A Direct Method of Solving (Linear or Nonlinear, Continuous or Discrete) Multicriteria Optimization Problems

Une méthode directe de résolution de problèmes d’optimisation multicritères (linéaires ou non linéaires, continus ou discrets)

A Thesis Submitted to the Division of Graduate Studies
of the Royal Military College of Canada
by

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In Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy

April 2018

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Acknowledgements

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The Royal Military College of Canada, 2018

My sincerest thanks go to my original supervisor, Professor Gheorghe Isac who passed away in 2010. He set the topic for me. The crux of the topic is putting in full into real applications and finding problems and solving the problems during research. His rigorous instruction, affable personality and assiduous study left a deep impression on me.

The author would especially like to thank Professor Jack Brimberg for solicitous guidance, sincere friendship, and the many hours spent discussing research in his office. I also thank him for providing some needed financial support.

Thanks to the head of the Mathematics Department of the Royal Military College of Canada, Alain Gosselin for suggestions of terminologies of inferior boundary and superior boundary, and discussing classification of application of MOP as further research in Chapter 9.

Thanks to Associate Registrar (Graduate Studies), Royal Military College of Canada, Shelagh Corbett for giving me a great moral support and help.

Thanks to DRDC, Department of National Defence, Ottawa, for its financial support.

Thanks to the Writing Centre of the Royal Military College of Canada, Terry Loveridge and Dale Tracy for helping with writing form and style in my second language (English).

Thanks to my examining committee for their work.

To Jiong Wang

Abstract

Considering the multicriteria optimization problem (MOP) and max multicriteria optimization problem (M_MOP) into real applications regardless of the image being continuous or discrete, linear or nonlinear, the present thesis explores a method, referred here as the direct method, of solving these problems in m criterion space $m \in I^+$.

The author assigns solving MOP and M_MOP to the three following stages: recording mapping characteristics stage, solving multiobjective problem stage and obtaining feasible solution stage which inverses the solution obtained in the second stage in consideration of the mapping characteristics in the first stage.

Based on the main goal of MOP and M_MOP being solving the multiobjective problem, most of the direct method focuses on the direct solving of the multiobjective problem. In solving a multiobjective problem, the author determines the (inferior and superior) boundary of the image, inferior_1 boundary set and superior_1 boundary set which are subsets of the boundary, and then filters out two subsets from inferior_1 boundary set and superior_1 boundary to obtain the nondominated sets.

The author defines inferior and superior boundaries, and inferior_1 and superior_1 boundary sets that fit both the discrete and continuous spaces, and are functional. The proof, in this thesis, shows these definitions to be identical with the classical definition of boundary when the set is dense.

In the theory, the thesis completes definitions, proofs and syllogisms that are needed to develop the direct method of solving MOP and M_MOP. In practice, since most $MOPs$ and M_MOPs are numerical in nature, the author develops an algorithm, designs code in MALTAB and uses the software in several examples. The results of running several examples (criterion space: 3, 4, 8, 10 and number of points: 147, 551, 8539, 69779) are listed in Chapter 8 and Appendix II. This research opens up applications of MOP in many fields such as management, engineering, medical science and so on, and also establishes a solid foundation for further research in the multicriteria optimization problem and max multicriteria optimization problem field.

Résumé

Considérant le problème d'optimisation multicritère (MOP) et le problème d'optimisation multicritère maximal (M_MOP) dans des applications réelles indépendamment du fait que l'image soit continue ou discrète, linéaire ou non linéaire, cette thèse explore une méthode, appelée ici méthode directe, pour résoudre ce problème dans l'espace à m critères $m \in I^+$.

La méthode présentée résout le MOP et le M_MOP en trois étapes: : l'enregistrement des caractéristiques de cartographie, la résolution d'un problème multi-objectif et l'obtention d'une solution réalisable en inversant la solution obtenue dans la seconde étape en tenant compte des caractéristiques de cartographie de la première.

Sur la base de l'objectif principal du MOP et du M_MOP la plus grande partie de l'algorithme se concentre sur la résolution directe du problème multi-objectif. Afin de résoudre le problème multiobjectif, l'algorithme identifie les bornes (inférieures et supérieures) de l'image, l'ensemble formant la borne inferior_1 et l'ensemble formant la borne superior_1 qui sont des sous-ensembles de la frontière, puis filtre deux sous-ensembles des ensembles formant les bornes inferior_1 et superior_1 afin d'identifier l'ensemble composé des points non-dominés.

L'auteur définit les frontières inférieures et supérieures, et les ensembles des bornes inferior_1 et superior_1 applicables à la fois aux espaces discrets et continus et qui sont fonctionnels. Cette thèse démontre que ces définitions sont identiques à la définition classique de frontière lorsque l'ensemble est dense.

Dans la partie théorie, cette thèse présente les définitions et les preuves nécessaires à la méthode directe de résolution du MOP et du M_MOP. En pratique, puisque la plupart des $MOPs$ et des M_MOPs sont de nature numérique, l'auteur développe un algorithme, conçoit du code dans MALTAB et utilise le logiciel dans plusieurs exemples. Les résultats de l'exécution de ces exemples (espace critère: 3, 4, 8, 10 et nombre de points: 147, 551, 8539, 69779) sont énumérés au chapitre 8 et à l'annexe II. Cette recherche rend possible l'application du MOP dans plusieurs domaines tels que, en autres, la gestion, l'ingénierie et la science médicale. Elle établit également une base solide pour la poursuite des recherches dans le domaine de l'optimisation multicritère et de l'optimisation multicritère maximale.

Contents

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List of Special Symbols

The symbols listed below are followed by brief statement of their meaning and by the number of the page on which they are defined

MOP	Multicriteria Optimization Problem	2
M_MOP	Max Multicriteria Optimization Problem	2
$\mathbf{x}=[x_1,\cdots,x_q]^T$	a decision in $q\in\mathbf{I}^+$ dimensional space	3
$\mathbf{X}$	feasible set	3
$\mathbf{f}(\mathbf{x})=\left[f_1(\mathbf{x}),\ \cdots,\ f_m(\mathbf{x})\right]^T$	a criteria or function vector with m criteria	3
$\mathbf{F}^m=\left\{\mathbf{f}(\mathbf{x})=\left[f_1(\mathbf{x}),\ \cdots,\ f_m(\mathbf{x})\right]^T:\mathbf{x}\in\mathbf{X}\right\}$	an image set in $m\in\mathbf{I}^+$ criterion space	3
$\hat{\mathbf{X}}$	solution set	4
$\mathbf{I}^+=\{1,2,\cdots\}$	is the set of natural numbers	6
$\mathcal{N}(\mathbf{S})$	the total number of members in set $\mathbf{S}$	6
$\mathbf{\Phi}=\{\}$	is the empty set	6
$\phi\triangleq(\ )\ or\ (\ ,\ )\ or\ (\ ,\ ,\ )\ or\cdots$	is null point in one, two, three, …dimensional space	6
$\phi\in\mathbf{\Phi}$		6
$\hat{\mathbf{x}}$	a solution of optimal	7
$\mathbf{f}(\hat{\mathbf{x}})$	non-dominated point	7
$\mathbf{X}_{wE}$	weakly efficient set	8
$\mathbf{X}_E$	Pareto optimal set	8
$\mathbf{X}_{sE}$	strictly efficient set	8
$\mathbf{X}_{lex}$	lexicographic optimal set	8
$\mathbf{X}_{MO}$	max-order optimal set	8
$\mathbf{F}^m_{wE}$	weakly non-dominated set	8
$\mathbf{F}^m_E$	non-dominated set	8
$\mathbf{F}^m_{sE}$	strictly non-dominated set	8
$\mathbf{F}^m_{lex}$	lexicographic non-dominated set	8
$\mathbf{F}^m_{MO}$	max-order non-dominated set	8
$\mathbf{X}_{M\_wE}$	max weakly efficient set	8
$\mathbf{X}_{M\_E}$	max efficient set	8
$\mathbf{X}_{M\_sE}$	max strictly efficient set	8
$\mathbf{F}^m_{M\_wE}$	max weakly non-dominated set	8
$\mathbf{F}^m_{M\_E}$	max non-dominated set	8

Chapter 1 A Brief History of Multicriteria Optimization

The concept of non-inferiority in the context of economics is credited to Francis Y. Edgeworth (1845-1926) and Vilfredo Pareto (1848-1923). Since then, multiobjective optimization has permeated the fields of management, economics, the social and physical sciences, engineering and design, and has developed at a rapidly increasing pace.

In one sense, the multicriteria problem was forced on the economist by having to deal with the aspirations of several consumers. The criteria are generally the utilities of individual consumers, and it was Edgeworth in 1881 who first successfully defined an optimum to such a multi-utility problem in the context of two consumers, P and π:

> A small step is required where: P and π do not increase together but that, while one increases, the other decreases.

Some years later, in 1906, Pareto took the more direct approach of ordering the decision set directly and, subsequently, defining an optimum for n consumers in the following manner:

> We will say that the members of a collectivity enjoy maximum ophelimity in a certain position when it is impossible to find a way of moving from that position very slightly in such a manner that the ophelimity enjoyed by each of the individuals of the collectivity increases or decreases. That is to say, any small displacement in departing from that position necessarily has the effect of increasing the ophelimity which certain individuals enjoy, and decreasing that which others enjoy, of being agreeable to some and disagreeable to others, see p3~p4 in [22].

This statement has been the basis for terming optimal decisions defined in this manner as Pareto optimal. Edgeworth's statement is more related to what is now termed a multicriteria problem. With this in mind, as well as historical precedent, such optima should more appropriately be termed Edgeworth-Pareto optima.

Also, the multicriteria problem was eventually united with the inception of game theory by Borel in 1921.

The optimal solution (decision) of each multicriteria problem is a set (decision set). The optimal solution, "best", has traditionally referred to the decision that minimized or maximized a single criterion in the sciences and has referred to the simultaneous optimization of several criteria in economics.

Simultaneously, the ordering is related to obtaining a decision set. The subject of ordering, however, also has a separate early mathematical history with the consideration of ordered sets by Cantor and Hausdorff. Their work on set theory inspired the extension of the natural properties of the real number system, such as total orderings, to more abstract sets.

Along with acceptance of the concept of "Multicriteria Optimization", applications have occurred not only in the field of economics, but also other fields such as engineering

[14],[20],[21], Natural Science [6], [12], [18], [23], [30], Chemistry [1], [9], and food processing [10], [17], for example, where the criterion spaces tend to be in 2 and 3 dimensions only. The applications have not involved the field of large scale management (with high-dimensional criterion spaces) such as human resource management, insurance, investment, and so on.

Exploring method of solving the Multicriteria Optimization Problem (MOP) and Max Multicriteria Optimization Problem (M_MOP) is an important area of study within the field of mathematics. For a recent example, see the popular book, "Multicriteria Optimization" [22], that systematically expounds the concept of the Multicriteria Optimization Problem and collects existing solution methods, or "Nonlinear Multiobjective Optimization" [15] that also covers in detail this vast field. Many mathematicians have published on various solution approaches and applications, e.g., see [2], [4], [5], [7], [13], [16], [19], [24] through [29].

Chapter 2 Introduction of Multicriteria Optimization

2.1 Multicriteria Optimization Problem (MOP)

A commercial example of a multicriteria optimization problem relates to car insurance. Car insurance companies keep information of insurance members: name, sex, age, location, and the years of driving, records of driving, and so on. All members construct a set $\mathbf{X}$, called the feasible set and $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}),\ \cdots,\ f_m(\mathbf{x})]^T$ is the mapping of member $\mathbf{x} \in \mathbf{X}$, such as sex ~ $f_1(\mathbf{x})$, age ~ $f_2(\mathbf{x})$, location ~ $f_3(\mathbf{x})$, the years of driving ~ $f_4(\mathbf{x})$, records of driving ~ $f_5(\mathbf{x})$, ….., called an image point, shown in Figure 1.

$$name\,\mathbf{x} \rightarrow characteristics = \begin{bmatrix} sex \\ age \\ location \\ years\ of\ driving \\ records\ of\ driving \\ \vdots \\ \vdots \end{bmatrix} \overset{\text{quantify}}{\Leftrightarrow} \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ f_3(\mathbf{x}) \\ f_4(\mathbf{x}) \\ f_5(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$$

Figure 1: Mapping of feasible point

All the points construct an image: $\mathbf{F} = \{\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}),\ \cdots,\ f_m(\mathbf{x})]^T : \mathbf{x} \in \mathbf{X}\}$ in the m dimensional criterion space. There is a minimal acceptance criteria set in the existing image $\mathbf{F}$. An applicant can be accepted as a member by the criteria set. Actually, a greatest acceptance criteria set would be one of the important policies for a car insurance company.

Another typical example of a multicriteria optimization problem is designing an automatic control system for a flying vehicle for some purposes. The derivation of a parameter vector is one of the main tasks in design. Let $\mathbf{x} = [x_1, \cdots, x_q]^T$ be a parameter vector, where $x_1, \cdots, x_q$ are components of the vector. All possible parameters construct a set: $\mathbf{X} = \{\mathbf{x} = [x_1, \cdots, x_q]^{T^T} : x_i \in [a_i, b_i], a_i, b_i \in R,\ a_i \le b_i; i = 1, \cdots, q\}$. For a given control strategy, the flying track depends on selected parameter vector $\mathbf{x}$, where the flying track, $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}),\ \cdots,\ f_m(\mathbf{x})]^T$, is an image point of the parameter vector. We would find the best flying orbit from the flying orbit set, image $\mathbf{F}^m = \{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in \mathbf{X}\}$, by obtaining the best parameter vector $\mathbf{x}$. Mostly, we cannot obtain only an image point such that every component of the point is best. On the other hand, we may obtain a set of best orbits, thus obtaining a set of best parameter vectors as our optimization solution. By the way, if we still cannot find a satisfactory orbit in the obtained set, obviously, we should consider a new control strategy or structure for the flying vehicle.

Let $\mathbf{X}$ be a military organization and $\mathbf{F}^m = \{\mathbf{f}(\mathbf{x}) : \mathbf{x} \in \mathbf{X}\}$ be a set of military qualities, where $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})]^T$ contains the military qualities of member $\mathbf{x}$ of the military organization. A commander (command system) selecting a team from staff $\mathbf{X}$ to fulfill a special task may also be considered a multicriteria optimization problem. It is a team optimization according to the special task.

We would sum up in mathematical language:

A multicriteria optimization problem, called *MOP* for short, involves a search for the set of optimal decisions of a certain type (such as Pareto optimal) in the given feasible (decision) set in decision space based on the image (objective) of the feasible set in criterion space.

The diagram in Figure 2, visualizes a typical multicriteria optimization problem.

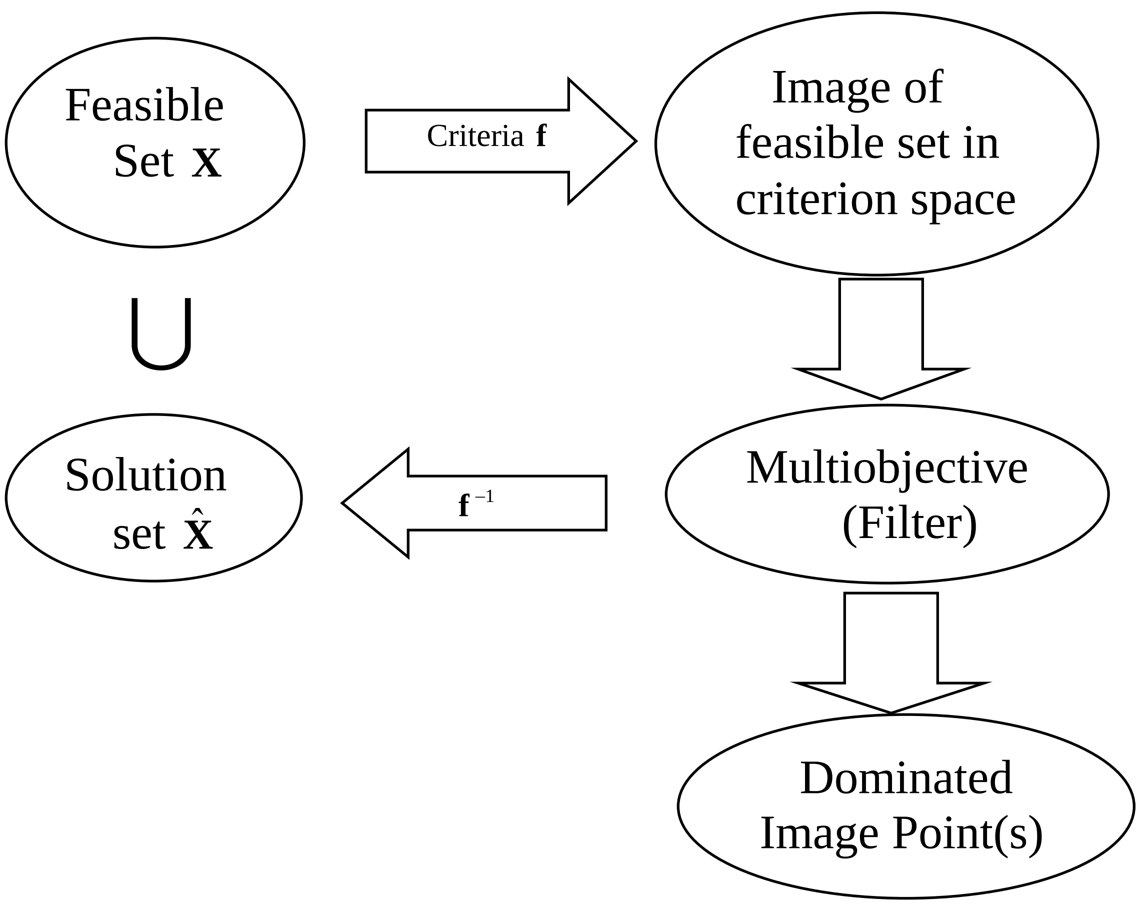

Figure 2: A typical multicriteria optimization problem

The filter is selected based on a selected definition of optimality, such as the case of Pareto optimality. We see that MOP is concerned with three main factors: feasible set, the image of the feasible set and filter. **The multicriteria optimization determines which part of the image is retained; then the set of optimal solutions is given by the inverse of this subset of image points**.

Since every image point is a multi-dimensional point, there may be a special order between two different image points. The popular names of the orders will be given in the next section.

