and deciding when to place an order for replenishment of stock).

There exists a special class of the stochastic processes called the Markovian stochastic processes (Markov processes). The Markov processes inherit the basic properties of the stochastic processes but have an additional consideration known as the Markov property: The probability of an outcome in the future only depends on the probability of the present outcome and not that of the past. For example, the Markov property states that given a car engine with a mileage of 120,000 kilometers, the probability of this engine lasting for another 50,000 kilometers is the same as the probability of the same engine lasting for 50,000 kilometers from the time it was first built. When comparing the two probabilities, the previous mileage on the engine (i.e. the past) is simply forgotten when considering additional mileage from the present to the future. The two well known d.f.'s that possess the Markov property are the exponential and geometric distributions. The Markov property is often referred to as 'the forgetfulness property' due to its tendency to ignore the past. Interestingly, the Markov processes are a powerful tool when deducing predictions from a limited amount of information. It enables a great degree of simplification in the queueing theory when considering the services times to follow the exponential or geometric distribution. The Markov processes can be further divided into four sub-processes:

Table 20: Classification of the Markov processes

	Discrete-state	Continuous-state
Discrete-time	Discrete parameter Markov chains	Discrete parameter Markov processes
Continuous-time	Continuous parameter Markov chains	Continuous parameter Markov processes

The Markov chains are Markov processes whose state space is discrete. In the continuous parameter Markov chains, the transition of a r.v. from a state to another can be probabilistically described in terms of the Chapman-Kolmogorov equation. Similarly in the discrete parameter Markov chains, the transition of a r.v. from a state to another can be probabilistically described in terms of the global balance equation.

A.2 Continuous probability theory

Assume that there is a continuous r.v., say T, such that it has a p.d.f. $f(t), (0 \leq t < \infty)$. The n-th moment of T is defined as $E[T^n] = \int_0^\infty t^n f(t)dt\,, (n \geq 1)$.

A.2.1 Laplace transform and Laplace-Stieltjes transform

As indicated in Chaudhry and Templeton [17], applying L.T. in the continuous probability theory transforms a p.d.f. into a L.T. In defining L.T., assume that there is a continuous r.v. T with a p.d.f. $f(t), (t \geq 0)$. Its L.T. is defined as

$$\bar{f}(\omega) = E[e^{-\omega T}] = \int_0^\infty e^{-\omega t} f(t)dt$$

where $\bar{f}(0) = 1$ and $\bar{f}(\omega)$ is an analytic function in the half-place $\mathrm{Re}(\omega) > \omega_0, (\omega_0 \leq 0)$ since $0 \leq \bar{f}(\omega) \leq 1$ for $\omega \geq 0$. L.T. is a useful tool in the continuous probability theory due to its ability to express useful information in a fairly simple form. When $\bar{f}(\omega)$ is inverted to $f(t)$, the procedure is known as the inverse L.T. and defined as

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\omega t}\bar{f}(\omega)d\omega$$

where the contour is any vertical line $\omega = a$ so that $\bar{f}(\omega)$ has no singularities on, or to the right of it.

The L-S.T is considered to be more general than the L.T. as it encompasses a wider class of r.v.'s than the simple L.T. The definition of the L-S.T. is as follows: Let T be a non-negative r.v. with a c.d.f. $F(t) = f(t \leq T)$, then the L-S.T. of $F(t)$ is defined as

$$\bar{f}_T(\omega) = \int_0^\infty e^{-\omega t}dF(t)$$

with $\mathrm{Re}(\omega) \geq 0$. The integral on the right-hand side of the definition of L-S.T. is known as the Stieltjes integral. In addition, the L-S.T. of $F(t)$ becomes the L.T. of $f(t)$ if $f(t) = {dF(t)}/{dt}$ exists.

A.3 Discrete probability theory

Assume that there is a discrete r.v., say M, such that it has a p.m.f. $f_m = P(M = m), (0 \leq m < \infty)$. The n -th moment of M is defined as $E[M^n] = \sum_{m=0}^{\infty} m^n f_m\,, (n \geq 1)$.

A.3.1 Generating function and probability generating function

Let $\{u_n\}$ be a sequence of real numbers. If $U(z) = \sum_{n=0}^{\infty} u_n z^n$ converges in some interval $|z| < z_0, (0 \leq z_0 \leq \infty)$, then $U(z)$ is called the g.f. of the sequence $\{u_n\}$ (see Hunter [27] for details). Here, z_0 is a unique number called the radius of convergence such that

- $\sum_{n=0}^{\infty} u_n z^n$ converges (absolutely) for $|z| < z_0$
- $\sum_{n=0}^{\infty} u_n z^n$ diverges for $|z| > z_0$
- $\sum_{n=0}^{\infty} u_n z^n$ converges uniformly for $|z| \leq \theta$, where $\theta < z_0$

Since a g.f. transforms a sequence into a power series (a procedure also known as the z-transform) an inverse g.f. returns a power series back into a sequence.

In introducing the g.f. in the discrete probability theory, let there be a discrete r.v. V. The $U(z)$ becomes a p.g.f. of V if and only if u_n satisfies the following the following conditions:

- $u_n = P(V = n), (n \geq 0)$
- $0 \leq u_n \leq 1, (n \geq 0)$
- $\sum_{n=0}^{\infty} u_n = 1$

When all three of the above conditions are met, $U(z)$ becomes the p.g.f. of V, such that

$$U(z) = E[z^V], (|z| \leq 1)$$

In this regard, a p.g.f. is always a g.f. but a g.f. is not always a p.g.f. In addition, the p.g.f is a power series that has advantages over its p.m.f. counterpart when obtaining moments of a r.v.. For instance, the moments of a discrete r.v. are easy to derive from a p.g.f. as illustrated by the following property:

$$U^{(r)}(1) = \lim_{z \to 1^-} \frac{d^r U(z)}{dz^r} = \frac{d^r}{dz^r} E[z^V]|_{z=1}, (r \geq 1)$$

where $U^{(r)}(1)$ is the r-th derivative of the p.g.f. of the r.v. V evaluated at $z = 1$. This can be used to find various parameters such as the mean, variance, and moments of V. Heavy-tailed probability distributions have non-closed form or non-existent p.g.f.'s.

APPENDIX B

In this appendix, we explain the basic constructs of a queueing model, as well, introduce some theorems that are commonly used when analyzing both continuous and discrete-time queues.

B.1 Basics of queueing theory and Kendall's notation

The queueing theory analyzes the properties that surround a queueing model. In this thesis, the term 'queueing system', 'queueing model', or simply 'queues' are synonymous terms that refer to the mathematical construct that composes of the servers, customers in the servers, and customers in queue (if any). Queueing models can be described as the mathematical models that describe the process of customers arriving for service, waiting for service (if service is not immediately available), and receiving of service, followed by leaving the system once service is complete. In the context of this thesis (i.e. bulk-arrival multi-server queues), a 'system' refers to the space that includes the queue of customers and the servers with customers under service. A queue refers to only the space which includes the queue of customers. As well, a customer is a generic term that refers to any element (person, product, packet, etc) that participates in a queueing system. Queueing models can be described in terms of the Kendall's notation:

$$A_n^{X_n}(t)/B_n^{a,b}/c/N$$

where

$A_n(t)$: Inter-arrival time distribution with arrival rate depending on t and n (if t and n in $A_n(t)$ are missing, it means the arrival rate λ_n is a constant λ).

X_n: Arrival group size distribution with group size probability depending on n (if n in X_n is missing, it means the group size probability is independent of n). In this thesis, the terms 'group,' 'bulk,' and 'batch' are synonymously used.

B_n: Service time distribution with service rate depending on n (if n in B_n is missing, it means the service rate μ_n is a constant μ).

a: Quorum for service group.

b: Capacity for service group.

c: Number of servers.

N: Finite-buffer. The queues with Kendall's notation that entail N are referred to as finite-buffer queues; otherwise they are referred to as infinite-buffer queues.

The traffic intensity (ρ) is a parameter that is uniquely defined for each queueing model. In finite-buffer queues $\rho > 0$ whereas in infinite-buffer queues $0 < \rho < 1$. Lastly, the term 'queue-length distribution' and 'steady-state p.m.f. of the number of customers in the system' are used synonymously in this thesis.

B.2 Linear difference equations

The linear difference equations frequently arise in the theory of queues (see Chaudhry and Templeton [17]). In particular, an equation of the type

$$a_0 f_{x+n} + a_1 f_{x+n-1} + \ldots + a_{n-1} f_{x+1} + a_n f_x = b_x, (x = 1,2, \ldots)$$

is called a non-homogeneous linear difference equation of order n where a_i are the known constants, f_i are the unknown functions to be determined, and b_x is the given function of x. If $b_x = 0$ for all x then it is called the homogeneous linear difference equation with constant coefficients. A general solution to the above non-homogenous equation consists of two parts:

1. A linear combination of all solutions to the homogeneous equation
2. A particular solution to the non-homogeneous equation

The solution to the homogeneous part of the equation proceeds along the following lines. Letting $f_x = Cz^x$ in the homogeneous equation leads to

$$a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0$$

The above expression is an n-th degree polynomial that has n roots (real or complex, distinct or coincident). This expression, unique for each queueing model, is also called the characteristic equation. Assuming that the roots of the characteristic equation are distinct, the general solution of the homogeneous part can be written as

$$f_x = \sum_{j=1}^{n} C_j z_j^x$$

For more details on linear difference equations, readers may refer to Chaudhry and Templeton [17].

B.3 Rouché's theorem

If $f(z)$ and $g(z)$ are the functions of z, which are analytic inside and on a closed contour C, and if $|f(z)| < |g(z)|$ on C, then $g(z)$ and $g(z) + f(z)$ have the same number of roots inside C.

B.4 Random biased sampling

The random biased sampling is a technique in queueing theory that is often used on two particular occasions: When developing a relation between the queue-length distributions at pre-arrival and random-time epochs (see e.g., Kim [31]), as well, when expressing the random position of a customer within an incoming batch (see Burke [5]).

The random biased sampling can be best explained using the renewal theory. The renewal theory deals with the study of renewal processes. A process $\{N(t), t \geq 0\}$ whose state space belongs to a denumerable set $\{0,1,2, \ldots\}$ and for which the inter-arrival times $U_n = \sigma_n' - \sigma_{n-1}', n = 1,2,3, \ldots, (\sigma_0' = 0)$ between successive arrivals (or occurrences) are positive i.i.d.r.v.'s, is called a renewal process. Let the renewals occur at instants of time σ_1', σ_2', ..., and suppose that $U_n = \sigma_n' - \sigma_{n-1}', n = 2,3, \ldots$ are i.i.d.r.v.'s with the c.d.f.

$$A(u) = P(U_n \leq u), (n \geq 2)$$

and mean

$$a = E[U_n], (n \geq 2)$$

Further, let $U_1 = \sigma_1' - \sigma_0'$ be independent of other U's and

$$A_1(u) = P(U_1 \leq u)$$

By letting U_1 to be independent of the remaining U's, the renewal process $\{N(t), t \geq 0\}$ becomes a modified (also called delayed or general) renewal process with

$$A_1(u) = \frac{1}{a} \int_0^u [1 - A(x)]\, dx$$

where $a = \int_0^\infty u dA(u)$. The above expression can be interpreted as the relation between the c.d.f.'s of the inter-batch-arrival times at random and pre-arrival time epochs in continuous-time queues.

Another application of the random biased sampling in queueing theory is explained. As introduced by Burke [5], using the random biased sampling, we can find the distribution of the position of the random customer within an incoming group of size X. The distribution of the size of the group in which the random position falls within should be proportional to hb_h where $P(X = h) = b_h, (h \geq 1)$, and is thus given by

$ha_h/\bar{b}$ where $\bar{b} = \sum_{h=1}^{\infty} hb_h$. Consequently, if the position of the random customer within the arrival group is J, then

$$P(J = j) = \sum_{h=j}^{\infty} \frac{b_h}{\bar{b}}, (j = 1,2, \dots)$$

The above relation can also be interpreted as the relation between the steady-state p.m.f.'s of the inter-batch-arrival times at pre-arrival and random time epochs in discrete-time queues.

APPENDIX C

The purpose of this appendix is to present a wide range of materials that are needed in explaining the solution procedures and numerical results in the main body of this thesis (Chapters 2, 3, 4, and 5).

C.1 Supplementary materials on $GI^X/M/c$ queues

C.1.1 Transition probabilities of $GI^X/M/c$ queues

The transition probabilities of $GI^X/M/c$ queues can be grouped into two different matrices: $[P_{i,j}]_{c\leq r}$ when $c \leq r$ and $[P_{i,j}]_{c>r}$ when $c > r$

$$[P_{i,j}]_{c\leq r} = \begin{bmatrix} 1-\sum_{j=1}^{\infty} P_{0,j} & P_{0,1} & \cdots & P_{0,c} & \cdots & P_{0,r-1} & P_{0,r} & 0 & 0 & 0 & \cdots \\ 1-\sum_{j=1}^{\infty} P_{1,j} & P_{1,1} & \cdots & P_{1,c} & \cdots & P_{1,r-1} & P_{1,r} & P_{1,r+1} & 0 & 0 & \cdots \\ 1-\sum_{j=1}^{\infty} P_{2,j} & P_{2,1} & \cdots & P_{2,c} & \cdots & P_{2,r-1} & P_{2,r} & P_{2,r+1} & P_{2,r+2} & 0 & \cdots \\ 1-\sum_{j=1}^{\infty} P_{3,j} & P_{3,1} & \cdots & P_{3,c} & \cdots & P_{3,r-1} & P_{3,r} & P_{3,r+1} & P_{3,r+2} & P_{3,r+3} & \cdots \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Figure 6: The transition probabilities of $GI^X/M/c$ queues when $c \leq r$

$$[P_{i,j}]_{c>r} = \begin{bmatrix} 1-\sum_{j=1}^{\infty} P_{0,j} & \cdots & P_{0,c-1} & 0 & 0 & 0 & \cdots \\ 1-\sum_{j=1}^{\infty} P_{1,j} & \cdots & P_{1,c-1} & 0 & 0 & 0 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \\ 1-\sum_{j=1}^{\infty} P_{c-r-1,j} & \cdots & P_{c-r-1,c-1} & 0 & 0 & 0 & \cdots \\ 1-\sum_{j=1}^{\infty} P_{c-r,j} & \cdots & P_{c-r,c-1} & P_{c-r,c} & 0 & 0 & \cdots \\ 1-\sum_{j=1}^{\infty} P_{c-r+1,j} & \cdots & P_{c-r+1,c-1} & P_{c-r+1,c} & P_{c-r+1,c+1} & 0 & \cdots \\ 1-\sum_{j=1}^{\infty} P_{c-r+2,j} & \cdots & P_{c-r+2,c-1} & P_{c-r+2,c} & P_{c-r+2,c+1} & P_{c-r+2,c+2} & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Figure 7: The transition probabilities of $GI^X/M/c$ queues when $c > r$