2.2 Some Orders, Addresses and Implications

The most popular orders, their addresses and implications are listed in this section. We list some orders that are generally used on R^m as follows.

Let $\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_n \end{pmatrix} = \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,m} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ y_{n,1} & y_{n,2} & & y_{n,m} \end{pmatrix}$ be an order of n points in R^m, where $m \geq 1$ and

if the Order – Notation	*definition*	*call the order as (Address)*
$\mathbf{y}_i < \mathbf{y}_{i+1}$	$y_{i,j} < y_{i+1,j}$, $j = 1, \cdots, m$	strict component– wise order
$\mathbf{y}_i \leq \mathbf{y}_{i+1}$	$\begin{cases} y_{i,l} = y_{i+1,l} & \exists \text{some } l \in \{1, \cdots, m\} \\ y_{i,j} < y_{i+1,j} & j \neq l,\ j \in \{1, \cdots, m\} \end{cases}$	component– wise order
$\mathbf{y}_i \leqq \mathbf{y}_{i+1}$	$\begin{cases} \mathbf{y}_k = \mathbf{y}_{k+1} & \exists \text{some } k \in \{1, \cdots, i\} \\ \mathbf{y}_l \leq \mathbf{y}_{l+1} & l \neq k, l \in \{1, \cdots, i\} \end{cases}$	weak component– wise order
$\mathbf{y}_i \leq_{\text{lex}} \mathbf{y}_{i+1}$	$\begin{cases} \exists\, l \in \{1, \cdots, m-1\} \\ y_{i,j} = y_{i+1,j}, \text{ when } j = 1, \cdots, l \\ y_{i,l+1} < y_{i+1,l+1} \end{cases}$	lexicographic order
$\mathbf{y}_i \leq_{MO} \mathbf{y}_{i+1}$	$\max\limits_{j=1,\cdots,m} y_{i,j} \leq \max\limits_{j=1,\cdots,m} y_{i+1,j}$	max - order

where $i \in \{1, \cdots, n-1\}$ and $\mathbf{y}_k = \mathbf{y}_{k+1}$, *i.e.*, $y_{k,j} = y_{k+1,j}$ *for* $j = 1, \cdots, m$.

The implications of each order below is shown

Order – Notation	*is called*
$\mathbf{y} \leqq \mathbf{y}^*$	point $\mathbf{y}$ weakly dominates point $\mathbf{y}^*$
$\mathbf{y} \leq \mathbf{y}^*$	point $\mathbf{y}$ dominates point $\mathbf{y}^*$
$\mathbf{y} < \mathbf{y}^*$	point $\mathbf{y}$ strictly dominates point $\mathbf{y}^*$
$\mathbf{y} \leq_{\text{lex}} \mathbf{y}^*$	point $\mathbf{y}$ lexicographically dominates point $\mathbf{y}^*$
$\mathbf{y} \leq_{MO} \mathbf{y}^*$	point $\mathbf{y}$ max-orderly dominates point $\mathbf{y}^*$

2.3 Postulates and Notation

The notation used in this thesis is summarized below:

$A \underline{\underline{\Delta}} B -$ B is defined as A.

$\mathcal{N}(\mathbf{S}) -$ the total number of members in set $\mathbf{S}$.

$\mathbf{I}^+ = \{1, 2, \cdots\}$ is the set of natural numbers.

$\mathbf{\Phi} = \{\} -$ is the empty set.

$\phi \underline{\underline{\Delta}} (\ ) \ or \ (\ ,\ ) \ or \ (\ ,\ ,\ ) \ or \cdots -$ is null point in one, two, three, …dimensional space.

$\phi \in \mathbf{\Phi}$.

$\mathbf{X} -$ the set of all the decisions.

If $\mathbf{X}$ is discrete,

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_H \end{bmatrix}, \text{ where } H \in \mathbf{I}^+ + 1.$$

$\mathbf{x}_i = [x_{i,1} \cdots x_{i,q}] \in \mathbf{X} \subseteq R^q$ is a decision in $q \in \mathbf{I}^+$ dimensional space.

$\mathbf{f} -$ a criteria or function vector with m criteria or function components.

$\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}) \ \cdots \ f_m(\mathbf{x})] \in R^m$, $\mathbf{x} \in \mathbf{X} -$ an image point of $\mathbf{x}$ in $m \in \mathbf{I}^+$ space.

$\mathbf{F}^m = \{\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}) \ \cdots \ f_m(\mathbf{x})] : \mathbf{x} \in \mathbf{X}\} \subset R^m -$ an image set in $m \in \mathbf{I}^+$ criterion space.

2.4 Formulating the Multiobjective Optimization Problem (MOP) and the Max Multiobjective Optimization Problem (M_MOP)

In mathematical terms, a classical multicriteria optimization problem can be formulated as,

$$\min_{\mathbf{x} \in \mathbf{X}} \mathbf{f}(\mathbf{x}), \tag{2.1}$$

where

$\mathbf{f} : \mathbf{X} \to R^m, \mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}) \ \ f_2(\mathbf{x}) \ \cdots \ f_m(\mathbf{x})], m \geq 1,$

$\mathbf{X} \subset R^q$, $q \geq 1$ is the feasible set,

$f_j(\mathbf{x})$ is objective function j,
$j = 1, 2, \cdots, m$.

Since the objective functions may be maximized in some applications, we follow definition (2.1) to define the max multicriteria optimization problem (M_MOP) as

$$\max_{\mathbf{x}\in\mathbf{X}} \mathbf{f}(\mathbf{x}), \tag{2.2}$$

where

$\mathbf{f}: \mathbf{X} \to R^m, \mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}) \quad f_2(\mathbf{x}) \quad \cdots \quad f_m(\mathbf{x})], m \geq 1$,
$\mathbf{X} \subset R^q$, $q \geq 1$ is the feasible set,
$f_j(\mathbf{x})$ is objective function j,
$j = 1, 2, \cdots, m$.

Set $\mathbf{F}^m = \{\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}) \quad f_2(\mathbf{x}) \quad \cdots \quad f_m(\mathbf{x})] : \mathbf{x} = [x_1, \cdots, x_q] \in \mathbf{X}\}$ is referred to as the image of the feasible set $\mathbf{X}$, or criterion space from which the criterion values are taken.

2.5 Classification and Relations of the Solutions of the Multicriteria Optimization Problem (MOP)

2.5.1 Classification of Solution of MOP

We list the main classes of solution of MOP as follows [22]

a feasible solution $\hat{\mathbf{x}} \in \mathbf{X}$ is called	*if there is no other $\mathbf{x} \in \mathbf{X}$ such that*
weakly efficient,	$\mathbf{f}(\mathbf{x}) < \mathbf{f}(\hat{\mathbf{x}})$.
efficient or Pareto optimal,	$\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\hat{\mathbf{x}})$.
strictly efficient,	$\mathbf{f}(\mathbf{x}) \leqq \mathbf{f}(\hat{\mathbf{x}})$.
lexicographic optimal,	$\mathbf{f}(\mathbf{x}) \leq_{\text{lex}} \mathbf{f}(\hat{\mathbf{x}})$.
max - order optimal,	$\mathbf{f}(\mathbf{x}) \leq_{MO} \mathbf{f}(\hat{\mathbf{x}})$.

Also we denote

if $\hat{\mathbf{x}} \in \mathbf{X}$ *is*	$\{all\ \hat{\mathbf{x}} \in \mathbf{X}\}$	point $\mathbf{f}(\hat{\mathbf{x}})$	$\{all\ \mathbf{f}(\hat{\mathbf{x}}) \in \mathbf{F}^m\}$
weakly efficient	set $\mathbf{X}_{wE}$	weakly non-dominated	set $\mathbf{F}^m_{wE}$
efficient or Pareto optimal	set $\mathbf{X}_E$	non-dominated	set $\mathbf{F}^m_E$
strictly efficient	set $\mathbf{X}_{sE}$	strictly non-dominated	set $\mathbf{F}^m_{sE}$
lexicographic optimal	set $\mathbf{X}_{lex}$	lexicographic non-dominated	set $\mathbf{F}^m_{lex}$
max-order optimal	set $\mathbf{X}_{MO}$	max-order non-dominated	set $\mathbf{F}^m_{MO}$

The main classes of solution of M_MOP are listed as follows:

a feasible solution $\hat{\mathbf{x}} \in \mathbf{X}$ *is called*	*if there is no other* $\mathbf{x} \in \mathbf{X}$ *such that*
max weakly efficient,	$\mathbf{f}(\mathbf{x}) > \mathbf{f}(\hat{\mathbf{x}})$.
max efficient,	$\mathbf{f}(\mathbf{x}) \geq \mathbf{f}(\hat{\mathbf{x}})$.
max strictly efficient,	$\mathbf{f}(\mathbf{x}) \geqq \mathbf{f}(\hat{\mathbf{x}})$.

Also note that

if $\hat{\mathbf{x}} \in \mathbf{X}$ *is*	$\{all\ \hat{\mathbf{x}} \in \mathbf{X}\}$	point $\mathbf{f}(\hat{\mathbf{x}})$	$\{all\ \mathbf{f}(\hat{\mathbf{x}}) \in \mathbf{F}^m\}$
weakly max efficient	set $\mathbf{X}_{M\_wE}$	max weakly non-dominated	set $\mathbf{F}^m_{M\_wE}$
max efficient	set $\mathbf{X}_{M\_E}$	max non-dominated	set $\mathbf{F}^m_{M\_E}$
strictly max efficient	set $\mathbf{X}_{M\_sE}$	max strictly non-dominated	set $\mathbf{F}^m_{M\_sE}$

Aside from these classes, there are four typical definitions of properly efficient, known as properly efficient in Geoffrion's sense, in Borwein's sense and in Benson's sense as well as in Kuhn –Tucker's sense. They were derived in year 1968, 1977, 1979, and 1951 respectively.

Properly Efficient (in Geoffrion's sense) A feasible solution $\hat{\mathbf{x}} \in \mathbf{X}$ is called properly efficient in Geoffrion's sense, if it is efficient and if there is a real number $M > 0$ such that for all i and $\mathbf{x} \in \mathbf{X}$ satisfying $f_i(\mathbf{x}) < f_i(\hat{\mathbf{x}})$ there exists an index j with $f_j(\hat{\mathbf{x}}) < f_j(\mathbf{x})$ such that

$$\frac{f_i(\hat{\mathbf{x}}) - f_i(\mathbf{x})}{f_j(\mathbf{x}) - f_j(\hat{\mathbf{x}})} \leq M .$$

Properly Efficient (in Borwein's sense) A feasible solution $\hat{\mathbf{x}} \in \mathbf{X}$ is called properly efficient in Borwein's sense, if

$$T_{\mathbf{Y}+R^m_{\geqq}}(\mathbf{f}(\hat{\mathbf{x}})) \cap (-R^m_{\geqq}) = \{0\} ,$$

where the tangent cone of $\mathbf{Y} + R^m_{\geqq}$ at $\mathbf{f}(\hat{\mathbf{x}}) \in \mathbf{Y} + R^m_{\geqq}$ is

$$T_{\mathbf{Y}+R^m_{\geqq}}(\mathbf{f}(\hat{\mathbf{x}})) = \{\mathbf{d} \in R^m : \exists\{t_k\} \subset R^+, \{\mathbf{y}^k\} \subset \mathbf{Y} + R^m_{\geqq} \ s.t. \lim_{\mathbf{y}^k \to \mathbf{f}(\hat{\mathbf{x}})} t_k(\mathbf{y}^k - \mathbf{f}(\hat{\mathbf{x}})) = \mathbf{d}\}$$

and

$$R^m_{\geqq} = \{\mathbf{y} = [y_1 \cdots, y_m]^T \in R^m : y_i \geq 0, i = 1, \cdots, m\}.$$

Properly Efficient (in Benson's sense) A feasible solution $\hat{\mathbf{x}} \in \mathbf{X}$ is called properly efficient in Benson's sense, if

$$\mathrm{cl}\left(\mathrm{cone}\left(\mathbf{Y} + R^m_{\geqq} - \mathbf{f}(\hat{\mathbf{x}})\right)\right) \cap (-R^m_{\geqq}) = \{0\},$$

where the conical hull $\mathbf{Y} + R^m_{\geqq} - \mathbf{f}(\hat{\mathbf{x}})$ is

$$\mathrm{cone}\left(\mathbf{Y} + R^m_{\geqq} - \mathbf{f}(\hat{\mathbf{x}})\right) = \{\alpha\mathbf{y} : \alpha \geq 0, \mathbf{y} \in \mathbf{Y} + R^m_{\geqq} - \mathbf{f}(\hat{\mathbf{x}})\} = \bigcup_{\alpha \geq 0} \alpha\left(\mathbf{Y} + R^m_{\geqq} - \mathbf{f}(\hat{\mathbf{x}})\right).$$

Properly Efficient (in Kuhn –Tucker's sense) A feasible solution $\hat{\mathbf{x}} \in \mathbf{X}$ is called properly efficient in Kuhn –Tucker's sense, if it is efficient and if there is no $\mathbf{d} \in R^n$ satisfying

$$\begin{aligned}
&\left\langle \nabla f_k(\hat{\mathbf{x}}), \mathbf{d} \right\rangle \leq 0 \quad \forall k = 1, \cdots, m \\
&\left\langle \nabla f_i(\hat{\mathbf{x}}), \mathbf{d} \right\rangle < 0 \quad \text{for some } i \in \{1, \cdots, m\} \\
&\left\langle \nabla g_j(\hat{\mathbf{x}}), \mathbf{d} \right\rangle \leq 0 \quad \forall j \in \{p : g_p(\hat{\mathbf{x}}) = 0, \ p = 1, \cdots, m\},
\end{aligned}$$

where constraint functions $g_j, \ j = 1, \cdots, m$ are continuously differentiable.

Properly Efficient (in Kuhn –Tucker's sense) considers the multiobjective programme

$$\min \mathbf{f}(\mathbf{x})$$

$\mathbf{x} \in \mathbf{X}$,
$\mathbf{X}$: feasible set. Now we add constraint $\mathbf{g}(\mathbf{x}) \leq 0$,

where $\mathbf{f} : R^n \to R^m$ and $\mathbf{g} : R^n \to R^m$.

The corresponding point $\mathbf{f}(\hat{\mathbf{x}})$ is called **properly non-dominated** in Geoffrion's sense, Borwein's sense, Benson's sense, and Kuhn –Tucker's sense, respectively.

2.5.2 The Relations of Solutions of *MOP*

We find that (according to [22])

- $\mathbf{X}_{sE} \subset \mathbf{X}_{E} \subset \mathbf{X}_{wE}$ (pp. 39).
- $\mathbf{X}_{lex} \subset \mathbf{X}_{E}$ (pp. 129).
- $\mathbf{X}_{MO} \subset \mathbf{X}_{E}$ (pp. 132).

- $\{\hat{\mathbf{x}} \in \mathbf{X} : \hat{\mathbf{x}} \textit{ is properly efficient in Geoffrion's sense}\}$
  $= \{\hat{\mathbf{x}} \in \mathbf{X} : \hat{\mathbf{x}} \textit{ is properly efficient in Benson's sense}\}$ (pp. 55)
  $\subset \{\hat{\mathbf{x}} \in \mathbf{X} : \hat{\mathbf{x}} \textit{ is properly efficient in Borwein's sense}\} \subset \mathbf{X}_{E}$ (pp. 54).

- A differentiable MOP,
  $\{\hat{\mathbf{x}} \in \mathbf{X} : \hat{\mathbf{x}} \textit{ is properly efficient in Geoffrion's sense}\}$
  $\subset \{\hat{\mathbf{x}} \in \mathbf{X} : \hat{\mathbf{x}} \textit{ is properly efficient in Kuhn and Tucker's sense}\}$ (pp. 57).

- $\{\hat{\mathbf{x}} \in \mathbf{X} : \hat{\mathbf{x}} \textit{ is properly efficient in Kuhn and Tucker's sense}\}$
  $\subset \{\hat{\mathbf{x}} \in \mathbf{X} : \hat{\mathbf{x}} \textit{ is properly efficient in Geoffrion's sense}\}$, if $f_k, g_j : R^n \to R$ are convex, continuously differentiable functions (pp. 58).

Chapter 3 Popular Methods for Accomplishing Multiobjective Optimization MOP in Finite-Dimensional Euclidean Space

There are a large variety of methods for accomplishing multiobjective optimization in finite-dimensional Euclidean space. These methods are classified as scalarizing and non-scalarizing methods. The typical methods in these two classes will be described in the following sections.

3.1 Scalarization Method

The scalarizing method is a traditional method for solving MOP of the Pareto class. The method involves formulating a single objective optimization problem that is related to the MOP

$$\min_{\mathbf{x}\in\mathbf{X}} \mathbf{f}(\mathbf{x}) = \min_{\mathbf{x}\in\mathbf{X}} \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$$

by means of a real-valued scalarizing function typically being a function of the objective functions of the MOP, auxiliary scalar or vector variables, and/or scalar or vector parameters. Sometimes the feasible set of the MOP is additionally restricted by new constraint functions related to the objective functions of the MOP and/or the new variables introduced.

The scalarizing methods typically include the Weighted Sum Method, the $\varepsilon-$Constraint method, the Hybrid Method, the Elastic Constraint Method, Benson's Method and Compromise Solutions. These comprise the most popular techniques for solving MOP.

The Weighted Sum Method is perhaps the "simplest" method that solves multicriteria optimization problem (2.1) by solving a single objective problem of the type:

$$\min_{\mathbf{x}\in\mathbf{X}} \left(\langle \boldsymbol{\lambda}, \mathbf{f}(\mathbf{x}) \rangle = \sum_{k=1}^{m} \lambda_k f_k(\mathbf{x}) \right), \tag{3.1}$$

where

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix} \in R^m \text{ and}$$

$\langle \boldsymbol{\lambda}, \mathbf{f}(\mathbf{x}) \rangle$ denotes the scalar product in R^m.

The main contributor to this method is Geoffrion (1968), who shows the relation of optimal solutions of (3.1) and (2.1):

- If $\hat{\mathbf{x}}$ is an optimal solution of the weighted sum optimization problem (3.1), then the following statements hold

  (1) If $\boldsymbol{\lambda} \in R^m_{\geq}$ then $\hat{\mathbf{x}}$ is a weakly efficient solution of MOP (2.1).