C.1.2 Proving the existence of roots

We prove that the characteristic equation of the model $GI^X/M/c$ has r roots inside the unit circle. In doing so, we rearrange (1) as

$$p_j^- = \sum_{h=1}^{r} b_h \sum_{i=j-h}^{\infty} p_i^- k_{i+h-j}\,, (j \geq 1)$$

Now substituting $p_j^- = Cz^j, (j \geq c)$ into above expression leads to (2) which is the underlying characteristic equation of $GI^X/M/c$ queues. To prove that (2) has r roots inside the unit circle $|z| = 1$, let us rewrite it as

$$z^r - \left(\sum_{h=1}^{r} b_h z^{r-h}\right) K(z) = 0$$

Now let

$$f(z) = z^r$$

and

$$g(z) = -\left(\sum_{h=1}^{r} b_h z^{r-h}\right) K(z)$$

Consider the absolute values of $f(z)$ and $g(z)$ on the circle $|z| = 1 - \delta$, where δ is positive and sufficiently small. This gives,

$$|f(z)| = (1-\delta)^r = 1 - \delta r + o(\delta),$$

and

$$|g(z)| \leq \sum_{h=1}^{r} b_h |z|^{r-h} K(|z|)$$

which leads to

$$= 1 - \delta(r - \mu_X) - \frac{c\mu}{\lambda}\delta + o(\delta)$$

or

$$= 1 - \delta r - \frac{c\mu}{\lambda}(1-\rho)\delta + o(\delta)$$

where $\rho = \frac{\lambda\mu_X}{c\mu}, (0 < \rho < 1)$. Thus, for $0 < \rho < 1$ and δ sufficiently small, $|f(z)| > |g(z)|$ on $|z| = 1 - \delta$. Because $f(z)$ and $g(z)$ satisfy the conditions of Rouché's theorem (see Appendix B.3), it follows that (2) has r roots inside the unit circle, since $f(z)$ has r roots inside $|z| = 1$.

C.1.3 Root finding program

Below program computes the roots of the characteristic equation of $GI^X/M/c$ queues involving heavy-tailed inter-batch-arrival times.

(1) with($plots$):

(2) $LC := 10^{-40}$:

(3) $\mu :=$

(4) $a := (t) \rightarrow$

(5) $b := (h) \rightarrow$

(6) $c :=$

(7) $r :=$

(8) $\Psi := 1$:

(9) while Re $\left[\int_0^\infty e^{-c\hat{\mu}t} \frac{(c\hat{\mu}t)^\Psi}{\Psi!} a(t)dt\right] > LC$ do

(10) $\quad\quad \Psi := \Psi + 1$:

(11) end do:

(12) if Re $\left[\int_0^\infty e^{-c\hat{\mu}t} \frac{(c\hat{\mu}t)^\Psi}{\Psi!} a(t)dt\right] \le LC$ then

(13) $\quad\quad \Psi := \Psi - 1$:

(14) end if:

(15) $y := z^r - expand\left(z^r * \sum_{h=1}^{r} b(h) * z^{-h} \sum_{n=0}^{\Psi}\left(\int_0^\infty e^{-c\mu t} \frac{(c\mu t)^n}{n!} a(t)dt\right) z^n\right)$

(16) $AllRoots := [fsolve(y, z, \{z = -1 - I..1 + I\}, complex)]$:

(17) $multiple(complexplot, [AllRoots, symbol = (diagonalcross, symbolsize = 10, style = point, labels = ["Re", "Im"], color = "Red"], [cos + sinI, -\pi..\pi, color = "SteelBlue"])$

To run the above program, $\mu, a(t), b(h), c$, and r must be defined. In line (2) of the program, the LC of 10^{-40} is used as the default value but it can be scaled depending on the desired degree of accuracy. In addition, the term Re $\left[\int_0^{\infty} e^{-c\hat{\mu}t} \frac{(c\hat{\mu}t)^{\Psi}}{\Psi!} a(t)dt\right]$ in line (9) and (12) of our program were chosen as the demarcation point to determine Ψ since it is the constant coefficient of the last term of z in $K_{\Psi}(z)$, hence LC stands for the Last Coefficient. Line (17) plots all r roots that are found within the contour of a unit circle $|z| = 1$.

C.1.4 Distinguishing between $c \leq r$ and $c > r$ in solving for p_j^-

We take an alternate approach by assuming two different solutions, $p_j^- = \sum_{h=1}^{r} C_h z_h^j, (j \geq r, c \leq r)$ and $p_j^- = \sum_{h=1}^{r} C_h z_h^j, (j \geq c, c > r)$, and then substitute each of them into (1). By doing so, it reveals additional mathematical properties of p_j^-: Readers will see that when $c \leq r$, the p_j^- can be completely expressed in terms of the roots of (2) (i.e. a geometric sum) whereas when $c > r$, the p_j^- can be partially expressed in terms of the roots of (2) (i.e. a partial geometric sum). This analytical finding was introduced by Chaudhry and Kim [13]. In the following, we first solve $\left(p_j^-, j \geq 0\right)$ when $c \leq r$ and then solve $\left(p_j^-, j \geq 0\right)$ when $c > r$.

C.1.4.1 First case: $c \leq r$

We assume that the $C_h, (1 \leq h \leq r)$ in $p_j^- = \sum_{h=1}^{r} C_h z_h^j, (j \geq r)$ also span the bound $(c \leq j \leq r-1)$. Thus, substituting $p_j^- = \sum_{h=1}^{r} C_h z_h^j$, $(c \leq j \leq r-1)$ into (1) leads to

$$\sum_{h=1}^{r} C_h\, z_h^j = \sum_{i=0}^{\infty}\sum_{h=1}^{r} C_h z_h^i \sum_{l=1}^{r} b_l\, k_{i+l-j}$$

$$= \sum_{h=1}^{r} C_h \sum_{l=1}^{r} b_l \left(\sum_{i=0}^{\infty} k_{i+l-j} z_h^i\right)$$

$$= \sum_{h=1}^{r} C_h \sum_{l=1}^{r} b_l \left(\sum_{n=l-j}^{\infty} k_n\, z_h^{n-l+j}\right)$$

which can be rearranged to

$$0 = \sum_{h=1}^{r} C_h z_h^j \left(\sum_{l=1}^{r} b_l\, z_h^{-l} \sum_{n=0}^{l-j-1} k_n\, z_h^n\right), (c \leq j \leq r-1) \tag{51}$$

Now letting $j = r-1$ in (51) gives

$$0 = \sum_{h=1}^{r} C_h z_h^{r-1} \left(\sum_{l=1}^{r} b_l\, z_h^{-l} \sum_{n=0}^{l-(r-1)-1} k_n\, z_h^n\right)$$

$$= \sum_{h=1}^{r} C_h z_h^{r-1} \left(\sum_{l=1}^{r-1} b_l\, z_h^{-l} \sum_{n=0}^{l-r} k_n\, z_h^n + b_r z_h^{-r} k_0\right)$$