  (2) If $\boldsymbol{\lambda} \in R^m_{>}$ then $\hat{\mathbf{x}}$ is an efficient solution of MOP (2.1), where $R^m_{>} = \{\mathbf{y} = [y_1 \cdots, y_m]^T \in R^m : y_i > 0, i = 1, \cdots, m\}$.

  (3) If $\boldsymbol{\lambda} \in R^m_{\geq}$ and $\hat{\mathbf{x}}$ is a unique optimal solution of (3.1) then $\hat{\mathbf{x}}$ is a strictly efficient solution of MOP (2.1).

The $\varepsilon -$ **Constraint method** is probably the best known technique to solve multicriteria optimization problems. There is no aggregation of criteria, instead only one of the original objectives is minimized, while the others are transformed to constraints. It was introduced by Haimes et al. (1971), and an extensive discussion can be found in Chankong and Haimes (1983).

The $\varepsilon -$ Constraint method to replace the multicriteria optimization problem (2.1) is given by:

$$\min_{\mathbf{x} \in \mathbf{X}} f_j(\mathbf{x}) \tag{3.2}$$
$$\textit{subject to } f_k(\mathbf{x}) \leq \varepsilon_k \quad k = 1, \cdots, m, \quad k \neq j,$$

where

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_m \end{bmatrix} \in R^m .$$

The optimal solutions of (3.2) problems are at least weakly efficient. This method works for general problems; no convexity assumption is needed. It has been shown that

- If $\hat{\mathbf{x}}$ is an optimal solution of (3.2) for some j, then $\hat{\mathbf{x}}$ is a weakly efficient solution of MOP (2.1).
- If $\hat{\mathbf{x}}$ is a unique optimal solution of (3.2) for some j, then $\hat{\mathbf{x}}$ is a strictly efficient solution (and therefore efficient solution) of MOP (2.1).

The Hybrid Method combines the weighted sum method with the $\varepsilon-$Constraint method. Let $\mathbf{x}^0$ be an arbitrary feasible point for a MOP.

$$\begin{aligned}&\min \sum_{k=1}^{m} \lambda_k f_k(\mathbf{x}) \qquad (3.3)\\ &\textit{subject to } f_k(\mathbf{x}) \le f_k(\mathbf{x}^0) \quad k = 1, \cdots, m\\ &\mathbf{x} \in \mathbf{X}\end{aligned}$$

where

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix} \in R^m_{\geq}.$$

A theorem relating the solutions of (3.3) and (2.1), given by Guddat et al. (1985), shows:

- A feasible solution $\mathbf{x}^0 \in \mathbf{X}$ with $\boldsymbol{\lambda} \in R^m_{>}$ is an optimal solution of problem (3.3) if and only if $\mathbf{x}^0$ is an efficient solution of (2.1).

Because there are no results on properly efficient solutions from the $\varepsilon-$constraint method, which is also difficult to solve in practice due to the added constraints $f_k(\mathbf{x}) \le \varepsilon_k$, Ehrgott and Ryan [8] introduced "relaxing" of these constraints by allowing them to be violated and penalizing any violation in the objective function to develop the elastic constraint scalarization.

The Elastic Constraint Method

$$\begin{aligned}&\min \left(f_j(\mathbf{x}) + \sum_{k \neq j} \mu_k s_k \right) \qquad (3.4)\\ &\textit{subject to } f_k(\mathbf{x}) - s_k \le \varepsilon_k, \quad k \neq j\\ &s_k \geq 0, \quad k \neq j\\ &\mathbf{x} \in \mathbf{X}\end{aligned}$$

where

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{bmatrix} \in R^m,\ \mu_k \geq 0, k \neq j\ ,\ \mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{bmatrix} \in R^m,\ \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_m \end{bmatrix} \in R^m .$$

It is shown that (pp. 103, [22]):

- If $(\hat{\mathbf{x}}, \hat{\mathbf{s}})$ is an optimal solution of (3.4) with $\boldsymbol{\mu} \geq \mathbf{0}$. Then $\hat{\mathbf{x}}$ is a weakly efficient solution of MOP (2.1).
- If $\hat{\mathbf{x}}$ is a unique optimal solution of (3.4), then $\hat{\mathbf{x}}$ is a strictly efficient solution of MOP (2.1).

Benson's Method was formulated by Benson [3]. The idea is to choose some initial feasible solution $\mathbf{x}^0 \in \mathbf{X}$ and, if it is not itself efficient, then produce a dominating solution that is. To do so, nonnegative deviation variables are defined, and their sum is maximized. This results in a solution $\mathbf{x}$ dominating $\mathbf{x}^0$, if one exists, and the objective ensures that it is efficient by pushing $\mathbf{x}$ as far from $\mathbf{x}^0$ as possible in the feasible region $\mathbf{X}$.

$$\begin{aligned} &\max \sum_{k=1}^{m} l_k \\ &\textit{subject to } f_k(\mathbf{x}^0) - l_k - f_k(\mathbf{x}) = 0 \quad k = 1, \cdots, m \\ &\mathbf{x} \in \mathbf{X} \end{aligned} \tag{3.5}$$

where

$$\mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{bmatrix} \in R^m_{\geqq} .$$

It is known that (pp. 107, [22])

- If (3.5) has optimal solution $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$ (and the optimal objective value is finite) then $\hat{\mathbf{x}}$ is an efficient solution of MOP (2.1).

Compromise Solutions given a distance measure

$$d : R^m \times R^m \to R_{\geqq} ,$$

the compromise programming problem is given by

$$\min_{\mathbf{x}\in\mathbf{X}} d(\mathbf{f}(\mathbf{x}),\mathbf{y}^I), \tag{3.6}$$

where

$$\mathbf{y}^I = \begin{bmatrix} y_1^I \\ y_2^I \\ \vdots \\ y_m^I \end{bmatrix},$$

$$y_k^I := \min_{\mathbf{x}\in\mathbf{X}} f_k(x).$$

The following basic results are established (pp. 111, [22]).

- If a norm $\|\cdot\|$ is monotonic and $\hat{\mathbf{x}}$ is an optimal solution of (3.6) then $\hat{\mathbf{x}}$ is a weakly efficient solution of MOP (2.1). If $\hat{\mathbf{x}}$ is a unique optimal solution of (3.6) then $\hat{\mathbf{x}}$ is an efficient solution of MOP (2.1).
- If a norm $\|\cdot\|$ is strictly monotonic and $\hat{\mathbf{x}}$ is an optimal solution of (3.6) then $\hat{\mathbf{x}}$ is an efficient solution of MOP (2.1).

The following definitions apply above:

a) A norm $\|\cdot\|: R^m \to R^m_{\geqq}$ is called monotone, if $\|\mathbf{y}^1\| \le \|\mathbf{y}^2\|$ holds for all $\mathbf{y}^1, \mathbf{y}^2 \in R^m$ if $|y_k^1| \le |y_k^2|, k = 1, \cdots, m$.

b) A norm $\|\cdot\|: R^m \to R^m_{\geqq}$ is called strictly monotone, if $\|\mathbf{y}^1\| < \|\mathbf{y}^2\|$ holds whenever $|y_k^1| \le |y_k^2|, k = 1, \cdots, m$ and $|y_j^1| < |y_j^2|$ for some j.

3.2 Nonscalarization Method

The nonscalarizing method, in general, includes Lexicographic Optimality, Max-Ordering Optimality and Lexicographic Max-Ordering Optimality. We enumerate these methods as follows.

Lexicographic Optimality, the lexicographic order is considered when comparing objective vectors in criterion space. As for efficiency, the model map is the identity map. The lexicographic optimization problem is written with a "lexmin" operator as:

$$lex\min_{\mathbf{x}\in\mathbf{X}} \mathbf{f}(\mathbf{x}), \tag{3.7}$$

where

$$\mathbf{f}:\mathbf{X}\to R^m,\ \mathbf{f}(\mathbf{x})=\begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix},\ m\geq 1.$$

A solution $\hat{\mathbf{x}}\in\mathbf{X}$ is called lexicographically optimal or a lexicographic solution if

$\mathbf{f}(\hat{\mathbf{x}})\leq_{lex}\mathbf{f}(\mathbf{x})$ for all $\mathbf{x}\in\mathbf{X}$.

The lexicographic solution is connected to the efficient solution by the proven statements (pp.129 and pp.131, [22]):

- If $\hat{\mathbf{x}}\in\mathbf{X}$ is lexicographically optimal then $\hat{\mathbf{x}}$ is an efficient solution of MOP.
- If $\hat{\mathbf{x}}\in\mathbf{X}$ is a unique lexicographically optimal of (3.7) for some $k\in\{1,\cdots,m\}$, then $\hat{\mathbf{x}}$ is a strictly efficient solution of MOP.

Max-Order Optimality. The MOP problem is written as follows:

$$\min_{\mathbf{x}\in\mathbf{X}}\ \max_{k=1,\cdots,m} f_k(\mathbf{x}). \tag{3.8}$$

A solution $\hat{\mathbf{x}}\in\mathbf{X}$ is max-order optimal or a max-order solution if there is no $\mathbf{x}\in\mathbf{X}$ such that $\max_{k=1,\cdots,m} f_k(\mathbf{x})<\max_{k=1,\cdots,m} f_k(\hat{\mathbf{x}})$. A weakly efficient solution of MOP can be constructed by a max-order solution of (3.8):

- An optimal solution of the max-ordering problem (3.8) is a weakly efficient solution of MOP (2.1) but not necessarily an efficient one of MOP (2.1).

Lexicographic Max-Order Optimality. This is a combination of max-ordering and lexicographic optimality, where the lexicographic order is applied to a non-increasingly ordered sequence of the objectives.

A lexicographic max-ordering optimization problem can be written as

$$\min_{\mathbf{x}\in\mathbf{X}} sort\ (\mathbf{f}(\mathbf{x})), \tag{3.9}$$

where

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}, \qquad sort(\mathbf{f}(\mathbf{x})) = \begin{bmatrix} \theta_1(\mathbf{x}) \\ \theta_2(\mathbf{x}) \\ \vdots \\ \theta_m(\mathbf{x}) \end{bmatrix},$$

$$\theta_i(\mathbf{x}) \in \{ f_1(\mathbf{x}),\ f_2(\mathbf{x}), \cdots,\ f_m(\mathbf{x})\},\ i = 1, 2, \cdots, m$$

and

$$\theta_1(\mathbf{x}) \geq \theta_2(\mathbf{x}) \geq \cdots \geq \theta_m(\mathbf{x}).$$

If $\hat{\mathbf{x}} \in \mathbf{X}$ satisfies

$$sort\ (\mathbf{f}(\hat{\mathbf{x}})) \leq_{lex} sort\ (\mathbf{f}(\mathbf{x})) \text{ for all } \mathbf{x} \in \mathbf{X}$$

then $\hat{\mathbf{x}} \in \mathbf{X}$ is a solution of (3.9) or a lexicographic max-ordering solution. It has been shown that (pp.136, [22]):

- If $\hat{\mathbf{x}} \in \mathbf{X}$ is a lexicographic max-ordering solution then $\hat{\mathbf{x}}$ is an efficient solution of *MOP* (2.1).
- If $\hat{\mathbf{x}} \in \mathbf{X}$ is a unique lexicographically optimal solution of (3.9) for some $k \in \{1, \cdots, m\}$, then $\hat{\mathbf{x}}$ is a strictly efficient solution of *MOP* (2.1).

Chapter 4 Comments on Existing Methods of Solving MOP, MOP Analysis and A Direct Method Exploration for Solving MOP and M_MOP

4.1 Comments on the Existing Methods of Solving MOP

Based on the published books and papers in the list of References, we find that the existing methods to solve MOP are generally stuck in a criterion space of only two dimensions, called bi-objective optimization, and the points from this on the space are small in number. Several papers [4] [5] [14] [26] consider three dimensional criterion space, but again the image points are few in number.

In the current world, we see that the criterion spaces of most MOPs are greater than three, and the points on the criterion space are huge no matter whether in the fields of management, science or engineering. Insurance (section 2.1), human resource management (of a country), investment, and data processing are a few examples. These published methods may not be suitable to solve this kind of MOP, with criterion space greater than three (e.g., may be 10, 20, …), and with huge numbers of image points in their criterion space. For example, an immigration system (management system) faces too many people, i.e., feasible set and the personal information, i.e., image in criterion space. In order to make an optimal (better) decision, the immigration system should collect more personal information, i.e., increasing the dimension of the criterion space. Scientific experimentation data processing systems also have huge numbers of image points in high criterion space. A flight test system will collect many kinds of data; the data will be treated by the scientific experimentation data processing system. These published methods do not consider these larger instances of MOP occurring in practice.

By the way, if we need to solve a MOP, we are unsure of using one existing method to solve the MOP over another, because we don't know which method is most suitable. None of these methods provides an application purview or has a clear concept of scene, and none of them can be said to be generally superior to all the others [15].

Some methods need the image of a feasible set being convex; it means that we should prove the image is convex at the beginning! If we can, we have already found the solution of a MOP.

After the exploration in chapters 4 and 5, we find the fact: **the solution set of the Multicriteria Optimization problem is the inverse of the partial boundary set of the image which is mapped by the feasible set**. **The key of solving the Multicriteria Optimization is finding the partial boundary of the image.**

The application fields of MOP will be broader if we can find a method that can get optimal solutions for all different purposes (filters) such as shown in section 2.1.

4.2 Direct Method Exploration for Solving MOP and M_MOP

By Figure 2, a MOP (or M_MOP) can be divided into the three stages: mapping stage, filtering stage and inversing stage. The mapping stage, besides recording the feasible set and image, records the mapping relationship; the filtering stage filters image points, i.e., solves the multiobjective problem; and the inversing stage inverses the filtered image points (in accordance with the record provided by mapping stage) to obtain the solution of MOP (or M_MOP). To begin, we give an example created by Göpfert and Nehse to show the working of the new method and obtain some hints in exploring this direct method of solving MOP or M_MOP.

Example 4.1 (Göpfert and Nehse [11]). Consider a bicriterion optimization problem with feasible set

$$\mathbf{X} = \left\{ \mathbf{x} = (x_1, x_2) \in R^2 \middle| \begin{array}{c} -1 \le x_1 \le 1, \\ -\sqrt{-x_1^2 + 1} < x_2 \le 0, \quad if\ -1 \le x_1 \le 0 \\ -\sqrt{-x_1^2 + 1} \le x_2 \le 0, \quad if\ \ 0 < x_1 \le 1 \end{array} \right\}$$

and objective function

$$\mathbf{f}(\mathbf{x} = (x_1, x_2)) = (f_1(\mathbf{x}), f_2(\mathbf{x})) = (x_1, x_2)\,.$$

The image of the feasible set is

$$\mathbf{F}^2 = \{\mathbf{f}(\mathbf{x}) = \mathbf{x} = (x_1, x_2) : \mathbf{x} \in \mathbf{X} \subset R^2\}.$$

The solution of the problem MOP (2.1) has non-dominated set Φ and the efficient set is Φ.

To find the solution, we individually assign the work to the three stages as the follows.

(1) Mapping stage records feasible set, image and mapping relationship of feasible point and image point.

In this example, $\mathbf{f}$ maps $\mathbf{X}$ onto $\mathbf{F}^2$, where $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x})) = \mathbf{x} = (x_1, x_2)$. See that $\mathbf{f}$ is a $1-1$ $(one-to-one)$, i.e., is injective mapping. The image of $\mathbf{X}$ is

$$\mathbf{F}^2 = \left\{ \mathbf{f} = (f_1, f_2) \in R^2 \middle| \begin{array}{c} -1 \le f_1 \le 1, \\ -\sqrt{-f_1^2 + 1} < f_2 \le 0, \quad if\ -1 \le f_1 \le 0 \\ -\sqrt{-f_1^2 + 1} \le f_2 \le 0, \quad if\ \ 0 < f_1 \le 1 \end{array} \right\}.$$

We set a two dimensional coordinate system: $of_1 f_2$ for the image. The image is shown in Figure 3. It is a dense half disk; there is no boundary down the quarter of the disk in the left of the half disk.

(2) The filtering stage filters all points of the image to find the non-dominated set. This job, in general, is called solving the multiobjective problem.

For a given feasible point $\mathbf{x} = (x_1, x_2) \in \mathbf{X}$, its image point is $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$. Observe in Figure 3 that we construct two sets that pass the image point:

$$\mathbf{F}_1^{\mathbf{L}}(\mathbf{x}) = \{(t, f_2(\mathbf{x})) : t \in R\} \cap \mathbf{F}^2 = \{(t, f_2) = (t, x_2) : t \in R\} \cap \mathbf{F}^2,$$

and

$$\mathbf{F}_2^{\mathbf{L}}(\mathbf{x}) = \{(f_1(\mathbf{x}), t) = (x_1, t) : t \in R\} \cap \mathbf{F}^2.$$

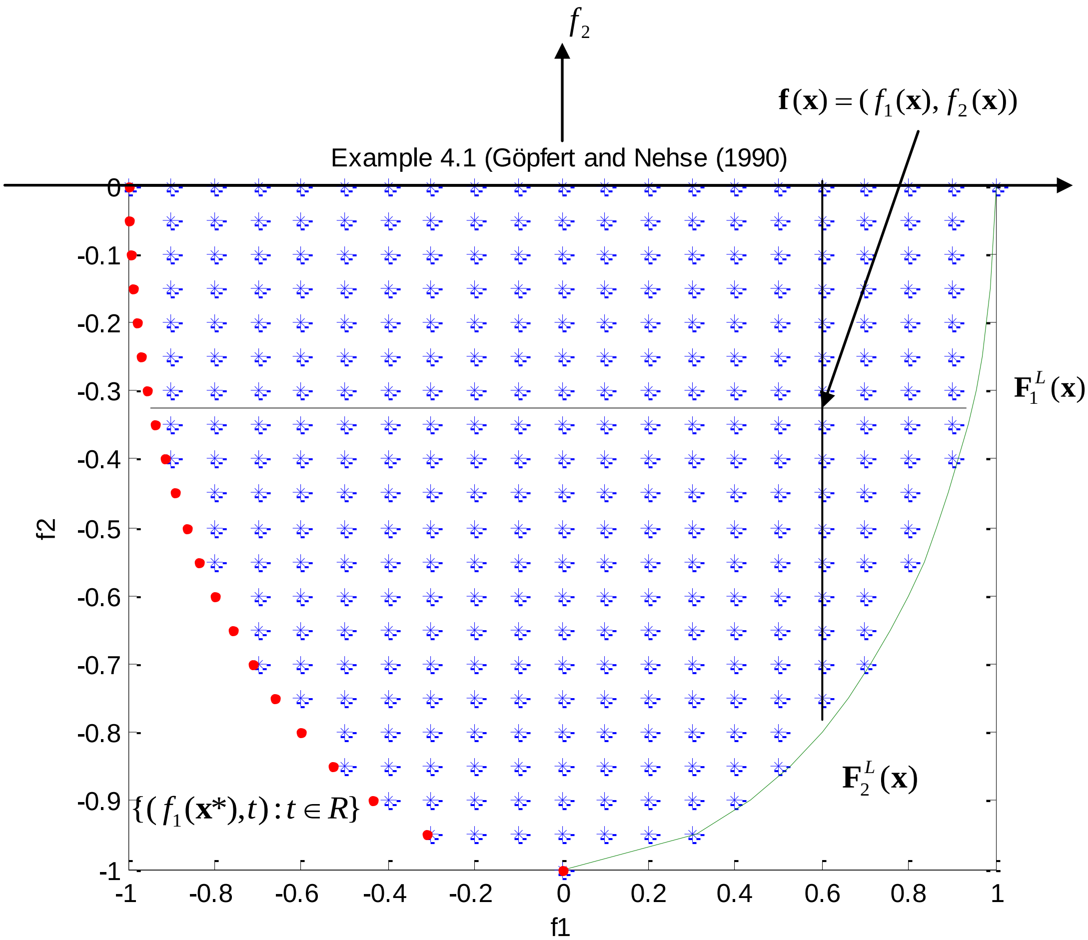

Figure 3: The image of feasible set $\mathbf{X}$

In geometry, the two sets may not be closed segments that are perpendicular to each other, where line $\{(t, f_2(\mathrm{x})): t \in R\}$ parallels axis of_1, line $\{(f_1(\mathrm{x}), t): t \in R\}$ parallels axis of_2 and the two lines both contain point $(f_1(\mathrm{x}), f_2(\mathrm{x}))$.