Since $\sum_{n<0} k_n z_h^n = 0$, the above expression simplifies to

$$= \sum_{h=1}^{r} C_h z_h^{r-1}\,(b_r k_0 z_h^{-r}),\ \ (b_r, k_0 > 0)$$

$$= \sum_{h=1}^{r} \frac{C_h}{z_h}$$

Similarly, by letting $j = r-2, \dots, c-1, c$, we obtain the following $(r-c)$ equations:

$$\sum_{h=1}^{r} \frac{C_h}{z_h} = \sum_{h=1}^{r} \frac{C_h}{z_h^2} = \cdots = \sum_{h=1}^{r} \frac{C_h}{z_h^{r-c-1}} = \sum_{h=1}^{r} \frac{C_h}{z_h^{r-c}} = 0 \tag{52}$$

Progressively, we further assume that the $C_h, (1 \le h \le r)$ in $p_j^- = \sum_{h=1}^{r} C_h z_h^j, (j \ge r)$ also span the bound $(1 \le j \le c-1)$. Thus we substitute $p_j^- = \sum_{h=1}^{r} C_h z_h^j, (1 \le j \le c-1)$ into $p_j^- = \sum_{i=0}^{\infty} p_i^- P_{i,j}$, such that

$$0 = \sum_{h=1}^{r} C_h z_h^j \left(\sum_{l=1}^{r} b_l \, z_h^{-l} \sum_{n=0}^{l-j-1} k_n \, z_h^n \right), (1 \le j \le c-1) \tag{53}$$

Now substituting $j = c-1, c-2, \dots, 1$ into (53), we obtain $(c-1)$ equations. Combining (52) and (53), we now have $(r-1)$ equations. However, to solve for the unknown $C_h, (1 \le h \le r)$, we require r equations. To get the r-th equation, so that the result is also true for $j = 0$, consider the sum of p_j^-

$$1 = \sum_{j=0}^{\infty} p_j^- = \sum_{j=0}^{\infty} \sum_{h=1}^{r} C_h \, z_h^j = \sum_{h=1}^{r} C_h \sum_{j=0}^{\infty} z_h^j$$

leading to

$$\sum_{h=1}^{r} \frac{C_h}{1-z_h} = 1 \tag{54}$$

Solving (52), (53), and (54) using available software such as MAPLE or Mathematica, we can easily compute the $C_h, (1 \le h \le r)$. Finally, the complete solution to $GI^X/M/c$ queues when $c \le r$ is

$$p_j^- = \sum_{h=1}^{r} C_h z_h^j, (j \ge 0) \tag{55}$$

C.1.4.2 Second case: $c > r$

In the case of $c > r$, we replace the entire tail distribution $\left(p_j^-, r < c \leq j\right)$ of (1) with (3). This is done by first expanding and rearranging (1) and letting $j = 0,1,2, \dots, c + r - 2$, which yields the following $c + r - 1$ equations.

$$0 = p_0^-\left(P_{0,0} - 1\right) + p_1^- P_{1,0} + p_2^- P_{2,0} + \dots + p_{c-1}^- P_{c-1,0} + \sum_{i=c}^{\infty}\left(\sum_{h=1}^{r} C_h z_h^i\right) P_{i,0}$$

$$0 = p_0^- P_{0,1} + p_1^-\left(P_{1,1} - 1\right) + p_2^- P_{2,1} + \dots + p_{c-1}^- P_{c-1,1} + \sum_{i=c}^{\infty}\left(\sum_{h=1}^{r} C_h z_h^i\right) P_{i,1} \tag{56}$$

$$\vdots$$

$$0 = p_0^- P_{0,c+r-2} + p_1^- P_{1,c+r-2} + p_2^- P_{2,c+r-2} + \dots + p_{c-1}^- P_{c-1,c+r-2} + \sum_{h=1}^{r} C_h \sum_{i=c}^{\infty}\left(z_h^i P_{i,c+r-2} - z_h^{c+r-2}\right)$$

The $(c + r)$-th equation is found from the normalizing condition, which is simply

$$1 = p_0^- + p_1^- + p_2^- + \cdots + \sum_{j=c}^{\infty}\left(\sum_{h=1}^{r} C_h z_h^j\right) \tag{57}$$

By solving (56) and (57), the boundary probabilities $\left(p_j^-, 0 \leq j < c\right)$ as well as the unknown constants $C_h, (1 \leq h \leq r)$ are determined. Hence the solution to $GI^X/M/c$ queues when $c > r$ is

$$p_j^- = \begin{cases} \text{determined above}, (0 \leq j < c) \\ \sum_{h=1}^{r} C_h z_h^j, \qquad (r < c \leq j) \end{cases} \tag{58}$$

C.2 Supplementary materials on $GI^X/Geo/c$ queues

C.2.1 Proving the existence of roots

The proof here runs parallel to that of Appendix C.1.2. First, by rearranging (21), we have

$$Q_j^- = \sum_{h=1}^{r} b_h \sum_{i=j-h}^{\infty} Q_i^- k_{i+h-j}\,, (j \geq 1)$$

Now substituting $Q_j^- = Cz^j, (j \geq c)$ into above expression leads to (22) which is the underlying characteristic equation of the $GI^X/Geo/c$ queue. To prove that (22) has r roots inside the unit circle $|z| = 1$, let us rewrite it as

$$z^r - \left(\sum_{h=1}^{r} b_h z^{r-h}\right) K(z) = 0$$

Now let

$$f(z) = z^r$$

and

$$g(z) = -\left(\sum_{h=1}^{r} b_h z^{r-h}\right) K(z)$$

Consider the absolute values of $f(z)$ and $g(z)$ on the circle $|z| = 1 - \delta$, where δ is positive and sufficiently small. This gives,

$$|f(z)| = (1-\delta)^r = 1 - \delta r + o(\delta),$$

and

$$|g(z)| \leq \sum_{h=1}^{r} b_h |z|^{r-h} K(|z|) = 1 - \delta(r - \bar{b}) - (\bar{a}c\mu)\delta + o(\delta)$$

$$= 1 - \delta r - (\bar{a}c\mu)(1-\rho)\delta + o(\delta)$$

Finally we have

$$|g(z)| < 1 - \delta r + o(\delta) = |f(z)|$$

where $\rho = \frac{\bar{b}}{\bar{a}c\mu}, (0 < \rho < 1)$. Thus, for $0 < \rho < 1$ and δ sufficiently small, $|f(z)| > |g(z)|$ on $|z| = 1 - \delta$. Because $f(z)$ and $g(z)$ satisfy the conditions of Rouché's theorem, it follows that (22) has r roots inside the unit circle, $|z| = 1$, since $f(z)$ has r roots inside $|z| = 1$.

C.2.2 Root finding program

Below program computes the roots of the characteristic equation of $GI^X/Geo/c$ queues involving heavy-tailed inter-batch-arrival times.

(1) with$(plots)$:

(2) $LC := 10^{-400}$:

(3) $\mu :=$

(4) $a := (m) \rightarrow$

(5) $b := (h) \rightarrow$

(6) $c :=$

(7) $r :=$

(8) $\Psi := 1$:

(9) while $\mu^{c*\Psi} * a(\Psi) > LC$ do

(10) $\quad \Psi := \Psi + 1$:

(11) end do:

(12) if $\mu^{c*\Psi} * a(\Psi) \le LC$ then

(13) $\quad \Psi := \Psi - 1$:

(14) end if:

(15) $y := z^r - expand(z^r * \sum_{h=1}^{r} b(h) * z^{-h} \sum_{m=1}^{\Psi} (\mu * z + 1 - \mu)^{c*m} * a(m))$

(16) $AllRoots := [fsolve(y, z, \{z = -1 - I..1 + I\}, complex)]$:

(17) $multiple(complexplot, [AllRoots, symbol = (diagonalcross, symbolsize = 10, style = point, labels = ["Re", "Im"], color = "Red"], [cos + sinI, -\pi..\pi, color = "SteelBlue"])$

To run the above program, $\mu, a(m), b(h), c$ and r must be defined. In line (2) of our program, the LC of 10^{-400} is used as the default value but it can be scaled depending on the desired degree of accuracy. In addition, the term $\mu^{c*\Psi}$ in line (9) and (12) of our program were chosen as the demarcation point to determine Ψ since it is the constant coefficient of the last term of z in $K_{\Psi}(z)$. Line (17) plots all r roots that are found within the contour of a unit circle $|z| = 1$.