The minimal points of the two (not closed) segments are described as

$$\mathbf{f}^{\mathbf{L}}_{1,\min}(\mathbf{x}) = \begin{cases} (f_{1,\min}, f_2(\mathbf{x})) & \textit{if } \min\left(\{(t, f_2(\mathbf{x})) = (t, x_2) : t \in R\} \cap \mathbf{F}^2\right) \text{exists} \\ \phi = (\ ,\ ) & \text{otherwise} \end{cases}$$

and

$$\mathbf{f}^{\mathbf{L}}_{2,\min}(\mathbf{x}) = \begin{cases} (f_1(\mathbf{x}), f_{2,\min}) & \textit{if } \min\left(\{(f_1(\mathbf{x}), t) = (x_1, t) : t \in R\} \cap \mathbf{F}^2\right) \text{exists}, \\ \phi = (\ ,\ ) & \text{otherwise.} \end{cases}$$

For all $\mathbf{x} = (x_1, x_2) \in \mathbf{X}$, we have the sets

$$\mathbf{F}_1^{\inf} \triangleq \left\{\mathbf{f}^{\mathbf{L}}_{1,\min}(\mathbf{x}) : \mathbf{x} = (x_1, x_2) \in \mathbf{X}\right\}$$

and

$$\mathbf{F}_2^{\inf} \triangleq \left\{\mathbf{f}^{\mathbf{L}}_{2,\min}(\mathbf{x}) : \mathbf{x} = (x_1, x_2) \in \mathbf{X}\right\}.$$

In this example,

$$\mathbf{F}_1^{\inf} = \Phi \text{ and } \mathbf{F}_2^{\inf} = \{(f_1, f_2) : f_2 = -\sqrt{-f_1^2 + 1}, \quad 0 < f_1 \le 1\}.$$

Observe, a non-dominated point, at least, is a non-dominated point in set $\mathbf{F}_1^{\inf}$ and also is a non-dominated point in set $\mathbf{F}_2^{\inf}$, i.e., the point should be in set:

$$\mathbf{F}^{\inf\_1} \triangleq \mathbf{F}_1^{\inf} \cap \mathbf{F}_2^{\inf}.$$

In our case $\mathbf{F}^{\inf\_1} = \Phi = \{\phi\}$. Hence our non-dominated set is Φ.

In fact, for every point $(f_1, f_2) \in \mathbf{F}_2^{\inf}$, there is some $\mathbf{x} = (x_1, x_2) \in \mathbf{X}$, such that

$$(f_1, f_2) \in \{(t, f_2(\mathbf{x})) = (t, x_2) : t \in R\} \cap \mathbf{F}^2.$$

If $(f_1, f_2) \in \mathbf{F}_2^{\inf}$ is a non-dominated image point then the image point is non-dominated in set $\{(t, f_2(\mathbf{x})) = (t, x_2) : t \in R\} \cap \mathbf{F}^2$.

(3) The inversing stage inverses the non-dominated set by the mapping recorded in the mapping stage.

Hence $\hat{\mathbf{X}}_E = \hat{\mathbf{X}}_{wE} = \hat{\mathbf{X}}_{sE} = \Phi$.

We replace "min" with "max" to find the solution to M_MOP (2.2) in the same way. Consider work in filtering stage.

$$\mathbf{F}_1^{\sup} \triangleq \{\mathbf{f}_{1,\max}^{\mathbf{L}}(\mathbf{x}) : \mathbf{x} = (x_1, x_2) \in \mathbf{X}\}$$

and

$$\mathbf{F}_2^{\sup} \triangleq \{\mathbf{f}_{2,\max}^{\mathbf{L}}(\mathbf{x}) : \mathbf{x} = (x_1, x_2) \in \mathbf{X}\},$$

where

$$\mathbf{f}_{1,\max}^{\mathbf{L}}(\mathbf{x}) = \begin{cases} (f_{1,\max}, f_2(\mathbf{x})) & \text{if } \max\left(\{(t, f_2(\mathbf{x})) = (t, x_2) : t \in R\} \cap \mathbf{F}^2\right) \text{exists}, \\ \phi = (\ ,\ ) & \text{otherwise}, \end{cases}$$
$$\mathbf{f}_{2,\max}^{\mathbf{L}}(\mathbf{x}) = \begin{cases} (f_1(\mathbf{x}), f_{2,\max}) & \text{if } \max\left(\{(f_1(\mathbf{x}), t) = (x_1, t) : t \in R\} \cap \mathbf{F}^2\right) \text{exists}, \\ \phi = (\ ,\ ) & \text{otherwise}. \end{cases}$$

Then we have

$$\mathbf{F}_1^{\sup} = \{(f_1, f_2) : f_2 = -\sqrt{-f_1^2 + 1},\quad 0 < f_1 \le 1\}$$

and

$$\mathbf{F}_2^{\sup} = \{(f_1, f_2) : f_1 \in [-1, 1],\ f_2 = 0\}.$$

Hence, we have the $\max -$ non-dominated set

$$\begin{aligned} \mathbf{F}^{\sup\_1} &\triangleq \mathbf{F}_1^{\sup} \cap \mathbf{F}_2^{\sup} \\ &= \{(f_1, f_2) : f_2 = -\sqrt{-f_1^2 + 1}, 0 < f_1 \le 1\} \cap \{(f_1, f_2) : f_1 \in [-1, 1],\ f_2 = 0\} \\ &= \{(1, 0)\}. \end{aligned}$$

By the inversing stage, we have $\hat{\mathbf{X}}_{M\_sE} = \{(1, 0)\}$.

All elements in $\mathbf{F}^{\inf\_1}$ and $\mathbf{F}^{\sup\_1}$ may not be non-dominated points if $\mathbf{F}^m$ is not a convex set.

This example illustrates several main points that draw attention to the new method as follows.

(a) The core work is in the filtering stage that is solving the multiobjective problem. Solving the multiobjective problem, in mathematics, is finding a subset (called the non-dominated set) of the image such that every point in the subset is a "non-dominated point". In this example, the non-dominated set, in geometry, looks to be located on the boundary of the image (geometrical characteristics). We guess that the non-dominated set has the same geometrical characteristics for a general MOP (or M_MOP). We will verify this later.

(b) In this example, the feasible set and image are dense. Because we know the definitions of boundary, open set, closed set, and so on we know the definition for dense set. However, the image of a general MOP (or M_MOP) may be dense or discrete or a mix with partial density and partial discrete. The existing definitions may not be suitable for these cases in operation and we have to give proper definitions.

(c) Since the mapping is a simple case: $1-1$, the mapping stage and inversing stage are much easier. In practical MOP (or M_MOP), there are multi mappings and each mapping may be $1-1$ or $2-1$ or $\cdots$. The mapping stage and inversing stage should take care of these cases.

(d) Example (4.1) is a continuous problem. Normally, we meet a MOP (or M_MOP) that is discrete and multi dimensional (>>3).

If these considerations can be solved, then the idea of the new method will be the same as this way of solving this example. As a derivation of this new method, we focus on obtaining the efficient, Pareto optimal and strictly efficient solutions for a multicriteria optimization problem. It follows from the idea that we can obtain other kinds of multicriteria optimization solutions such as lexicographic optimal.

Chapter 5 A Direct Method of Solving the Multicriteria Optimization Problem and Max Multicriteria Optimization Problem

The multicriteria optimization problem (MOP) given in (2.1) and the max multicriteria optimization problem (M_MOP) in (2.2) are re-written here as

$$\min_{\mathbf{x}\in\mathbf{X}} \mathbf{f}(\mathbf{x}), \tag{2.1}$$

$$\max_{\mathbf{x}\in\mathbf{X}} \mathbf{f}(\mathbf{x}), \tag{2.2}$$

where

$\mathbf{f} : \mathbf{X} \to R^m$, $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_m(\mathbf{x}))$, $m \geq 1$,

$\mathbf{X} \subset R^q$, $q \geq 1$ is the feasible set,

$f_j(\mathbf{x}) : R^q \to R$, $j = 1, \cdots, m$ are the objective functions.

The image of the feasible set is

$$\mathbf{F}^m \underline{\underline{\Delta}} \left\{ \mathbf{f}(\mathbf{x}) = \left(f_1(\mathbf{x}), \cdots, f_m(\mathbf{x}) \right) : \mathbf{x} = (x_1, \cdots, x_q) \in \mathbf{X} \right\}.$$

The main work of the direct method is solving a multiobjective problem to obtain its non-dominated set or max-non-dominated set. The exploration in Chapter 4 introduces the idea of a direct method of solving MOP and M_MOP. Strict mathematical definitions and proofs are required now to complete the method. We will present these in the following subsections.

5.1 Inferior Boundary, Superior Boundary and Boundary of Objective (Image)

Let $ol_1 \cdots ol_m$ be a m dimensional coordinate system. Then $\mathbf{f}(\mathbf{x}) = \left(f_1(\mathbf{x}), \cdots, f_m(\mathbf{x}) \right)$, $\mathbf{x} \in \mathbf{X}$ as an image point of $\mathbf{x}$ is located at $\left(f_1(\mathbf{x}), \cdots, f_m(\mathbf{x}) \right)$ in the coordinate system, where $f_j(\mathbf{x})$ is a value on axis ol_j, $j \in \{1, \cdots, m\}$. The set $\mathbf{F}^m$ could be dense, discrete or mixed. We have mentioned in chapter 4 that our familiar definitions of boundary, bounded and interior of a dense set are not suitable for a discrete set or mixed set. We should give operational definitions for an arbitrary set such as a dense set, discrete set and mixed set (dense and discrete set). The definitions should not clash with the classical definitions when the set is dense.

Point Line in Parallel with a Coordinate Axis Passing $\mathbf{F}^m$

Let $\mathbf{F}^m = \{\mathbf{f}(\mathbf{x}) : \mathbf{x} = (x_1, \cdots, x_q) \in \mathbf{X}\}$ be the image of the given feasible set $\mathbf{X} \subset R^q$, where

$$\mathbf{f} : \mathbf{X} \to R^m, \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})), m \geq 1,$$
$$f_j(\mathbf{x}) : R^q \to R, \; j = 1, \cdots, m.$$

For given $\mathbf{x} \in \mathbf{X}$, let parameter t replace component $f_j(\mathbf{x})$ of image point $\mathbf{f}(\mathbf{x})$, where $j \in \{1, \cdots, m\}$. We use notation

$$\mathbf{f}_j(\mathbf{x}, t) = \begin{cases} (t, f_2(\mathbf{x}), \cdots, f_m(\mathbf{x})) & j = 1, \\ (f_1(\mathbf{x}), \cdots, f_{j-1}(\mathbf{x}), t, f_{j+1}(\mathbf{x}), \cdots, f_m(\mathbf{x})) & j \in \{2, \cdots, m-1\}, \\ (f_1(\mathbf{x}), \cdots, f_{m-1}(\mathbf{x}), t) & j = m. \end{cases}$$

For given $t \in R$, $\mathbf{f}_j(\mathbf{x}, t)$ is a point in m dimensional space. Let t vary in R; then it forms, in geometry, a line in the space:

$$\mathbf{L}_j(\mathbf{x}, t) \underline{\underline{\Delta}} \{\mathbf{f}_j(\mathbf{x}, t) = (f_1(\mathbf{x}), \cdots, f_{j-1}(\mathbf{x}), t, f_{j+1}(\mathbf{x}), \cdots, f_m(\mathbf{x})) \in R^m : t \in R\}.$$

Since

$$\mathbf{f}_j(\mathbf{x}, t)\Big|_{t = f_j(\mathbf{x})} = \mathbf{f}(\mathbf{x}, f_j(\mathbf{x})) = (f_1(\mathbf{x}), \cdots, f_{j-1}(\mathbf{x}), f_j(\mathbf{x}), f_{j+1}(\mathbf{x}), \cdots, f_m(\mathbf{x})) \in \mathbf{L}_j(\mathbf{x}, t)$$

and for given $t, h \in R$ and $t \neq h$

$$\vec{\mathbf{r}}_j(\mathbf{x}, t-h) = \mathbf{f}_j(\mathbf{x}, t) - \mathbf{f}_j(\mathbf{x}, h) = \left[0, \cdots, \overset{j-1}{0}, \overset{j}{t-h}, \overset{j+1}{0}, \cdots, 0\right],$$

hence, we have

- Line $\mathbf{L}_j(\mathbf{x}, t)$ passes point $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_m(\mathbf{x}))$; in other words, the intersection set of $\mathbf{F}^m$ and line $\mathbf{L}_j(\mathbf{x}, t)$ is nonempty.
- Line $\mathbf{L}_j(\mathbf{x}, t)$ parallels with coordinate axis ol_i.

The intersection set of image $\mathbf{F}^m$ and line $\mathbf{L}_j(\mathbf{x}, t)$ is:

$$\mathbf{F}_j^{\mathbf{L}}(\mathbf{x}) \underline{\underline{\Delta}} \mathbf{F}^m \cap \mathbf{L}_j(\mathbf{x}, t), \; \mathbf{x} \in \mathbf{X}. \tag{5.1}$$

We call $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ as the intersection set of image $\mathbf{F}^m$ and line $\mathbf{L}_j(\mathbf{x},t)$ or j component line of $\mathbf{f}(\mathbf{x})$. We remark that set $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ may only include one point, itself ($\mathbf{f}(\mathbf{x})$). If there is a $\mathbf{x}^* \in \mathbf{X}$, $\mathbf{x}^* \neq \mathbf{x}$ and $\mathbf{f}(\mathbf{x}^*) \neq \mathbf{f}(\mathbf{x})$ such that $\mathbf{f}(\mathbf{x}^*) = \left(f_1(\mathbf{x}^*), \cdots, f_m(\mathbf{x}^*)\right) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$, then there exists $j \in \{1, \cdots, m\}$ such that one of the following three cases holds.

$$j = 1 \text{ case:} \qquad \begin{cases} f_1(\mathbf{x}^*) \neq f_1(\mathbf{x}), \\ f_k(\mathbf{x}^*) = f_k(\mathbf{x}) \quad k \in \{2, \cdots, m\}, \end{cases}$$

$$j \in \{2, \cdots, m-1\} \text{ case:} \quad \begin{cases} f_j(\mathbf{x}^*) \neq f_j(\mathbf{x}), \\ f_k(\mathbf{x}^*) = f_k(\mathbf{x}) \quad k \in \{1, \cdots, j-1, j+1, \cdots, m\}, \end{cases}$$

$$j = m \text{ case:} \qquad \begin{cases} f_m(\mathbf{x}^*) \neq f_m(\mathbf{x}), \\ f_k(\mathbf{x}^*) = f_k(\mathbf{x}) \quad k \in \{1, \cdots, m-1\}. \end{cases}$$

See that if $\mathbf{f}(\mathbf{x}) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ and $\mathbf{f}(\mathbf{x}^*) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$, $\mathbf{x} \neq \mathbf{x}^*$ then $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x}) = \mathbf{F}_j^{\mathbf{L}}(\mathbf{x}^*)$. We mention that if mapping $\mathbf{f}$ is not $1 \to 1$ but $f_k(\mathbf{x}^*) = f_k(\mathbf{x})$, $k \in \{1, \cdots, m\}$ holds then $\mathbf{x}^* \neq \mathbf{x}$ should happen.

5.1.1 Definitions of Real Inferior, Imaginary Inferior, and Real Superior and Imaginary Superior Boundaries

Definition Let $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ be the j component line of $\mathbf{f}(\mathbf{x})$, where $\mathbf{x} \in \mathbf{X}$, $j \in \{1, \cdots, m\}$. We say that:

(a) $\mathbf{f}_j(\mathbf{x}, f_j(\mathbf{x})) = \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ is a real inferior boundary point of set $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ if for all $\mathbf{f}(\mathbf{x}^*) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$, $\mathbf{x}^* \in \mathbf{X}$, we have $f_j(\mathbf{x}) \leq f_j(\mathbf{x}^*)$. We denote point $\mathbf{f}(\mathbf{x})$ as $\mathbf{f}_j^{\inf, R}(\mathbf{x})$.