C.2.3 Deriving the characteristic equation: From $GI^X/Geo/c$ to $GI^X/M/c$ queues

We derive the characteristic equation of the $GI^X/M/c$ queue from that of the $GI^X/Geo/c$ queue. Let t be the inter-batch-arrival time of the $GI^X/M/c$ queue such that it is a non-zero real-number $(t \in R^*)$. It has a c.d.f. $A(t) = P(\hat{T} \le t), (dA(t)/dt \equiv a(t), t > 0)$ with the mean $E[\hat{T}]$ and arrival rate $\hat{\lambda}$ such that $E[\hat{T}] = {}^1/_{\hat{\lambda}}$. Similarly, let $\hat{Y}$ be the exponential service time of a server with mean $E[\hat{Y}]$ and service rate $\hat{\mu}, (0 < \hat{\mu} < 1)$ such that $E[\hat{Y}] = {}^1/_{\hat{\mu}}$. In the $GI^X/Geo/c$ queue, T is divided into m time slots and multiplying m by a very small real number (say Δ) results in a continuous inter-batch-arrival time t. Based on this notion, we have $t = \Delta m$ (similarly, $E[\hat{T}] = \Delta E[T]$ and $E[\hat{Y}] = \Delta E[Y]$).

To derive the characteristic equation of the $GI^X/M/c$ queue we consider the $k_{i+h-j}, (c \le j \le i + h)$ from Subsection 4.2.2 in a form that is prior to being conditioned on a_m which is

$$\binom{cm}{i+h-j} \mu^{i+h-j} (1-\mu)^{cm-(i+h-j)}, (m \ge 1, 1 \le h \le r, i \ge 0, c \le j \le i + h,)$$

Substituting $m = {}^t/_{\Delta}$ and $\mu = \Delta\hat{\mu}$ into the above equation gives

$$\frac{\left({}^{ct}/_{\Delta}\right)!}{(i+h-j)!\left[{}^{ct}/_{\Delta} - (i+h-j)\right]!} (\Delta\hat{\mu})^{i+h-j} (1-\Delta\hat{\mu})^{{}^{ct}/_{\Delta}-(i+h-j)}$$

To transform the k_{i+h-j} of the $GI^X/Geo/c$ queue into $\hat{k}_{i+h-j}$ of the $GI^X/M/c$ queue, taking the limit of the above result as $\Delta \to 0$ (so that the discrete time parameter becomes the continuous time parameter) gives

$$\lim_{\Delta \to 0} \frac{\left({}^{ct}/_{\Delta}\right)!}{(i+h-j)!\left[{}^{ct}/_{\Delta} - (i+h-j)\right]!} (\Delta\hat{\mu})^{i+h-j}(1-\Delta\hat{\mu})^{\left({}^{ct}/_{\Delta}\right)-(i+h-j)}$$

Multiplying and then dividing the above expression by $(ct)^{i+h-j}, (t > 0, c \geq 1, i+h \geq j)$ gives

$$\lim_{\Delta \to 0} \frac{\left({}^{ct}/_{\Delta}\right)!}{\left(\frac{ct}{\Delta}\right)^{i+h-j}\left[{}^{ct}/_{\Delta} - (i+h-j)\right]!} \frac{(ct\hat{\mu})^{i+h-j}}{(i+h-j)!}\left[(1-\Delta\hat{\mu})^{\frac{1}{\Delta}}\right]^{ct}(1-\Delta\hat{\mu})^{-(i+h-j)}$$

Since $i + h \geq j$ and using the fact that $\lim_{a\to\infty} \binom{a}{b}\left(\frac{1}{a^b}\right) = \frac{1}{b!}$, the above simplifies to

$$\frac{(c\hat{\mu}t)^{i+h-j}}{(i+h-j)!} e^{-c\hat{\mu}t}$$

We now condition the above expression on the p.d.f. $a(t), (t > 0)$ over $(0, \infty)$ such that it leads to

$$\hat{k}_{i+h-j} = \int_0^\infty \frac{(c\hat{\mu}t)^{i+h-j}}{(i+h-j)!} e^{-c\hat{\mu}t} a(t)dt\,, (1 \leq h \leq r)$$

or

$$\hat{k}_{i+h-j} = \int_0^\infty \frac{(c\hat{\mu}t)^{i+h-j}}{(i+h-j)!} e^{-c\hat{\mu}t} dA(t)\,, (1 \leq h \leq r)$$

where $A(t) = P(\hat{T} \leq t), (t > 0)$. The p.m.f. $\hat{k}_{i+h-j}$ has a p.g.f. $\hat{K}(z)$

$$\hat{K}(z) = \sum_{i=j-h}^{\infty} \hat{k}_{i+h-j} z^{i+h-j}\,, (1 \leq h \leq r, |z| \leq 1)$$

which can also be expressed as $\widehat{K}(z) = \int_0^\infty e^{-c\hat{\mu}(1-z)t} dA(t) = \tilde{a}\big(c\hat{\mu}(1-z)\big)$, where $\tilde{a}(s)$ is the L-S.T. $\tilde{a}(s) = \int_0^\infty e^{-st} dA(t)$. As the last step, replacing $\widehat{K}(z)$ in (22) with $\widehat{K}(z)$ leads to (2).

C.3 Additional numerical results

C.3.1 Computing the roots of the characteristic equation

In Table 21, 22, and 23 we consider the binomial, (1,10), and normalized Poisson batch size distributions, respectively. In the same tables, the traffic intensities are in a descending order and the maximum batch size includes both even and odd numbers. We find the roots of the characteristic equation of the $GI^X/M/c$ and $GI^X/Geo/c$ queues in Tables 21 and 22, whereas in Table 23 we focus on $GI^X/Geo/c$ queues with a larger r. We have purposely used $c = 5$ in all tables in order to isolate and study the effect of other input parameters. All computations were performed in MAPLE using the program in Appendices C.1.3 and C.2.2. All the roots were found up to the tenth decimal place and rounded to four decimal places. We have also embedded the figures of plotted roots in each table for visual illustration.