(b) $\mathbf{f}_j(\mathbf{x}, f_j(\mathbf{x})) = \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ is a real superior boundary point of set $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ if there exists $f_j(\mathbf{x}^*) \leq f_j(\mathbf{x})$ for all $\mathbf{f}(\mathbf{x}^*) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$, where $\mathbf{x}^* \in \mathbf{X}$. We denote the point as $\mathbf{f}_j^{\sup, R}(\mathbf{x})$.

(c) $\mathbf{f}_j(\mathbf{x}, t^\circ) = (f_1(\mathbf{x}), \cdots, f_{j-1}(\mathbf{x}), t^\circ, f_{j+1}(\mathbf{x}) \cdots f_m(\mathbf{x}))$ is an imaginary inferior boundary point of set $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ if there exists $t^\circ \in R$ such that $f_j(\mathbf{x}') > t^\circ$ for all $\mathbf{f}(\mathbf{x}') \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ and there exists $\mathbf{f}(\mathbf{x}^*) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ such that $f_j(\mathbf{x}^*) < t^\circ + \delta$, $\forall \delta > 0$, where $\mathbf{x}', \mathbf{x}^* \in \mathbf{X}$. Point $\mathbf{f}_j(\mathbf{x}, t^\circ) \notin \mathbf{F}^m$ is denoted as $\mathbf{f}_j^{\inf, I}(\mathbf{x})$.

(d) $\mathbf{f}_j(\mathbf{x}, t^\circ) = (f_1(\mathbf{x}), \cdots, f_{j-1}(\mathbf{x}), t^\circ, f_{j+1}(\mathbf{x}) \cdots f_m(\mathbf{x}))$ is an imaginary superior boundary point of set $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ if there exists $t^\circ \in R$ such that $f_j(\mathbf{x}') < t^\circ$ for all $\mathbf{f}(\mathbf{x}') \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ and there exists $\mathbf{f}(\mathbf{x}^*) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ such that $f_j(\mathbf{x}^*) > t^\circ - \delta$ for $\forall \delta > 0$, where $\mathbf{x}', \mathbf{x}^* \in \mathbf{X}$. Denote the point as $\mathbf{f}_j^{\sup, I}(\mathbf{x})$.

Since only two of the four cases (a, b, c, d) apply for any set $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$, where $\mathbf{x} \in \mathbf{X}$ and $j \in \{1, \cdots, m\}$, we stipulate that the remaining two all are ϕ. By these definitions, we remark that

- $\forall \mathbf{f}(\mathbf{x}') \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$, where $\mathbf{x}', \mathbf{x} \in \mathbf{X}$ then

$$\begin{aligned}
&\mathbf{f}_j^{\inf, R}(\mathbf{x}) = \mathbf{f}_j^{\inf, R}(\mathbf{x}') \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x}), \\
&\mathbf{f}_j^{\sup, R}(\mathbf{x}) = \mathbf{f}_j^{\sup, R}(\mathbf{x}') \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x}), \\
&\mathbf{f}_j^{\inf, I}(\mathbf{x}) = \mathbf{f}_j^{\inf, I}(\mathbf{x}') \notin \mathbf{F}_j^{\mathbf{L}}(\mathbf{x}), \\
&\mathbf{f}_j^{\sup, I}(\mathbf{x}) = \mathbf{f}_j^{\sup, I}(\mathbf{x}') \notin \mathbf{F}_j^{\mathbf{L}}(\mathbf{x}).
\end{aligned}$$

- $\mathbf{f}_j^{\inf, R}(\mathbf{x}) = \mathbf{f}_j(\mathbf{x}, f_j(\mathbf{x})) = \mathbf{f}_j^{\sup, R}(\mathbf{x})$ may hold for some $\mathbf{x} \in \mathbf{X}$ and $j \in \{1, \cdots, m\}$ for given $\mathbf{F}^m$.

All boundary points that parallel with coordinate axis ol_j, $j \in \{1, \cdots, m\}$ are classified and collected as follows.

the real inferior boundary set	$\mathbf{F}_j^{\inf, R} \triangleq \{\mathbf{f}(\mathbf{x}) : \mathbf{f}(\mathbf{x}) = \mathbf{f}_j^{\inf, R}(\mathbf{x}), \mathbf{x} \in \mathbf{X}\}$,
the real superior boundary set	$\mathbf{F}_j^{\sup, R} \triangleq \{\mathbf{f}(\mathbf{x}) : \mathbf{f}(\mathbf{x}) = \mathbf{f}_j^{\sup, R}(\mathbf{x}), \mathbf{x} \in \mathbf{X}\}$,
the imaginary inferior boundary set	$\mathbf{F}_j^{\inf, I} \triangleq \{\mathbf{f}_j^{\inf, I}(\mathbf{x}) : \mathbf{f}(\mathbf{x}) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x}), \mathbf{x} \in \mathbf{X}\}$,
the imaginary superior boundary set	$\mathbf{F}_j^{\sup, I} \triangleq \{\mathbf{f}_j^{\sup, I}(\mathbf{x}) : \mathbf{f}(\mathbf{x}) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x}), \mathbf{x} \in \mathbf{X}\}$.

Hence

the real inferior boundary set of $\mathbf{F}^m$ is $\qquad \mathbf{F}^{\inf, R} \triangleq \bigcup_{j=1}^{m} \mathbf{F}_j^{\inf, R}$, the real superior boundary set of $\mathbf{F}^m$ is $\mathbf{F}^{\sup,R} \underline{\underline{\Delta}} \bigcup_{j=1}^{m} \mathbf{F}_j^{\sup,R}$,

the imaginary inferior boundary set of $\mathbf{F}^m$ is $\mathbf{F}^{\inf,I} \underline{\underline{\Delta}} \bigcup_{j=1}^{m} \mathbf{F}_j^{\inf,I}$,

the imaginary superior boundary set of $\mathbf{F}^m$ is $\mathbf{F}^{\sup,I} \underline{\underline{\Delta}} \bigcup_{j=1}^{m} \mathbf{F}_j^{\sup,I}$.

It is clear that all cases may exist for given set $\mathbf{F}^m$. For convenience, we call

$\mathbf{F}^{\inf} \underline{\underline{\Delta}}\, \mathbf{F}^{\inf,R} \bigcup \mathbf{F}^{\inf,I}$ as inferior boundary set of $\mathbf{F}^m$,
$\mathbf{F}^{\sup} \underline{\underline{\Delta}}\, \mathbf{F}^{\sup,R} \bigcup \mathbf{F}^{\sup,I}$ as superior boundary set of $\mathbf{F}^m$,

$\mathbf{F}^{b,R} \underline{\underline{\Delta}}\, \mathbf{F}^{\inf,R} \bigcup \mathbf{F}^{\sup,R}$ as real boundary set of $\mathbf{F}^m$,
$\mathbf{F}^{b,I} \underline{\underline{\Delta}}\, \mathbf{F}^{\inf,I} \bigcup \mathbf{F}^{\sup,I}$ as imaginary boundary set of $\mathbf{F}^m$,

$\mathbf{F}^{b} \underline{\underline{\Delta}}\, \mathbf{F}^{\inf} \bigcup \mathbf{F}^{\sup} = \left(\mathbf{F}^{\inf,R} \bigcup \mathbf{F}^{\inf,I}\right) \bigcup \left(\mathbf{F}^{\sup,R} \bigcup \mathbf{F}^{\sup,I}\right)$ as boundary set of $\mathbf{F}^m$.

It is easy to find that the following cases may exist for given $\mathbf{F}^m$:

$\mathbf{F}^{\inf,R} \bigcap \mathbf{F}^{\sup,R} \neq \Phi$,
$\mathbf{F}^{\inf,R} = \Phi$,
$\mathbf{F}^{\sup,R} = \Phi$,
$\mathbf{F}^{\inf,R} \bigcup \mathbf{F}^{\sup,R} = \Phi$.

5.1.2 Real Inferior_1 and Imaginary Inferior_1 Boundaries, and Real Superior_1 and Imaginary Superior_1 Boundaries

Definition Let $\mathbf{F}_j^{\inf,R}$, $\mathbf{F}_j^{\inf,I}$, and $\mathbf{F}_j^{\sup,R}$ and $\mathbf{F}_j^{\inf,I}$ be (real and imaginary) inferior boundary sets and superior boundary sets of image set $\mathbf{F}^m$, $j = 1, \cdots, m$; we define:

the real inferior_1 boundary set of $\mathbf{F}^m$: $\mathbf{F}^{\inf\_1,R} \underline{\underline{\Delta}} \bigcap_{j=1}^{m} \mathbf{F}_j^{\inf,R}$,

the real superior_1 boundary set of $\mathbf{F}^m$: $\mathbf{F}^{\sup\_1,R} \underline{\underline{\Delta}} \bigcap_{j=1}^{m} \mathbf{F}_j^{\sup,R}$, the real _1 boundary set of $\mathbf{F}^m$: $\mathbf{F}^{b\_1,R} \triangleq \mathbf{F}^{\inf\_1,R} \bigcup \mathbf{F}^{\sup\_1,R}$,

the imaginary inferior_1 boundary set of $\mathbf{F}^m$: $\mathbf{F}^{\inf\_1,I} \triangleq \bigcap_{j=1}^{m} \mathbf{F}_j^{\inf,I}$,

the imaginary superior_1 boundary set of $\mathbf{F}^m$: $\mathbf{F}^{\sup\_1,I} \triangleq \bigcap_{j=1}^{m} \mathbf{F}_j^{\sup,I}$,

the imaginary _1 boundary set of $\mathbf{F}^m$: $\mathbf{F}^{b\_1,I} \triangleq \mathbf{F}^{\inf\_1,I} \bigcup \mathbf{F}^{\sup\_1,I}$,

the inferior_1 boundary set of $\mathbf{F}^m$: $\mathbf{F}^{\inf\_1} \triangleq \mathbf{F}^{\inf\_1,R} \bigcup \mathbf{F}^{\inf\_1,I}$,

the superior_1 boundary set of $\mathbf{F}^m$: $\mathbf{F}^{\sup\_1} \triangleq \mathbf{F}^{\sup\_1,R} \bigcup \mathbf{F}^{\sup\_1,I}$,

the _1 boundary set of $\mathbf{F}^m$: $\mathbf{F}^{b\_1} \triangleq \mathbf{F}^{b\_1,R} \bigcup \mathbf{F}^{b\_1,I}$.

Obviously,

- $\mathbf{F}^{\inf\_1,R} \subseteq \mathbf{F}^{\inf,R} \subseteq \mathbf{F}^m$,
- $\mathbf{F}^{\inf\_1,I} \subseteq \mathbf{F}^{\inf,I} \subseteq \mathbf{F}^m$,
- $\mathbf{F}^{\sup\_1,R} \subseteq \mathbf{F}^{\sup,R} \subseteq \mathbf{F}^m$,
- $\mathbf{F}^{\sup\_1,I} \subseteq \mathbf{F}^{\sup,I} \subseteq \mathbf{F}^m$.

We conclude that $\mathbf{F}^m$ is a real boundary set if and only if $\mathbf{F}^{b,I} = \mathbf{F}^{\inf,I} \bigcup \mathbf{F}^{\sup,I} = \Phi$.

We give an example in Figure 4 below of $\mathbf{F}^2$ in two dimensions to illustrate $\mathbf{F}_1^{\inf,R}$, $\mathbf{F}_2^{\inf,R}$, $\mathbf{F}_1^{\sup,R}$, $\mathbf{F}_2^{\sup,R}$, $\mathbf{F}^{\inf\_1,R}$, $\mathbf{F}^{\sup\_1,R}$.

$\mathbf{F}_1^{\inf,R}$ is the curve degab, $\mathbf{F}_2^{\inf,R}$ is the curve abc,
$\mathbf{F}_1^{\sup,R}$ is the curve bcd, $\mathbf{F}_2^{\sup,R}$ is the curve cde,
$\mathbf{F}^{\inf\_1,R} = \mathbf{F}_1^{\inf,R} \bigcap \mathbf{F}_2^{\inf,R}$ is the curve ab,
$\mathbf{F}^{\sup\_1,R} = \mathbf{F}_1^{\sup,R} \bigcap \mathbf{F}_2^{\sup,R}$ is the curve cd.

Note that $\mathbf{F}_1^{\inf,I} = \mathbf{F}_2^{\inf,I} = \mathbf{F}_1^{\sup,I} = \mathbf{F}_2^{\sup,I} = \mathbf{F}^{b,I} = \Phi$.

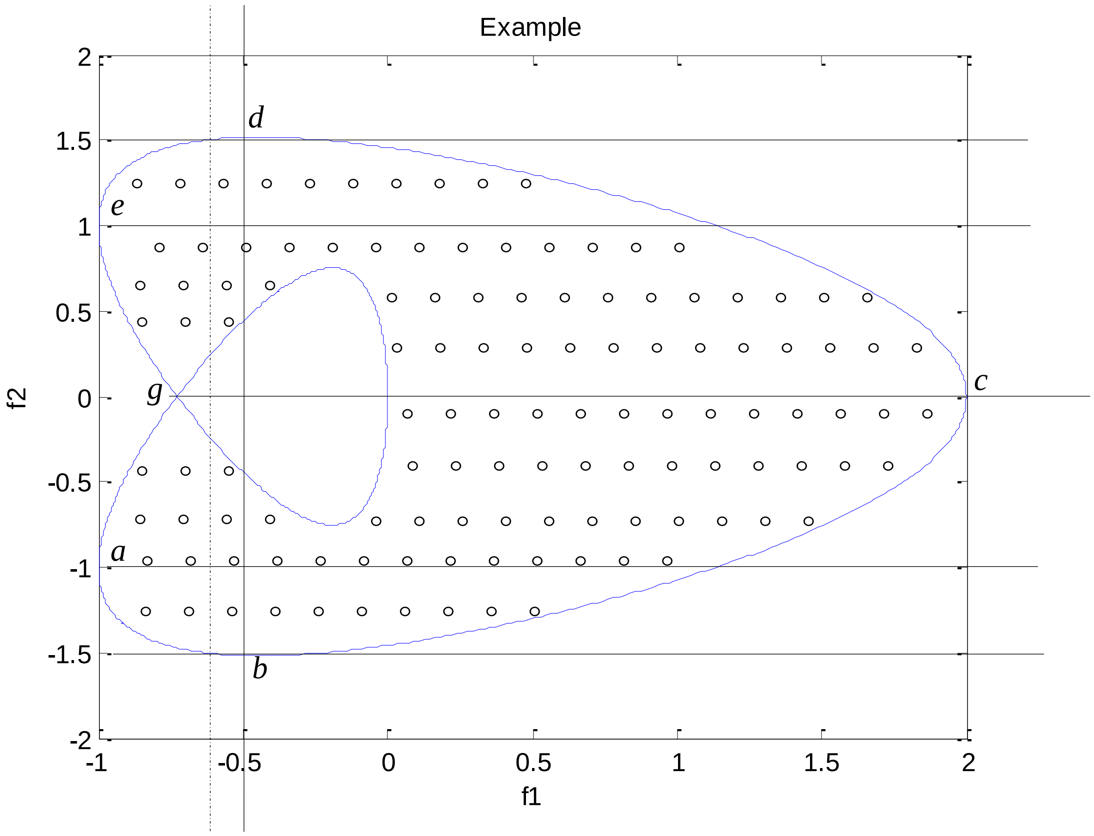

Figure 4: Illustration of $\mathbf{F}_1^{\inf,R}$, $\mathbf{F}_2^{\inf,R}$, $\mathbf{F}_1^{\sup,R}$, $\mathbf{F}_2^{\sup,R}$, $\mathbf{F}^{\inf\_1,R}$, $\mathbf{F}^{\sup\_1,R}$

Theorem 5.1 If $\mathbf{F}^m$ is dense and a real boundary set then the real boundary is identical to the boundary in real analysis.

Proof

(Only consider $\mathbf{f}_j^{\inf,R}(\mathbf{x}) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ which is a real inferior boundary point of $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ (or $\mathbf{F}^m$). Other cases can be proved in the same way)

Review the classical definition of boundary in real analysis: $\mathbf{y}$ is a boundary point of set $\mathbf{Z}$ if there are points $\mathbf{u} \in \mathbf{Z}$ and $\mathbf{v} \notin \mathbf{Z}$ in $N_\delta(\mathbf{y})$ (it is an open set called δ-nbhd, its center is $\mathbf{y}$ and the radius is δ) for any $\delta > 0$ [31].

Let $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ be the j component line of point $\mathbf{f}(\mathbf{x})$, where $\mathbf{x}$ is an arbitrary point in the feasible set $\mathbf{X}$ and $j \in \{1, \cdots, m\}$. Assuming there exists a real inferior boundary point $\mathbf{f}_j^{\inf,R}(\mathbf{x}) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$, we prove $\mathbf{f}_j^{\inf,R}(\mathbf{x})$ is a boundary point in real analysis.

i) $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x}) \neq \{\mathbf{f}(\mathbf{x})\}$, i.e., $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ is not only a singleton

Let $\mathbf{f}_j^{\text{inf}, R}(\mathbf{x}) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ be a real inferior boundary point. By Definition 5.1.1, there is $\mathbf{x}^* \in \mathbf{X}$, $\mathbf{f}_j^{\text{inf}, R}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^*) = \left(f_1(\mathbf{x}^*), \cdots, f_m(\mathbf{x}^*)\right) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ such that

$$\begin{cases} f_k(\mathbf{x}^*) = f_k(\mathbf{x}) = f_k(\mathbf{x}') & k \neq j, \\ \quad f_k(\mathbf{x}^*) \leq f_k(\mathbf{x}') & k = j, \end{cases}$$

for all $\mathbf{f}(\mathbf{x}') \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x}), \mathbf{x}' \in \mathbf{X}$. Since $\mathbf{F}^m$ is a dense set, $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ is a dense set. For any given $\delta > 0$, $\exists \delta' > 0$ and $\delta' < \delta$ such that

$$\begin{aligned} \mathbf{f}(\mathbf{x}') &= \left(f_1(\mathbf{x}'), \cdots, f_m(\mathbf{x}')\right) \\ &= \left(f_1(\mathbf{x}), \cdots, f_{j-1}(\mathbf{x}), f_j(\mathbf{x}^*) + \frac{\delta'}{2}, f_{j+1}(\mathbf{x}), \cdots, f_m(\mathbf{x})\right) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x}). \end{aligned}$$

By $\left\|\mathbf{f}(\mathbf{x}') - \mathbf{f}_j^{\text{inf}, R}(\mathbf{x})\right\| = \left\|\mathbf{f}(\mathbf{x}') - \mathbf{f}(\mathbf{x}^*)\right\| = \frac{\delta'}{2}$, we have

$$\mathbf{f}(\mathbf{x}') \in N_{\delta'}(\mathbf{f}_j^{\text{inf}, R}(\mathbf{x})) = N_{\delta'}(\mathbf{f}(\mathbf{x}^*) \subset N_{\delta}(\mathbf{f}_j^{\text{inf}, R}(\mathbf{x})) = N_{\delta}(\mathbf{f}(\mathbf{x}^*).$$

Consider point

$$\begin{aligned} \bar{\mathbf{f}} &= \left(\bar{f}_1, \cdots, \bar{f}_m\right) \\ &= \left(f_1(\mathbf{x}), \cdots, f_{j-1}(\mathbf{x}), f_j^{\text{inf}, R}(\mathbf{x}) - \frac{\delta'}{2}, f_{j+1}(\mathbf{x}), \cdots, f_m(\mathbf{x})\right) \notin \mathbf{F}_j^{\mathbf{L}}(\mathbf{x}). \end{aligned}$$

But $\left\|\bar{\mathbf{f}} - f_j^{\text{inf}, R}(\mathbf{x})\right\| = \left\|\mathbf{f}(\mathbf{x}') - \mathbf{f}(\mathbf{x}^*)\right\| = \frac{\delta'}{2}$, i.e., $\bar{\mathbf{f}} \in N_{\delta'}(\mathbf{f}_j^{\text{inf}, R}(\mathbf{x})) \subset N_{\delta}(\mathbf{f}_j^{\text{inf}, R}(\mathbf{x}))$.

Thus $\mathbf{f}_j^{\text{inf}, R}(\mathbf{x}) \in \mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ is a boundary point in real analysis.

ii) $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x}) = \{\mathbf{f}(\mathbf{x})\}$, i.e., $\mathbf{F}_j^{\mathbf{L}}(\mathbf{x})$ is a singleton

In this case: $\mathbf{f}_j^{\text{inf}, R}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$; by $\mathbf{F}^m$ being a dense set, $\exists\, \mathbf{f}(\mathbf{x}') \in \mathbf{F}^m$ such that $\mathbf{f}(\mathbf{x}') \in N_{\delta'}(\mathbf{f}_j^{\text{inf}, R}(\mathbf{x})) \subset N_{\delta}(\mathbf{f}_j^{\text{inf}, R}(\mathbf{x}))$. As in i), we consider a point

$$\bar{\mathbf{f}} = \left(\bar{f}_1, \cdots, \bar{f}_m\right) = \left(f_1(\mathbf{x}), \cdots, f_{j-1}(\mathbf{x}), f_j^{\inf, R}(\mathbf{x}) - \frac{\delta'}{2}, f_{j+1}(\mathbf{x}), \cdots, f_m(\mathbf{x}) \right).$$

By the Definition 5.1.1 and $\left\|\bar{\mathbf{f}} - f_j^{\inf, R}(\mathbf{x})\right\| = \dfrac{\delta'}{2} < \delta$, we obtain

$$\bar{\mathbf{f}} \notin \mathbf{F}_j^{\mathbf{L}}(\mathbf{x}) \text{ and } \bar{\mathbf{f}} \in N_{\delta'}(\mathbf{f}_j^{\inf, R}(\mathbf{x})) \subset N_{\delta}(\mathbf{f}_j^{\inf, R}(\mathbf{x})).$$

It follows from the definition of a boundary point of a set in real analysis that $\mathbf{f}_j^{\inf, R}(\mathbf{x})$ is a boundary point of set $\mathbf{F}^m$.

By combining of i) and ii), the proof is complete. □

Definition of Bounded Set Let $\mathbf{F}^m = \{\mathbf{f}(\mathbf{x}) = \left(f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})\right) : \mathbf{x} \in \mathbf{X}\}$ be the image of feasible set $\mathbf{X}$. If there exists $M \in R^+$ ($M < +\infty$) such that $\left| f_j(\mathbf{x}) \right| \le M$, for all $\mathbf{x} \in \mathbf{X}$ and $j = 1, \cdots, m$, then we say that set $\mathbf{F}^m$ is bounded.

5.2 The Relations of Non-dominated Set and Real Boundary Set

Theorem 5.2 Let $\mathbf{F}^{\inf\_1, R}$ be a real inferior_1 boundary set of set $\mathbf{F}^m$, and $\hat{\mathbf{F}}$ be a type of (a weakly non-dominated set or non-dominated set or strictly) non-dominated set of set $\mathbf{F}^m$, then $\hat{\mathbf{F}} \subseteq \mathbf{F}^{\inf\_1, R}$.

Proof

(1) $\hat{\mathbf{x}} \in \mathbf{X}_E$

Let $\hat{\mathbf{x}} \in \mathbf{X}_E$ be an arbitrary efficient point and $\mathbf{f}(\hat{\mathbf{x}}) = \left(f_1(\hat{\mathbf{x}}), \cdots, f_m(\hat{\mathbf{x}})\right) \in \hat{\mathbf{F}} \subseteq \mathbf{F}^m$ be a non-dominated point. For point $\mathbf{f}(\hat{\mathbf{x}})$ and $j \in \{1, \cdots, m\}$, there is j component line $\mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}})$, where

$$\mathbf{f}(\hat{\mathbf{x}}) \in \mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}}) = \mathbf{F}^m \cap \mathbf{L}_j(\hat{\mathbf{x}}, t),$$

and

$$\mathbf{L}_j(\hat{\mathbf{x}}, t) \underline{\underline{\Delta}} \left\{\mathbf{f}_j(\hat{\mathbf{x}}, t) = \left(f_1(\hat{\mathbf{x}}), \cdots, f_{j-1}(\hat{\mathbf{x}}), t, f_{j+1}(\hat{\mathbf{x}}), \cdots, f_m(\hat{\mathbf{x}})\right) \in R^m : t \in R\right\}.$$

Not to lose the general case, we assume that there is more than one point in set $\mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}})$.

Suppose $\mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}})$ is an open set. By $\mathbf{f}(\hat{\mathbf{x}}) \in \mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}})$, $\exists \delta > 0$ such that $\mathbf{f}(\mathbf{x}') \in N_\delta(\mathbf{f}(\hat{\mathbf{x}})) \subset \mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}})$, where

$$\mathbf{f}(\mathbf{x}') \neq \mathbf{f}(\hat{\mathbf{x}}),$$
$$\mathbf{f}(\mathbf{x}') = \left(f_1(\mathbf{x}'), \cdots f_j(\mathbf{x}'),\ \cdots,\ f_m(\mathbf{x}')\right),$$
$$\begin{cases} f_k(\mathbf{x}') = f_k(\hat{\mathbf{x}}) & k \neq j, \\ f_k(\mathbf{x}') < f_k(\hat{\mathbf{x}}) & k = j. \end{cases}$$

Since $\mathbf{f}(\hat{\mathbf{x}})$ is a non-dominated point, inequality $f_j(\mathbf{x}') < f_j(\hat{\mathbf{x}})$ does not hold. It means that $\mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}})$ is not an open set and (by Definition 5.1.1 (a)) $\mathbf{f}(\hat{\mathbf{x}})$ is an inferior point of set $\mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}})$, i.e.,

$$\mathbf{f}(\hat{\mathbf{x}}) = \mathbf{f}_j^{\inf,R}(\hat{\mathbf{x}}) \in \mathbf{F}_j^{\inf,R}, \quad j = 1, \cdots, m.$$

Hence

$$\mathbf{f}(\hat{\mathbf{x}}) \in \bigcap_{j=1}^{m} \mathbf{F}_j^{\inf,R} = \mathbf{F}^{\inf\_1,R}.$$

(2) $\hat{\mathbf{x}} \in \mathbf{X}_{sE}$

Let $\hat{\mathbf{x}} \in \mathbf{X}_{sE}$ be an arbitrary strictly efficient point and $\mathbf{f}(\hat{\mathbf{x}}) \in \hat{\mathbf{F}} \subseteq \mathbf{F}^m$ be a strictly non-dominated point, where $\mathbf{f}(\hat{\mathbf{x}}) = \left(f_1(\hat{\mathbf{x}}), \cdots,\ f_m(\hat{\mathbf{x}})\right)$. Following proof (1), we obtain

$$\mathbf{f}(\hat{\mathbf{x}}) \in \bigcap_{j=1}^{m} \mathbf{F}_j^{\inf,R} = \mathbf{F}^{\inf\_1,R}.$$

(3) $\hat{\mathbf{x}} \in \mathbf{X}_{wE}$

If $\hat{\mathbf{x}} \in \mathbf{X}_{wE}$, then there at least exists $\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2 \in \mathbf{X}_{wE}$ such that $\mathbf{f}(\hat{\mathbf{x}}^1) = \mathbf{f}(\hat{\mathbf{x}}^2) \in \hat{\mathbf{F}} \subset \mathbf{F}^m$, i.e., the images of $\hat{\mathbf{x}}^1$ and $\hat{\mathbf{x}}^2$ locate on the same position in m dimension space. In this case, we have

$$\mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}}^1) = \mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}}^2),$$
$$\mathbf{L}_j(\hat{\mathbf{x}}^1, t) = \mathbf{L}_j(\hat{\mathbf{x}}^2, t),$$
$$\mathbf{f}(\hat{\mathbf{x}}^1) \in \mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}}^1) = \mathbf{F}^m \cap \mathbf{L}_j(\hat{\mathbf{x}}^1, t),$$

$$\mathbf{f}(\hat{\mathbf{x}}^2) \in \mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}}^2) = \mathbf{F}^m \cap \mathbf{L}_j(\hat{\mathbf{x}}^2, t).$$

Suppose $\mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}}^1) = \mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}}^2)$ is an open set. As in the proof (1), $\mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}}^1) = \mathbf{F}_j^{\mathbf{L}}(\hat{\mathbf{x}}^2)$ is either an open set or a closed set and

$$\mathbf{f}(\hat{\mathbf{x}}^1) = \mathbf{f}(\hat{\mathbf{x}}^2) = \mathbf{f}_j^{\inf,R}(\hat{\mathbf{x}}^1) = \mathbf{f}_j^{\inf,R}(\hat{\mathbf{x}}^2) \in \mathbf{F}_j^{\inf,R}, \quad j = 1, \cdots, m.$$

And then we have

$$\mathbf{f}(\hat{\mathbf{x}}^1) = \mathbf{f}(\hat{\mathbf{x}}^2) \in \bigcap_{j=1}^{m} \mathbf{F}_j^{\inf,R} = \mathbf{F}^{\inf\_1,R}.$$

In the same way, we can prove this case: $\mathbf{f}(\hat{\mathbf{x}}^1) = \cdots = \mathbf{f}(\hat{\mathbf{x}}^h) \in \hat{\mathbf{F}} \subseteq \mathbf{F}^m$, where $\hat{\mathbf{x}}^1, \cdots, \hat{\mathbf{x}}^h \in \mathbf{X}_{wE}$, $h \in I^+$ and $\hat{\mathbf{x}}^i \neq \mathbf{x}^k$ when $i \neq k, i, k \in \{1, \cdots, h\}$ such that
$\mathbf{f}(\hat{\mathbf{x}}^1) = \cdots = \mathbf{f}(\hat{\mathbf{x}}^h) \in \bigcap_{j=1}^{m} \mathbf{F}_j^{\inf,R} = \mathbf{F}^{\inf\_1,R}$.

It follows from (1), (2) and (3) that $\hat{\mathbf{F}} \subseteq \mathbf{F}^{\inf\_1,R} \subseteq \mathbf{F}^m$. Hence, Theorem 5.2 is proven. □

Using an analogous proof which has been omitted, we have a parallel with Theorem 5.2:

Theorem 5.3 Assume $\mathbf{F}^{\sup\_1,R}$ is real superior_1 boundary set of set $\mathbf{F}^m$, and $\hat{\mathbf{F}}^{\max}$ is a max weakly non-dominated set or max non-dominated set or max strictly non-dominated set of set $\mathbf{F}^m$. Then $\hat{\mathbf{F}}^{\max} \subseteq \mathbf{F}^{\sup\_1,R}$.

In addition, we have the following theorem (without proof).

Theorem 5.4 If $\mathbf{F}^m$ is convex set then $\mathbf{F}^{\inf\_1,R}$ is a type of (a weakly non-dominated set or non-dominated set or strictly) non-dominated set of set $\mathbf{F}^m$ and $\mathbf{F}^{\sup\_1,R}$ is a type of max (a weakly non-dominated set or non-dominated set or strictly) non-dominated set of set $\mathbf{F}^m$.

For a general (discrete, mixed and non-convex) set, to find a type of non-dominated set and a type of max non-dominated set, we have to go further after obtaining $\mathbf{F}^{\inf\_1,R}$ and $\mathbf{F}^{\sup\_1,R}$.

Definition 5.2.1 Let $\mathbf{F}^{\inf\_1,R}$ be the real inferior_1 set of set $\mathbf{F}^m$. For arbitrary real inferior point $\mathbf{f} = \{f_1, \cdots, f_m\} \in \mathbf{F}^{\inf\_1,R}$, we define set

$$\mathbf{F}_{\leq} f_j = \left\{\mathbf{f}' = (f_1', \cdots, f_m') : f_j' \leq f_j, \mathbf{f}' \in \mathbf{F}^{\inf\_1,R}\right\}, \; j = 1, \cdots, m.$$

Of course, $\mathbf{F}_{\leq} f_j \subseteq \mathbf{F}^{\inf\_1,R}$.

Theorem 5.5 Let $\mathbf{F}^{\inf\_1,R}$ be the real inferior_1 set of set $\mathbf{F}^m$ and $\hat{\mathbf{F}}$ be a type of non-dominated set of set $\mathbf{F}^m$. Point $\mathbf{f} = \{f_1, \cdots, f_m\} \in \mathbf{F}^{\inf\_1,R}$ is a type of non-dominated point if and only if

$$\bigcap_{j=1}^{m} \mathbf{F}_{\leq} f_j = \{\mathbf{f}\}.$$

Proof

We only prove $\mathbf{f} = \{f_1, \cdots, f_m\} \in \mathbf{F}^{\inf\_1,R}$ is a non-dominated point if and only if

$$\bigcap_{j=1}^{m} \mathbf{F}_{\leq} f_j = \{\mathbf{f}\}.$$

"$\Rightarrow$" Let $\mathbf{f} = \{f_1, \cdots, f_m\} \in \mathbf{F}^{\inf\_1,R}$ be a type of non-dominated point. It follows from Theorem 5.2 that

$$\mathbf{f} = \{f_1, \cdots, f_m\} \in \hat{\mathbf{F}} \subseteq \mathbf{F}^{\inf\_1,R} \subseteq \mathbf{F}^m.$$

For each component of $\mathbf{f}$, by Definition 5.2.1, there exists set

$$\mathbf{F}_{\leq} f_j = \left\{\mathbf{f}' = \left(f_1', \cdots, f_m'\right) : f_j' \leq f_j, \mathbf{f}' \in \mathbf{F}^{\inf\_1,R}\right\}$$

for each $j \in \{1, \cdots, m\}$. Since $\mathbf{f} = \{f_1, \cdots, f_m\} \in \mathbf{F}^{\inf\_1,R}$, by Definition 5.2.1, we have

$$\mathbf{f} \in \mathbf{F}_{\leq} f_j$$

for each $j \in \{1, \cdots, m\}$. Hence $\mathbf{f} \in \bigcap_{j=1}^{m} \mathbf{F}_{\leq} f_j$, i.e., $\bigcap_{j=1}^{m} \mathbf{F}_{\leq} f_j \neq \Phi$.

Let $\mathbf{f}' = \left(f_1', \cdots, f_m'\right) \in \bigcap_{j=1}^{m} \mathbf{F}_{\leq} f_j$ and $\mathbf{f}' \neq \mathbf{f}$. By definition 5.2.1, the following inequality holds for all $j = 1, \cdots, m$.

$$f_j' \leq f_j.$$

That is

$$\begin{cases} f_j' = f_j & \text{for some of } j \in \{1, \cdots, m\} \\ f_j' < f_j & \text{remaining } j \end{cases}$$

i.e., $\mathbf{f}'$ dominates $\mathbf{f}$, which is a contradiction. Here, we only have

$$\bigcap_{j=1}^{m} \mathbf{F}_{\leq} f_j = \{\mathbf{f}\}.$$

"$\Leftarrow$" Let $\bigcap_{j=1}^{m} \mathbf{F}_{\leq} f_j \neq \Phi$ and $\bigcap_{j=1}^{m} \mathbf{F}_{\leq} f_j = \{\mathbf{f}\}$ hold. In accordance with Theorem 5.2, all non-dominated points are in the real inferior_1 boundary set $\mathbf{F}^{\inf\_1,R}$. By Definition 5.2.1, some points in set $\mathbf{F}^{\inf\_1,R}$ can dominate point $\mathbf{f}$ only in set $\bigcap_{j=1}^{m} \mathbf{F}_{\leq} f_j$. By $\bigcap_{j=1}^{m} \mathbf{F}_{\leq} f_j = \{\mathbf{f}\}$, no such point can dominate point $\mathbf{f}$ except itself, i.e., $\mathbf{f}$ is a non-dominated point. □

Parallel with Definition 5.2.1, we have

Definition 5.2.2 Let $\mathbf{F}^{\sup\_1,R}$ be the real superior_1 set of set $\mathbf{F}^m$. For arbitrary $\mathbf{f} \in \mathbf{F}^{\sup\_1,R}$, we define set

$$\mathbf{F}_{\geq} f_j = \{\mathbf{f}' = (f_1', \cdots, f_m') : f_j' \geq f_j, \mathbf{f}' \in \mathbf{F}^{\sup\_1,R}\}, \ j = 1, \cdots, m.$$

Similar to Theorem 5.5, the following theorem holds.