Table 21: The roots of the characteristic equations with the binomial batch size distribution $b_h = \binom{r}{h-1} p^{h-1} q^{r-h+1}$, $(p = 0.45, q = 0.55, r = 21)$ and $\rho = 0.8$

	$GI^X/Geo/5$		$GI^X/M/5$	
GI \ z_h	SLN	$Weibull[2]$	$Pareto[4.5,2]$	$IG[2,3]$
z_1	0.9656	0.9786	0.8031	0.9453
z_2	0.5625 − 0.4433i	0.0977 − 0.5157i	0.3845 − 0.3232i	0.5494 − 0.4076i
z_3	0.5625 + 0.4433i	0.0977 + 0.5157i	0.3845 + 0.3232i	05494 + 0.4076i
z_4	02578 − 0.5013i	0.3205 − 0.5482i	0.1938 − 0.3462i	0.2830 − 0.4770i
z_5	0.2578 + 0.5013i	0.3205 + 0.5482i	0.1938 + 0.3462i	0.2830 + 0.4770i
z_6	0.0669 − 0.4518i	0.6354 − 0.4463i	0.0655 − 0.3256i	0.0970 − 0.4537i
z_7	0.0669 + 0.4518i	0.6354 + 0.4463i	0.0655 + 0.3256i	0.0970 + 0.4537i

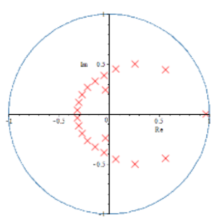
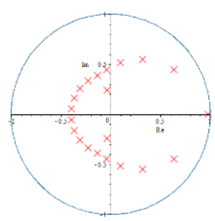
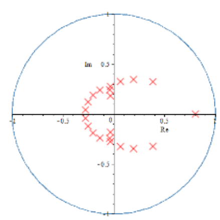
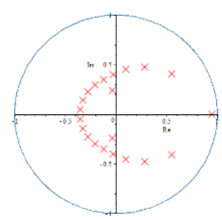
z_8	$-0.0364 - 0.2412i$	$-0.3959 - 0.0553i$	$-0.0337 - 0.2760i$	$-0.0364 - 0.2412i$
z_9	$-0.0364 + 0.2412i$	$-0.3959 + 0.0553i$	$-0.0337 + 0.2760i$	$-0.0364 + 0.2412i$
z_{10}	$-0.0496 - 0.3859i$	$-0.3663 - 0.1622i$	$-0.2536 - 0.1213i$	$-0.3530 - 0.0487i$
z_{11}	$-0.0496 + 0.3859i$	$-0.3663 + 0.1622i$	$-0.2536 + 0.1213i$	$-0.3530 + 0.0487i$
z_{12}	$-0.1391 - 0.3290i$	$-0.3087 - 0.2578i$	$-0.2789 - 0.0417i$	$-0.3275 - 0.1428i$
z_{13}	$-0.1391 + 0.3290i$	$-0.3087 + 0.2578i$	$-0.2789 + 0.0417i$	$-0.3275 + 0.1428i$
z_{14}	$-0.2149 - 0.2670i$	$-0.2270 - 0.3348i$	$-0.1367 - 0.2398i$	$-0.2071 - 0.2973i$
z_{15}	$-0.2149 + 0.2670i$	$-0.2270 + 0.3348$	$-0.1367 + 0.2390i$	$-0.2071 + 0.2973i$
z_{16}	$-0.3160 - 0.0388i$	$-0.1330 - 0.3877i$	$-0.2052 - 0.1893i$	$-0.2779 - 0.2275i$
z_{17}	$-0.3160 + 0.0388i$	$-0.1330 + 0.3877i$	$-0.2052 + 0.1893i$	$-0.2779 + 0.2275i$
z_{18}	$-0.3012 - 0.1166i$	$-0.0414 - 0.4447i$	$-0.0332 - 0.1820i$	$-0.1219 - 0.3499i$
z_{19}	$-0.3012 + 0.1166i$	$-0.0414 + 0.4447i$	$-0.0332 + 0.1820i$	$-0.1219 + 0.3499i$
z_{20}	$-0.2688 - 0.1941i$	$-0.0364 - 0.2412i$	$-0.0406 - 0.2424i$	$-0.0282 - 0.4006i$
z_{21}	$-0.2688 + 0.1941i$	$-0.0364 + 0.2412i$	$-0.0406 + 0.2424i$	$-0.0282 + 0.4006i$
Ψ	91	144	49	81
LC	10^{-110}	10^{-520}	10^{-4}	10^{-4}
$\lvert z \rvert = 1$				

Table 22: The roots of the characteristic equations with the (1,10) batch size distribution $b_1 = 0.9, b_{10} = 0.1$, and $\rho = 0.25$

	$GI^X/Geo/5$		$GI^X/M/5$	
GI \ z_h	SLN	$Weibull[2]$	$Pareto[4.5,2]$	$IG[2,3]$
z_1	0.7408	0.7820	0.6065	0.7086
z_2	$0.0974 - 0.5902i$	$0.5448 - 0.4547i$	$0.0209 - 0.4062i$	$0.4544 - 0.4278i$
z_3	$0.0974 + 0.5902i$	$0.5448 + 0.4547i$	$0.0209 + 0.4062i$	$0.4544 + 0.4278i$
z_4	$0.4798 - 0.4518i$	$0.1663 - 0.6423i$	$0.3137 - 0.3790i$	$0.1048 - 0.5568i$
z_5	$0.4798 + 0.4518i$	$0.1663 + 0.6423i$	$0.3137 + 0.3790i$	$0.1048 + 0.5568i$
z_6	-0.4953	-0.6026	-0.3262	-0.5033
z_7	$-0.4344 - 0.2646i$	$-0.5021 - 0.3460i$	$-0.1765 - 0.3123i$	$-0.2121 - 0.4860i$
z_8	$-0.4344 + 0.2646i$	$-0.5021 + 0.3460i$	$-0.1765 + 0.3123i$	$-0.2121 + 0.4860i$
z_9	$-0.2377 - 0.4915i$	$-0.2221 - 0.5901i$	$-0.2895 - 0.1665i$	$-0.4275 - 0.2780i$
z_{10}	$-0.2377 + 0.4915i$	$-0.2221 + 0.5901i$	$-0.2895 + 0.1665i$	$-0.4275 + 0.2780i$

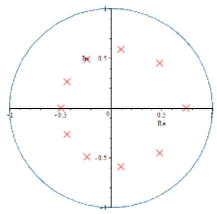
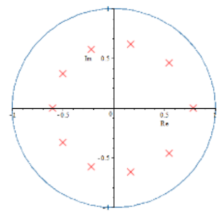
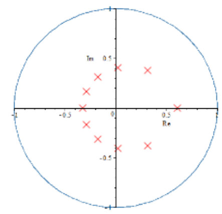
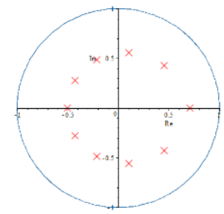
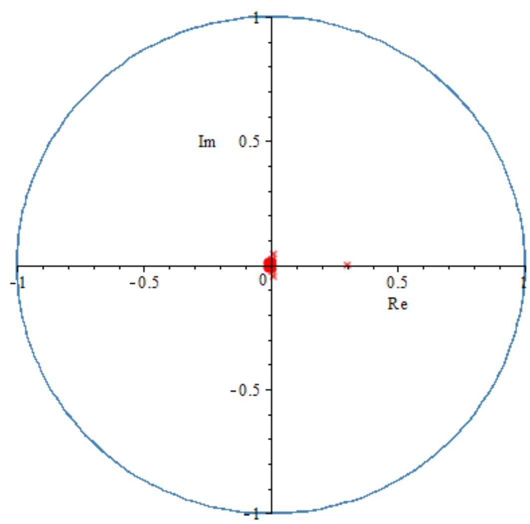
Ψ	15	49	22	34
LC	10^{-40}	10^{-240}	10^{-3}	10^{-3}
$\lvert z\rvert = 1$				

Table 23: The roots of the characteristic equation with the normalized Poisson batch size distribution $b_h = \left(\frac{1}{\sum_{l=1}^{r}\frac{P^l}{l!}}\right)\frac{P^h}{h!}$, $(P = 0.3, r = 50)$ and $\rho = 0$