Theorem 5.6 Let $\mathbf{F}^{\sup\_1,R}$ be the superior_1 set of set $\mathbf{F}^m$. Point $\mathbf{f} = (f_1, \cdots, f_m) \in \mathbf{F}^{\sup\_1,R} \subseteq \mathbf{F}^m$ is a type of max non-dominated point if and only if

$$\bigcap_{j=1}^{m} \mathbf{F}_{\geq} f_j = \{\mathbf{f}\}.$$

All non-dominated points and max non-dominated points can be written as

$$\mathbf{F}_{no} = \left\{ \mathbf{f} : \mathbf{f} \in \mathbf{F}^{\inf\_1,R} \subseteq \mathbf{F}^m, \bigcap_{j=1}^{m} \mathbf{F}_{\leq} f_j = \{\mathbf{f}\} \right\}$$

and

$$\mathbf{F}_{M\_no} = \left\{ \mathbf{f} : \mathbf{f} \in \mathbf{F}^{\sup\_1,R} \subseteq \mathbf{F}^m, \bigcap_{j=1}^{m} \mathbf{F}_{\geq} f_j = \{\mathbf{f}\} \right\}.$$

5.3 The Relations of Non-Dominated Set, Mapping and the Solution of Multicriteria Optimization Problem

From the definitions of weakly efficient, efficient and strictly efficient in section 2.4.1, we have the following theorem.

Theorem 5.7 Let $\mathbf{F}_{no} = \left\{ \mathbf{f} : \mathbf{f} \in \mathbf{F}^{\inf\_1,R} \subseteq \mathbf{F}^m, \bigcap_{j=1}^{m} \mathbf{F}_{\leq} f_j = \{\mathbf{f}\} \right\}$ be the non-dominated set of set $\mathbf{F}^m$ then

(1) if $\mathbf{F}_{no} = \Phi$ then there is no solution of the MOP,

(2) if $\mathbf{F}_{no} = \{\mathbf{f}(\hat{\mathbf{x}})\}$, i.e., $\mathbf{f}(\hat{\mathbf{x}})$ is the only element of $\mathbf{F}_{no}$, and if the mapping at $\hat{\mathbf{x}}$ is "one to one", then inversing $\mathbf{f}(\hat{\mathbf{x}})$ to obtain $\{\hat{\mathbf{x}}\}$ is $\mathbf{X}_{sE}$, i.e., $\hat{\mathbf{x}}$ is a strictly efficient point, and otherwise $\{\mathbf{x} : \mathbf{f}(\mathbf{x}) = \mathbf{f}(\hat{\mathbf{x}}), \mathbf{x} \in \mathbf{X}\}$ is $\mathbf{X}_{wE}$, i.e., a weakly efficient set,

(3) if there is more than one element in set $\mathbf{F}_{no}$, if there is "one to one" for each element in $\mathbf{F}_{no}$, then $\{\mathbf{x} : \mathbf{f}(\mathbf{x}) \in \mathbf{F}_{no}\} = \mathbf{X}_E$ otherwise set $\{\mathbf{x} : \mathbf{f}(\mathbf{x}) \in \mathbf{F}_{no}\}$ is $\mathbf{X}_{wE}$.

Similar to Theorem 5.7, the following theorem holds.

Theorem 5.8 Let $\mathbf{F}_{M\_no} = \left\{ \mathbf{f} : \mathbf{f} \in \mathbf{F}^{\sup\_1,R} \subseteq \mathbf{F}^m, \bigcap_{j=1}^{m} \mathbf{F}_{\geq} f_j = \{\mathbf{f}\} \right\}$ be max non-dominated set of set $\mathbf{F}^m$ then

(1) if $\mathbf{F}_{M\_no} = \Phi$ then there is no solution of the M_MOP,

(2) if $\mathbf{F}_{M\_no} = \{\mathbf{f}(\hat{\mathbf{x}})\}$, i.e., $\mathbf{f}(\hat{\mathbf{x}})$ is the only element of $\mathbf{F}_{M\_no}$ and if the mapping at $\hat{\mathbf{x}}$ is "one to one" then inversing $\mathbf{f}(\hat{\mathbf{x}})$ to obtain $\{\hat{\mathbf{x}}\}$ is $\mathbf{X}_{M\_sE}$ i.e., $\hat{\mathbf{x}}$ is a max strictly efficient, and otherwise set $\{\mathbf{x} : \mathbf{f}(\mathbf{x}) = \mathbf{f}(\hat{\mathbf{x}}), \mathbf{x} \in X\}$ is $\mathbf{X}_{M\_wE}$, i.e., a max weakly efficient set,

(3) if there is more than one element in set $\mathbf{F}_{M\_no}$, if there is a "one to one" relationship for each element in $\mathbf{F}_{M\_no}$, then set $\{\mathbf{x} : \mathbf{f}(\mathbf{x}) \in \mathbf{F}_{M\_no}\}$ is $\mathbf{X}_{M\_E}$ and otherwise it is $\mathbf{X}_{M\_wE}$.

Chapter 6 A Direct Method of Solving the Discrete Multicriteria Optimization Problem

When we encounter multicriteria optimization problems, most are a discrete set, i.e., the feasible set and its image are both discrete. The mapping varies: some are "one to one" and some are "two to one", and so on. It is very beneficial to provide a proven algorithm for solving the discrete case. Referring to Chapter 5, we have:

Theorem Let $\mathbf{F}^m \subset R^m$ be a discrete and bounded mapping set of feasible set $\mathbf{X}$ then $\mathbf{F}^m$ has a real boundary set.

6.1 Description of Discrete Case

In the discrete case, following Chapter 5, there are procedures to determine a boundary set and then obtain a non-dominated set. In this chapter, we will follow these procedures to find a non-dominated set and also construct a useful and operational algorithm.

Let $\mathbf{F}^m = \{\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})) : \mathbf{x} \in \mathbf{X}\} = \{\mathbf{f}_1, \cdots, \mathbf{f}_n\}$, $m, n \in \mathbf{I}^+$ be the image set with discrete elements $\mathbf{f}_i$, $i \in \{1, \cdots, n\}$, where $\mathbf{f}_i = (f_{i,1}, \cdots, f_{i,m})$, $\mathbf{X} = \{\mathbf{x}_1, \cdots, \mathbf{x}_H\}$, $\mathbf{x} = (x_1, \cdots, x_q)$, $H, q \in I^+$, where m is the number of components of each image point. We can imagine every component as a pixel of an image point then each image point is constructed by m pixels. Write $\mathbf{F}^m$ as the image matrix of feasible set $\mathbf{X}$, $\mathbf{r}$ as mapping characteristic number matrix and $\mathbf{xr}$ as the status matrix:

$$\mathbf{F}^m = \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_i \\ \vdots \\ \mathbf{f}_n \end{bmatrix} = \begin{bmatrix} \mathbf{f}(\mathbf{x}_1^*) \\ \vdots \\ \mathbf{f}(\mathbf{x}_i^*) \\ \vdots \\ \mathbf{f}(\mathbf{x}_n^*) \end{bmatrix} = \begin{bmatrix} f_{1,1} & \cdots & f_{1,j} & \cdots & f_{1,m} \\ \vdots & & \vdots & & \vdots \\ f_{i,1} & \cdots & f_{i,j} & \cdots & f_{i,m} \\ \vdots & & \vdots & & \vdots \\ f_{n,1} & \cdots & f_{n,j} & \cdots & f_{n,m} \end{bmatrix}_{n\times m},$$

$$\mathbf{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_i \\ \vdots \\ r_n \end{bmatrix}_{n\times 1} \quad \text{and} \quad \mathbf{xr} = \begin{bmatrix} xr_{1,1} & \cdots & xr_{1,j} & \cdots & xr_{1,\overline{m}} \\ \vdots & & \vdots & & \vdots \\ xr_{i,1} & \cdots & xr_{i,j} & \cdots & xr_{i,\overline{m}} \\ \vdots & & \vdots & & \vdots \\ xr_{n,1} & \cdots & xr_{n,j} & \cdots & xr_{n,\overline{m}} \end{bmatrix}_{n\times \overline{m}} \qquad (6.1)$$

where

$\mathbf{x}_i^* \in \mathbf{X}, i \in \{1, \cdots, n\}$, and $\mathbf{f}(\mathbf{x}_i^*) \neq \mathbf{f}(\mathbf{x}_k^*)$ when $i \neq k, i, k \in \{1, \cdots, n\}$,

$\mathbf{f}_i = [f_{i,1}, \cdots, f_{i,m}]$ is a point of the image with m pixels located at row i in $\mathbf{F}^m$,

$\mathbf{f}_i \neq \mathbf{f}_l$ when $i \neq l$, i.e., $\exists$ some $j \in \{1, \cdots, m\}$ such that $f_{i,j} \neq f_{l,j}$,

$r_i = \mathcal{N}(\{\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_i^*) : \mathbf{x} \in \mathbf{X}\}$ is the number of total elements with the same image point in a feasible set,

$\overline{m} = \max\{r_1, \cdots, r_n\}$,

$$\begin{cases} \{xr_{i,1}, \cdots, xr_{i,r_i}\} = \{\mathbf{x} \in \mathbf{X} : \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_i^*)\} \\ xr_{i,j} = 0 \ if \ r_i < j \leq \overline{m} \end{cases}.$$

By all the notations, we have

$$r_i \geq 1, \ i \in \{1, 2, \cdots, n\},$$

$$\sum_{i=1}^{n} r_i = H,$$

$$n \leq H.$$

Matrix $\mathbf{r}$ records mapping characteristics: multi or one. Matrix $\mathbf{xr}$ will be filled during forming matrix $\mathbf{F}^m$. It is clear that if $\mathbf{f}_k, k \in \{1, \cdots, n\}$ is a non-dominated point then we can find the optimal solution set $\{xr_{k,1}, \cdots, xr_{k,r_k}\}$ in $\mathbf{xr}$. It is an easy matter to form matrix $\mathbf{r}$ and matrix $\mathbf{xr}$; we will not be concerned with forming matrix $\mathbf{xr}$ in this thesis.

6.2 Finding Out $\mathbf{F}^{\inf\_1,R}$ and $\mathbf{F}^{\sup\_1,R}$

It follows from section 5.1that we should find $\mathbf{F}_j^L$ for each image point then find $\mathbf{F}_j^{\inf,R}$ and $\mathbf{F}_j^{\sup,R}$, and after that find $\mathbf{F}^{\inf\_1,R}$ and $\mathbf{F}^{\sup\_1,R}$. We will reach the goal step by step. At the first, we derive a series of matrices from the original matrix $\mathbf{F}^m$ as the following shows.

6.2.1 Ordered Matrix and Permutation Matrices

Let

$$\mathbf{FM}^m = \begin{bmatrix} \mathbf{fm}_1 \\ \vdots \\ \mathbf{fm}_i \\ \vdots \\ \mathbf{fm}_n \end{bmatrix} = \begin{bmatrix} fm_{1,1} & \cdots & fm_{1,j} & \cdots & fm_{1,m} \\ \vdots & & & & \vdots \\ fm_{i,1} & \cdots & fm_{i,j} & \cdots & fm_{i,m} \\ \vdots & & & & \vdots \\ fm_{n,1} & \cdots & fm_{n,j} & \cdots & fm_{n,m} \end{bmatrix} \tag{6.2}$$

be the ordered matrix, where each column in $\mathbf{FM}^m = [fm_{i,j}]_{n\times m}$ is the ordered column from matrix $\mathbf{F}^m = [f_{i,j}]_{n\times m}$, i.e., for each $j \in \{1, 2, \cdots, m\}$, we have

$$fm_{i,j} \le fm_{k,j}, \; if \;\; i \le k \;\; for \; i,k \in \{1,2,\cdots,n\}.$$

If we consider that each component ($f_{i,j}$) of an image point in $\mathbf{F}^m$ is a pixel, then the column (pixel) in $\mathbf{FM}^m$ is of a brightness order of the color light. By the way, the effect of the image is not changed, i.e., image $\mathbf{F}^m$ and image $\mathbf{FM}^m$ are the same. We may call it image stability.

Let matrices

$$\mathbf{FM\_F}^m = \begin{bmatrix} fm\_f_{1,1} & \cdots & fm\_f_{1,j} & \cdots & fm\_f_{1,m} \\ \vdots & & & & \vdots \\ fm\_f_{i,1} & \cdots & fm\_f_{i,j} & \cdots & fm\_f_{i,m} \\ \vdots & & & & \vdots \\ fm\_f_{n,1} & \cdots & fm\_f_{n,j} & \cdots & fm\_f_{n,m} \end{bmatrix}_{n\times m} \qquad (6.3)$$

and

$$\mathbf{F\_FM}^m = \begin{bmatrix} f\_fm_{1,1} & \cdots & f\_fm_{1,j} & \cdots & f\_fm_{1,m} \\ \vdots & & & & \vdots \\ f\_fm_{i,1} & \cdots & f\_fm_{i,j} & \cdots & f\_fm_{i,m} \\ \vdots & & & & \vdots \\ f\_fm_{n,1} & \cdots & f\_fm_{n,j} & \cdots & f\_fm_{n,m} \end{bmatrix}_{n\times m} . \qquad (6.4)$$

be **mapping matrices** to build the relationships between $\mathbf{F}^m$ and $\mathbf{FM}^m$, that are $\mathbf{FM}^m \overset{\mathbf{FM\_F}^m}{\Rightarrow} \mathbf{F}^m$ and $\mathbf{F}^m \overset{\mathbf{F\_FM}^m}{\Rightarrow} \mathbf{FM}^m$, where $fm\_f_{i,j}$ shows the location of $fm_{i,j}$ in $\mathbf{F}^m$ and $f\_fm_{i,j}$ shows the location of $f_{i,j}$ in $\mathbf{FM}^m$. Matrices $\mathbf{FM\_F}^m$ and $\mathbf{F\_FM}^m$ would be constructed by given $\mathbf{F}^m$ and its (each column) ordered matrix $\mathbf{FM}^m$.

6.2.2 Classification Matrix

We classify each component of an image point in its component column in matrix $\mathbf{F}^m$, i.e., every component of an image point should be set by the class in its column.

Let

$$\mathbf{CF}^m = \begin{bmatrix} cf_{1,1} & \cdots & cf_{1,j} & \cdots & cf_{1,m} \\ \vdots & & \vdots & & \vdots \\ cf_{i,1} & \cdots & cf_{i,j} & \cdots & cf_{i,m} \\ \vdots & & \vdots & & \vdots \\ cf_{n,1} & \cdots & cf_{n,j} & \cdots & cf_{n,m} \end{bmatrix}_{n\times m} \qquad \mathbf{cF}^m = \begin{bmatrix} c_1 \\ \vdots \\ c_i \\ \vdots \\ c_n \end{bmatrix}_{n\times 1} \qquad (6.5)$$

be a component classification matrix, where $cf_{i,j}$ records the class of $f_{i,j}$ (j component of image point i) in $\mathbf{F}^m$. It identifies the class of each component of an image point. The principles of classification are:

$$cf_{i,j} = \begin{cases} 0 & f_{i,j} \neq f_{k,j} \text{ when } i \neq k, \\ s \in \{1, \cdots, s_j\} & \exists k \neq i \text{ such that } f_{i,j} = f_{k,j} \end{cases} \quad i, k \in \{1, \cdots, n\}, \qquad (6.6)$$

$cf_{i,j} = cf_{k,j}$ if $f_{i,j} = f_{k,j}$ when $k \neq i,\ i, k \in \{1, \cdots, n\}$,

$cf_{i,j} \in \{0, 1, \cdots, s_j\} \subset \{0, 1, \cdots, n\}$, $i \in \{1, \cdots, n\}$, $j \in \{1, \cdots, m\}$ and $s_j \leq n$,

$s_j + 1$ is the (maximum) number of classes in j column.

Every element in the component classification matrix $\mathbf{CF}^m$ can be determined by the ordered matrix $\mathbf{FM}^m$ and mapping matrix $\mathbf{FM\_F}^m$. In fact, for every element $fm_{k,j} \in \mathbf{FM}^m, \exists h$ and p such that

$$fm_{k,j} \in \{fm_{h,j},\ fm_{h+1,j}, \cdots,\ fm_{h+p,j}\},$$

where

$k \in \{h, \cdots, h+p\}$,

$j \in \{1, \cdots, m\}$,

$fm_{h,j} = fm_{h+1,j} = \cdots = fm_{h+p,j}$,

$h \in \{1, \cdots, n\}$,

$p \in \{0, 1, \cdots, n-h\}$.

If $p > 0$ then we call the set $\{fm_{h,j},\ fm_{h+1,j}, \cdots,\ fm_{h+p,j}\}$ as an equivalent group of the j column. We give a number to the group

$$\begin{cases} 0 & \text{if } p = 0, \\ s \in \mathbf{I}^+ & \text{otherwise.} \end{cases}$$

Number s depends on the issuing $p \neq 0$ sequence in column j, and we put the number s in the component classification matrix $\mathbf{CF}^m$ in accordance with the mapping matrix $\mathbf{FM\_F}^m$ at appropriate locations:

$$cf_{i,j} = s \text{ for all } i = fm\_f_{h,j},\ fm\_f_{h+1,j}, \cdots, fm\_f_{h+p,j}.$$

Since mappings $\mathbf{FM}^m \underset{\mathbf{F\_FM}^m}{\overset{\mathbf{FM\_F}^m}{\Leftrightarrow}} \mathbf{F}^m$ are "$one \Leftrightarrow one$", the component classification matrix $\mathbf{CF}^m$ will be filled completely. For convenience, $\mathbf{CF}^m$ is initialized with zero.