$Pareto^X[1]/Geo/5$							
z_1	0.3004	z_{20}	0.0072 − 0.0069i				
z_2	0.0007 − 0.0146i	z_{21}	0.0072 + 0.0069i				
z_3	0.0007 + 0.0146i	z_{22}	0.0073 − 0.0051i				
z_4	0.0012 − 0.0158i	z_{23}	0.0073 + 0.0051i				
z_5	0.0012 + 0.0158i	z_{24}	0.0101 − 0.0449i				
z_6	0.0025 − 0.0138i	z_{25}	0.0101 + 0.0449i				
z_7	0.0025 + 0.0138i	z_{26}	−0.0146				
z_8	0.0027 − 0.0235i	z_{27}	−0.0145 − 0.0018i				
z_9	0.0027 + 0.0235i	z_{28}	−0.0145 + 0.0018i				
z_{10}	0.0039 − 0.0127i	z_{29}	−0.0141 − 0.0036i				
z_{11}	0.0039 + 0.0127i	z_{30}	−0.0141 + 0.0036i				
z_{12}	0.0050 − 0.0115i	z_{31}	−0.0136 − 0.0053i				
z_{13}	0.0050 + 0.0115i	z_{32}	−0.0136 + 0.0053i				
z_{14}	0.0060 − 0.0101i	z_{33}	−0.0128 − 0.0069i	z_{39}	−0.0095 − 0.0111i	z_{45}	−0.0048 − 0.0138i
z_{15}	0.0060 + 0.0101i	z_{34}	−0.0128 + 0.0069i	z_{40}	−0.0095 + 0.0111i	z_{46}	−0.0048 + 0.0138i
z_{16}	0.0067 − 0.0086i	z_{35}	−0.0119 − 0.0085i	z_{41}	−0.0080 − 0.0122i	z_{47}	−0.0030 − 0.0143i
z_{17}	0.0067 + 0.0086i	z_{36}	−0.0119 + 0.0085i	z_{42}	−0.0080 + 0.0122i	z_{48}	−0.0030 + 0.0143i
z_{18}	0.0069 − 0.0032i	z_{37}	−0.0108 − 0.0099i	z_{43}	−0.0065 − 0.0131i	z_{49}	−0.0012 − 0.0146i
z_{19}	0.0069 + 0.0032i	z_{38}	−0.0108 + 0.0099i	z_{44}	−0.0065 + 0.0131i	z_{50}	−0.0012 + 0.0146i
$LC = 10^{-55}$				$N = 19$			

C.3.2 Numerical analysis

From Appendix C.3.1 and other additional numerical tests, we have deduced the following points:

- **Point 1**: While z_1 is always a positive real root (as proven by Kim and Choi [30]), when r is an odd number there will be $r-1$ imaginary roots, and when r is an even number, there is also a negative real root and $r-2$ imaginary roots. In addition, z_1 has the largest modulus out of all r roots inside $|z|=1$.
- **Point 2**: The imaginary roots of the characteristic equations always exist in complex conjugate pairs. If r is an odd number, there are $\frac{r-1}{2}$ pairs of complex conjugates whereas if r is an even number, there are $\frac{r-2}{2}$ pairs of complex conjugates.
- **Point 3**: The LC for the $GI^X/M/c$ queue must be significantly larger than the LC for the $GI^X/Geo/c$ queue in order to compute the roots of satisfying accuracy. This is due to the magnitude of $\mu^{c*\Psi}$ being much smaller than that of $\int_0^\infty e^{-c\hat{\mu}t}\frac{(c\hat{\mu}t)^\Psi}{\Psi!}dA(t)$ when $\mu=\hat{\mu}$.
- **Point 4**: In the $GI^X/M/c$ and $GI^X/Geo/c$ queues, we can verify the accuracy of roots by back-substituting any of the r roots into (2) or (22), respectively. While substituting into the characteristic equation with a root that is found at a higher Ψ leads to a value that is closer to 1, a moderately sized Ψ will find roots that are just as effective. The accuracy of roots can be verified in another way: Compute the queue-length distributions with the r roots found at a moderately sized Ψ and then use those roots to compute the left-hand sides of

(14) and (37). Since the right-hand sides of (14) and (37) are independent of roots, one can choose to use a higher Ψ to make the left-hand sides of (14) and (37) match their respective right-hand sides up to the desired decimal place.

- **Point 5**: As seen in Table 23, as $\bar{a} \to \infty$, the roots converge towards the origin. However, the value of $\mu, (0 < \mu < 1)$ also influences the plotting pattern of roots. Given the relation $\rho = {}^{\bar{b}}/_{\bar{a}c\mu}$, letting $\bar{a} = \infty$ leads to $\rho = 0$. However, making $\mu \to 0$ will have a counter effect and withhold the roots from converging towards the origin (see later in Figure 6 where $\bar{a} = \infty$ and $\mu = 10^{-6}$). On the other hand, a large μ (and c) coupled with $\rho = 0$ would even further the clustering of roots towards the origin. In either case, roots can be found.
- **Point 6:** The root z_1 nears 1 as ρ approaches 1: This behavior can be seen by observing the decrease in z_1 in the tables as we make $\rho = 0.8$, 0.25, and 0.
- **Point 7:** The magnitude of $\mu, (0 < \mu < 1)$ is directly proportional to the size of Ψ. While the size of Ψ is primarily determined by the size of LC, if LC is fixed at a certain value, a smaller μ will lead to smaller Ψ while a larger μ will lead to a larger Ψ. This is true since given the same power, an exponent with a smaller base will always be smaller than the other exponent with a larger base. In the $GI^X/Geo/c$ and $GI^X/M/c$ queues involving heavy-tailed inter-batch-arrival times the a_m and $a(t)$ decay at an extremely slow rate as $m, t \to \infty$ while μ^{cm} and $e^{-c\hat{\mu}t}\frac{(c\hat{\mu}t)^n}{n!}$ decay faster with smaller values of μ and $\hat{\mu}$, thus

requiring less number of iterations (Ψ) to reach LC. We have tested this concept on $Pareto^X[1]/Geo/5$ by finding the Ψ at various values of μ while letting $LC = 10^{-100}, b_h = \left(\frac{1}{\sum_{l=1}^{r}\frac{P^l}{l!}}\right)\frac{P^h}{h!}, (P = 0.1, r = 10)$, and $\rho = 0$:

Table 24: Relation between μ and Ψ

$Pareto^X[1]/Geo/5$	
μ	Ψ
0.1	19
0.2	27
0.3	36
0.4	48
0.5	63
0.6	86
0.7	123
0.8	196
0.9	413
0.99	4239

- **Point 8**: Different batch size distributions lead to different plotting patterns of roots. In Table 22, the (1,10) batch size distribution plotted an equal number of roots on each side of the imaginary axis (with the exception of $Pareto^X[4.5,2]/M/5$) whereas this was not the case in other tables. However, there is always an equal number of roots plotted on each side of the real axis (this is due to the complex conjugate pairing as per the Point 2).

- **Point 9**: When $\bar{a} = \infty$ in the $GI^X/Geo/c$ queue it results in $Q_0 = Q_0^o = P_0^o = P_0 = 1$. Intuitively, this can be understood as the system being empty (in steady-state) at the outside observer's and random time epochs since there are no arrivals while remaining customers continue to get served. This phenomenon can be proven as follows: Since $\bar{a} = \infty$ the relation $Q_j = \sum_{i=0}^{\infty} Q_i^- P_{i,j}^*, (j \geq 1)$ becomes $Q_j = 0, (j \geq 1)$ which results in $Q_0 = 1$. Another approach is by letting $\rho = 0$ in (37) which leads to $\sum_{j=0}^{c-1}(c - j)Q_j^o = c$. This indicates that the mean number of idle servers is c (i.e. an empty system). Therefore it can be simplified to $cQ_0^o = c$ or $Q_0^o = 1$. When $\lambda \to 0$ in $GI^X/M/c$ queues, the queue-length distribution at a random epoch (say $p_j, j \geq 0$) becomes $p_0 = 1$. As a remark, several relations between the performance measures of the $GI^X/M/c$ queue were are derived by Yao et al. [50]. If the inter-batch-arrival times follow heavy-tailed distributions with an infinite mean, some of their relations that involve ρ become invalid. As an example, Yao et al. [50] derive the relation

$$W_{q_1}^- = \frac{\rho}{\lambda \mu_X}\left(L_s^- + 1 - c + \sum_{k=0}^{c-1}(c - k - 1)p_k^-\right)$$

In this relation, when λ and $\rho \to 0$, the right-hand side becomes undefined while the left-hand side does not. The same phenomena can be observed when $\rho = 0$ in another relation that determines W_q^- which is the waiting-time-in-queue of the random customer within an incoming batch.

C.3.3 Extreme case

We have plotted the roots of the characteristic equation for the model $GI^X/Geo/5$ when $GI = Pareto[1]$. Doing so leads to $\bar{a} = \infty$ which results in $\rho = 0$. The batch size distribution is a binomial distribution $(p = 0.45, q = 0.55)$ with $r = 1{,}000$. We have used $\mu = 10^{-6}$ and $LC = 10^{-10{,}000}$ which led to $\Psi = 333$.

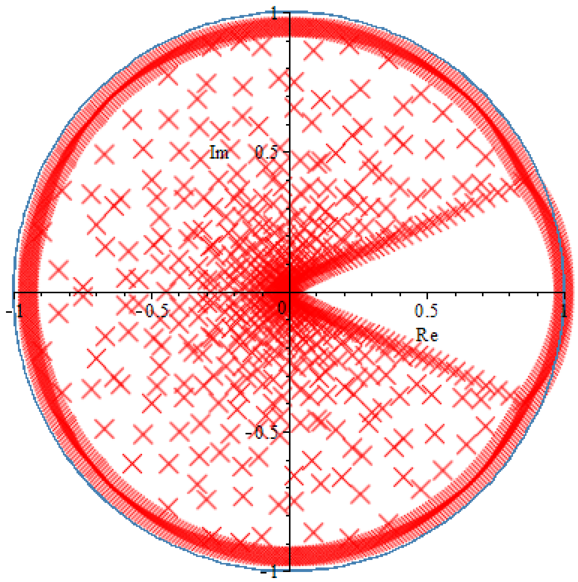


Figure 8: Plotting 1,000 roots

Using our MAPLE program from Appendix C.2.2 we have successfully plotted all 1,000 roots inside the unit circle with $z_1 = 0.9999959545$ (a smaller LC will lead to a z_1 that is closer to 1). As a remark, one may encounter the error message in MAPLE *"Length of output exceeds limit of 1000000"* as z_1 becomes very close to 1. This can be overcome by changing the default setting in MAPLE in the following manner: go to *'Tools'*, *'Options'*, *'Precision'*, and change the *'Limit expression length'* from 1,000,000 to 90,000,000 or greater. Doing so prevents MAPLE from rounding z_1 to 1 which is what is needed to compute the queue-length distributions in terms of roots.

C.3.4 Approximating the loss probability based on a single root

In the model $GI^X/M/c/N$, Kim and Choi [30] define the asymptotic loss probability as the probability of N customers in system at a pre-arrival time epoch as N tends to infinity. Their proposition is that the asymptotic loss probability of the $GI^X/M/c/N$ queue based on both partial and total rejections (i.e. $p_{\bar{N}}$) can be accurately approximated using the boundary probabilities and the roots of the characteristic equation of the $GI^X/M/c$ queue. In the context of this thesis, we express the proposition of Kim and Choi [30] as

$$p_N^- = C_1 z_1^N \text{ as } N \to \infty$$

where $z_1, (0 < z_1 < 1)$ is a root of (2) and C_1 is its corresponding constant coefficient in (5). Kim and Choi [30] proved that there is exactly a simple, positive root (z_1) of (2) that has the largest modulus among the r roots. However, it is evident that all constant terms $C_h, (1 \le h \le r)$ must be determined prior to identifying its corresponding constant coefficient C_1. This results in a phenomenon where all terms within (5) need to be determined prior to selecting the geometric term of interest. If this is the case, then the question arises as to why not use all the roots and the corresponding constants to enhance the accuracy of asymptotic approximations. Furthermore, since the numerical computation of (5) with a large j can be done easily, the approximation of p_N^- can be enhanced if we, instead of the proposition by Kim and Choi [30], express it as

$$p_N^- = \sum_{h=1}^{r} C_h z_h^N \text{ as } N \to \infty$$

where $N \to \infty$. Although $\sum_{h=1}^{r} C_h z_h^N \to C_1 z_1^N$ for $N \to \infty$, the emphasis is on the enhancement of Kim and Choi [30]'s method when approximating p_N^- at moderate values of N using already determined results. The numerical comparison between the proposition of Kim and Choi [30] and our expression is provided below in Table 25.

Consider the model $D^X/M/c/N$ with the same parameters as used in Table 2 except for the finite-buffer N. We compare the p_N^- at $N = 5,10,15,20,50,100,$ and 1000 using a geometric sum ($p_N^- = \sum_{h=1}^{r} C_h z_h^N$) and a geometric term ($p_N^- = C_1 z_1^N$), where the positive root of (2) is $z_1 = 0.3716$ and its corresponding constant C_1 is 0.7497. Relative errors between the two results are the absolute values of each difference such that $\delta_{error}(N) = |\sum_{h=2}^{r} C_h z_h^N|$.

Table 25: Comparing the geometric sum and geometric term

N	$\sum_{h=1}^{r} C_h z_h^N$	$C_1 z_1^N$	$\delta_{error}(N)$
5	$5.455309\text{x}10^{-3}$	$5.415048\text{x}10^{-3}$	$1.402611\text{x}10^{-4}$
10	$3.762469\text{x}10^{-5}$	$3.768008\text{x}10^{-5}$	$5.538531\text{x}10^{-8}$
15	$2.671454\text{x}10^{-7}$	$2.671261\text{x}10^{-7}$	$1.925054\text{x}10^{-11}$
20	$1.893737\text{x}10^{-9}$	$1.893743\text{x}10^{-9}$	$6.126980\text{x}10^{-15}$
50	$2.404079\text{x}10^{-22}$	$2.404079\text{x}10^{-22}$	$2.791419\text{x}10^{-36}$
100	$7.708950\text{x}10^{-44}$	$7.708950\text{x}10^{-44}$	$3.342722\text{x}10^{-71}$
1000	$9.904428\text{x}10^{-431}$	$9.904428\text{x}10^{-431}$	$6.841268\text{x}10^{-697}$

As expected, $\delta_{error}(N) \to 0$ as $N \to \infty$. Kim and Choi [30] state that a geometric term can approximate accurate results even for moderate size N, which we have shown at $N = 15, 20$, and 50. However, even more accurate approximations can be made by using a geometric sum instead of a geometric term. As a remark, the same phenomenon occurs when approximating the asymptotic loss probability of the $GI^X/Geo/c/N$ queue using the roots of (22) in the $GI^X/Geo/c$ queue.