$\mathbf{cF}^m$ is each image point classification matrix. It identifies the class of every image point. $\mathbf{cF}^m$ is also initialized at zero. At beginning, every element records the number of the joining components corresponding to the point:

$$c_i = \sum_{j=1}^{m} d_{i,j}, \tag{6.7}$$

where

$$d_{i,j} = \begin{cases} 1 & \text{if } cf_{i,j} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The following is a simple example to show how to form the component classification matrix at column $j \in \{1, \cdots, m\}$. Let $\mathbf{F}^{m-j}$ be the $j \in \{1, \cdots, m\}$ column in $\mathbf{F}^m$ with 10 elements and $\mathbf{FM}^{m-j}$ be its ordered column. The columns $\mathbf{F}^{m-j}$ and $\mathbf{FM}^{m-j}$ are connected by permutation columns $\mathbf{FM\_F}^{m-j}$ and $\mathbf{F\_FM}^{m-j}$. It shows as follows.

$$\mathbf{F}^{m-j}=\begin{bmatrix} f_{1,j}\\ f_{2,j}\\ f_{3,j}\\ f_{4,j}\\ f_{5,j}\\ f_{6,j}\\ f_{7,j}\\ f_{8,j}\\ f_{9,j}\\ f_{10,j}\end{bmatrix}=\begin{bmatrix}5\\3\\9\\17\\5\\5\\11\\9\\11\\7\end{bmatrix}\Rightarrow \mathbf{F\_FM}^{m-j}=\begin{bmatrix} f\_fm_{1,j}\\ f\_fm_{2,j}\\ f\_fm_{3,j}\\ f\_fm_{4,j}\\ f\_fm_{5,j}\\ f\_fm_{6,j}\\ f\_fm_{7,j}\\ f\_fm_{8,j}\\ f\_fm_{9,j}\\ f\_fm_{10,j}\end{bmatrix}=\begin{bmatrix}2\\1\\6\\10\\3\\4\\8\\7\\9\\5\end{bmatrix}\Rightarrow \mathbf{FM}^{m-j}=\begin{bmatrix} fm_{1,j}\\ fm_{2,j}\\ fm_{3,j}\\ fm_{4,j}\\ fm_{5,j}\\ fm_{6,j}\\ fm_{7,j}\\ fm_{8,j}\\ fm_{9,j}\\ fm_{10,j}\end{bmatrix}=\begin{bmatrix}3\\5\\5\\5\\7\\9\\9\\11\\11\\17\end{bmatrix},$$

$$\begin{bmatrix}5\\3\\9\\17\\5\\5\\11\\9\\11\\7\end{bmatrix}=\begin{bmatrix} f_{1,j}\\ f_{2,j}\\ f_{3,j}\\ f_{4,j}\\ f_{5,j}\\ f_{6,j}\\ f_{7,j}\\ f_{8,j}\\ f_{9,j}\\ f_{10,j}\end{bmatrix}=\mathbf{F}^{m-j}\Leftarrow\begin{bmatrix}2\\1\\5\\6\\10\\3\\8\\7\\9\\4\end{bmatrix}=\begin{bmatrix} fm\_f_{1,j}\\ fm\_f_{2,j}\\ fm\_f_{3,j}\\ fm\_f_{4,j}\\ fm\_f_{5,j}\\ fm\_f_{6,j}\\ fm\_f_{7,j}\\ fm\_f_{8,j}\\ fm\_f_{9,j}\\ fm\_f_{10,j}\end{bmatrix}=\mathbf{FM\_F}^{m-j}\Leftarrow\begin{bmatrix}3\\5\\5\\5\\7\\9\\9\\11\\11\\17\end{bmatrix}=\begin{bmatrix} fm_{1,j}\\ fm_{2,j}\\ fm_{3,j}\\ fm_{4,j}\\ fm_{5,j}\\ fm_{6,j}\\ fm_{7,j}\\ fm_{8,j}\\ fm_{9,j}\\ fm_{10,j}\end{bmatrix}=\mathbf{FM}^{m-j},$$

Then $\mathbf{CF}^{m-j}$ is obtained by constructed columns $\mathbf{FM}^{m-j}$ and $\mathbf{FM\_F}^{m-j}$. $\mathbf{CF}^{m-j}$ records every element in $\mathbf{F}^{m-j}$ located group. In this example, column $\mathbf{F}^{m-j}$ includes 4 classes: 0, 1, 2, 3. As an example, elements $f_{1,j}$, $f_{5,j}$ and $f_{6,j}$ are in class 1.

$$\mathbf{FM}^{m-j} = \begin{bmatrix} fm_{1,j} \\ fm_{2,j} \\ fm_{3,j} \\ fm_{4,j} \\ fm_{5,j} \\ fm_{6,j} \\ fm_{7,j} \\ fm_{8,j} \\ fm_{9,j} \\ fm_{10,j} \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 5 \\ 5 \\ 7 \\ 9 \\ 9 \\ 11 \\ 11 \\ 17 \end{bmatrix} \Rightarrow \begin{bmatrix} fm\_f_{1,j} \\ fm\_f_{2,j} \\ fm\_f_{3,j} \\ fm\_f_{4,j} \\ fm\_f_{5,j} \\ fm\_f_{6,j} \\ fm\_f_{7,j} \\ fm\_f_{8,j} \\ fm\_f_{9,j} \\ fm\_f_{10,j} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 6 \\ 10 \\ 3 \\ 8 \\ 7 \\ 9 \\ 4 \end{bmatrix} \Rightarrow \mathbf{CF}^{m-j} = \begin{bmatrix} cf_{1,j} \\ cf_{2,j} \\ cf_{3,j} \\ cf_{4,j} \\ cf_{5,j} \\ cf_{6,j} \\ cf_{7,j} \\ cf_{8,j} \\ cf_{9,j} \\ cf_{10,j} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \\ 1 \\ 3 \\ 2 \\ 3 \\ 0 \end{bmatrix}.$$

Obviously, we have

$$\mathbf{F}^{m-j} = \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ f_{3,j} \\ f_{4,j} \\ f_{5,j} \\ f_{6,j} \\ f_{7,j} \\ f_{8,j} \\ f_{9,j} \\ f_{10,j} \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 9 \\ 17 \\ 5 \\ 5 \\ 11 \\ 9 \\ 11 \\ 7 \end{bmatrix} \quad \text{is classified with} \quad \mathbf{CF}^{m-j} = \begin{bmatrix} cf_{1,j} \\ cf_{2,j} \\ cf_{3,j} \\ cf_{4,j} \\ cf_{5,j} \\ cf_{6,j} \\ cf_{7,j} \\ cf_{8,j} \\ cf_{9,j} \\ cf_{10,j} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \\ 1 \\ 3 \\ 2 \\ 3 \\ 0 \end{bmatrix}.$$

The numbers in $\mathbf{CF}^{m-j}$ depends on issuing $p \neq 0$ sequence in column j of $\mathbf{FM}^{m}$. Since $fm_{2,j} = fm_{3,j} = fm_{4,j} = 5$, the class is defined as 1, etc.

6.2.3 Additional Matrices

To make fast, convenient calculations and avoid repeating them, we design some additional matrices.

We design two matrices to record the starting position and the total members of every class in each column in $\mathbf{FM}^{m}$, the two matrices are notated as

$$\mathbf{Nb} = [n_{k,j}]_{ss \times m} \text{ and } \mathbf{TT} = [t_{k,j}]_{ss \times m}, \tag{6.8}$$

where

$ss = \max\{s_j : j = 1, \cdots, m\}$,

s_j is the number of classes in j column without "0" class,

$n_{k,j}$ records beginning row of k class in j column of ordered matrix $\mathbf{FM}^m$,

$t_{k,j}$ records total number of members of k class in j column of ordered matrix $\mathbf{FM}^m$.

s_j depends on the column. As in the example in 6.2.1, we see that

$$j \text{ column of } \mathbf{Nb} \text{ is } \begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix} \text{ and } j \text{ column of } \mathbf{TT} \text{ is } \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}.$$

In accordance with $\mathbf{FM\_F}^m$, we can find the locations in $\mathbf{F}^m$, as shown in the same example:

$$\mathbf{FM}^{m\_j} = \begin{bmatrix} fm_{1,j} \\ fm_{2,j} \\ fm_{3,j} \\ fm_{4,j} \\ fm_{5,j} \\ fm_{6,j} \\ fm_{7,j} \\ fm_{8,j} \\ fm_{9,j} \\ fm_{10,j} \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 5 \\ 5 \\ 7 \\ 9 \\ 9 \\ 11 \\ 11 \\ 17 \end{bmatrix} \Rightarrow \quad \mathbf{Nb} = \begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix} \quad \overset{\mathbf{FM}^{m\_j}}{\Rightarrow} \quad \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \quad \overset{\mathbf{F}^{m\_j}}{\Rightarrow} \quad \begin{bmatrix} 5 \\ 9 \\ 11 \end{bmatrix}$$

6.2.4 Determination of the Characteristic of Each Image Point

Applying the preparation in the above sections of this chapter, we can classify every point. As convenient, we divide all image points as four types: 0, 1 and 2 and m+1. Every image point will be one of the four types. We stipulate

$$\begin{cases} 0 & \text{if point } \mathbf{f}_i \text{ is a suspected non-dominated or max non-dominated point,} \\ 1 & \text{if point } \mathbf{f}_i \text{ is a suspected non-dominated point,} \\ 2 & \text{if point } \mathbf{f}_i \text{ is a suspected max non-dominated point,} \\ m+1 & \text{if point } \mathbf{f}_i \text{ is a dominated point.} \end{cases} \tag{6.9}$$

The number of each image point will be dropped into the (image) point classification matrix $\mathbf{cF}^m$. After dropping the numbers in $\mathbf{cF}^m$, we call matrix $\mathbf{cF}^m$ the characteristic matrix of the image point.

We should use the obtained information about the image point classification in matrix $\mathbf{cF}^m$ and the knowledge in chapter 5 to designate the type such as 0, 1, 2 or m+1 at every image point. Theorem 6.1 issued later lays the foundation for the discrete case.

Definition 6.1 Let $\mathbf{A} = (f_1^A, \cdots, f_m^A)$ and $\mathbf{B} = (f_1^B, \cdots, f_m^B)$ be two points in m dimension; we define

(1) $\|\mathbf{A} - \mathbf{B}\| = \sqrt{(f_1^A - f_1^B)^2 + \cdots + (f_m^A - f_m^B)^2}$ as the distance between $\mathbf{A}$ and $\mathbf{B}$.

(2) $\mathbf{AB} = [f_1^B - f_1^A, \cdots, f_m^B - f_m^A]$ as a vector with direction $\mathbf{A} \to \mathbf{B}$, $\overline{\mathrm{AB}}$ is the segment of $\mathbf{AB}$ and $\left|\overline{\mathrm{AB}}\right|$ is the length of segment $\overline{\mathrm{AB}}$.

We see that $\|\mathbf{A} - \mathbf{B}\| = \|\mathbf{B} - \mathbf{A}\| = \left|\overline{\mathrm{AB}}\right|$.

In accordance with analytic geometry, the line past two points $\mathbf{A} = (f_1^A, \cdots, f_m^A)$ and $\mathbf{B} = (f_1^B, \cdots, f_m^B)$ is $\mathbf{L}_{A,B} = \mathbf{A} + t(\mathbf{B} - \mathbf{A})$, where $t \in R$ is a parameter.

Definition 6.2 Let $\mathbf{L}^1$ and $\mathbf{L}^2$ be two lines in m dimension. Line $\mathbf{L}^1$ parallels with line $\mathbf{L}^2$ or line $\mathbf{L}^2$ parallels with line $\mathbf{L}^1$ if for two arbitrary points $\mathbf{A} = (f_1^A, \cdots, f_m^A) \in \mathbf{L}^1$, $\mathbf{B} = (f_1^B, \cdots, f_m^B) \in \mathbf{L}^1$, $\|\mathbf{A} - \mathbf{B}\| \neq 0$ and two arbitrary points $\mathbf{C} = (f_1^C, \cdots, f_m^C) \in \mathbf{L}^2$, $\mathbf{D} = (f_1^D, \cdots, f_m^D) \in \mathbf{L}^2$, $\|\mathbf{C} - \mathbf{D}\| \neq 0$ such that $\dfrac{\mathbf{A} - \mathbf{B}}{\|\mathbf{A} - \mathbf{B}\|} = \pm \dfrac{\mathbf{C} - \mathbf{D}}{\|\mathbf{C} - \mathbf{D}\|}$.

Definition 6.3 Let $\mathbf{A} = (f_1^A, \cdots, f_m^A)$, $\mathbf{B} = (f_1^B, \cdots, f_m^B)$, $\mathbf{C} = (f_1^C, \cdots, f_m^C)$ and $\mathbf{D} = (f_1^D, \cdots, f_m^D)$ be four points in m dimension, if $\mathbf{L}_{A,B}$ parallels with $\mathbf{L}_{C,D}$ then we say $\overline{\mathrm{AB}}$ parallels with $\overline{\mathrm{CD}}$.

Theorem 6.1 Let $\mathbf{F}^m$ be a discrete image in m dimension, $m \geq 2$ and $\mathbf{A} = (f_1^A, \cdots, f_m^A)$, $\mathbf{B} = (f_1^B, \cdots, f_m^B)$ be two different image points in the image. Assume $\overline{\mathrm{AB}}$ is a segment constructed by image point $\mathbf{A}$ and image point $\mathbf{B}$, then segment $\overline{\mathrm{AB}}$ parallels with $k \in \{1, \cdots, m\}$ axis of the m dimension coordination system, if and only if

$$\begin{cases} f_j^A = f_j^B, & \text{for all } j \neq k, \\ f_k^A \neq f_k^B. \end{cases}$$

Proof

"$\Rightarrow$" Let $\overline{\mathrm{AB}}$ parallel with $k \in \{1, \cdots, m\}$ axis of the m dimension coordination system. We construct two hyperplanes:

The hyperplane passes point $\mathbf{A} = (f_1^A, \cdots, f_m^A)$ is

$$P_A: \quad \vec{\mathbf{r}} \bullet (\mathbf{y} - \mathbf{A}) = 0 \quad \Rightarrow \quad y_k - f_k^A = 0$$

and the hyperplane passes point $\mathbf{B} = (f_1^B, \cdots, f_m^B)$ is

$$P_B: \quad \vec{\mathbf{r}} \bullet (\mathbf{y} - \mathbf{B}) = 0 \quad \Rightarrow \quad y_k - f_k^B = 0,$$

where

$\vec{\mathbf{r}} = [\overset{1}{0}, \cdots, \overset{k-1}{0}, \overset{k}{1}, \overset{k+1}{0}, \cdots, \overset{m}{0}]$ is an unit vector of k axis,
$\mathbf{y} = (y_1, \cdots, y_m)$ is a variable with m components.

The intersection point of hyperplane P_A and the axis (on the axis) is

$$\begin{cases} \vec{\mathbf{r}} \bullet (\mathbf{y} - \mathbf{A}) = 0 \\ \mathbf{y} = \mathbf{0} + t\mathbf{r} \end{cases} \Rightarrow \begin{cases} y_k - f_k^A = 0 \\ y_k = t \end{cases} \Rightarrow \mathbf{E} = (\overset{1}{0}, \cdots, \overset{k-1}{0}, \overset{k}{f_k^A}, \overset{k+1}{0}, \cdots, \overset{m}{0})$$

and the intersection point of hyperplane P_B and the axis (on the axis) is

$$\begin{cases} \vec{\mathbf{r}} \bullet (\mathbf{y} - \mathbf{B}) = 0 \\ \mathbf{y} = \mathbf{0} + t\mathbf{r} \end{cases} \Rightarrow \begin{cases} y_k - f_k^b = 0 \\ y_k = t \end{cases} \Rightarrow \mathbf{F} = (\overset{1}{0}, \cdots, \overset{k-1}{0}, \overset{k}{f_k^B}, \overset{k+1}{0}, \cdots, \overset{m}{0}),$$

where $\mathbf{0} = [\overset{1}{0}, \cdots, \overset{m}{0}]$ and $t \in R$. Since P_A and P_B have the same normal vector, P_A parallels with P_B. Then segment $\overline{\mathrm{AE}}$ parallels with segment $\overline{\mathrm{BF}}$. Hence $|\overline{\mathrm{AB}}| = |\overline{\mathrm{EF}}|$. Since

$$|\overline{\mathrm{AB}}| = \| \mathbf{A} - \mathbf{B} \| = \sqrt{(f_1^A - f_1^B)^2 + \cdots + (f_m^A - f_m^B)^2} \text{ and}$$
$$|\overline{\mathrm{EF}}| = \| \mathbf{F} - \mathbf{E} \| = \sqrt{(f_k^A - f_k^B)^2},$$

we have

$$(f_1^A - f_1^B)^2 + \cdots + (f_{k-1}^A - f_{k-1}^B)^2 + (f_{k+1}^A - f_{k+1}^B)^2 + \cdots + (f_m^A - f_m^B)^2 = 0,$$

that is

$$\begin{cases} f_j^A = f_j^B & when\ j \neq k \\ f_k^A \neq f_k^B \end{cases}.$$

To be sure points $\mathbf{A}$ and $\mathbf{B}$ are different, condition $f_k^A \neq f_k^B$ should hold.

$"\Leftarrow"$ We have gotten the intersection points of hyperplanes, $\mathbf{E}$ and $\mathbf{F}$, by the previous calculation. Then we obtain

$$\left|\overline{\mathrm{AE}}\right| = \| \mathbf{A} - \mathbf{E} \| = \sqrt{(f_1^A)^2 + \cdots + (f_{k-1}^A)^2 + 0 + (f_{k+1}^A)^2 + \cdots + (f_m^A)^2},$$
$$\left|\overline{\mathrm{BF}}\right| = \| \mathbf{B} - \mathbf{F} \| = \sqrt{(f_1^B)^2 + \cdots + (f_{k-1}^B)^2 + 0 + (f_{k+1}^B)^2 + \cdots + (f_m^B)^2}.$$

Consider

$$\begin{aligned} \Delta d &= \left|\overline{\mathrm{AE}}\right|^2 - \left|\overline{\mathrm{BF}}\right|^2 \\ &= (f_1^A)^2 + \cdots + (f_{k-1}^A)^2 + 0 + (f_{k+1}^A)^2 + \cdots + (f_m^A)^2 \\ &\quad - \left((f_1^B)^2 + \cdots + (f_{k-1}^B)^2 + 0 + (f_{k+1}^B)^2 + \cdots + (f_m^B)^2\right). \end{aligned}$$

By $f_j^A = f_j^B$ when $j \neq k$, $\Delta d = 0$ and then $\left|\overline{\mathrm{AE}}\right| = \left|\overline{\mathrm{BF}}\right|$. An arbitrary point on line $L_{\mathbf{A},\mathbf{B}}$ can be written as

$$\mathbf{y}^{(L)} = \mathbf{A} + t^{(L)}(\mathbf{B} - \mathbf{A}), t^{(L)} \in R,$$

where

$$\mathbf{y}^{(L)} = \{y_1^L, \cdots, y_m^L\},$$
$$y_j^L = f_j^A + t(f_j^B - f_j^A),\ j \in (1, \cdots, m\}.$$

By $\begin{cases} f_j^A = f_j^B & \text{when } j \neq k \\ f_k^A \neq f_k^B \end{cases}$, then

$$y_j^L = \begin{cases} f_j^A & j \neq k \\ f_k^A + t^{(L)}(f_k^B - f_k^A) & j = k \end{cases}.$$

Then the hyperplane passing point $\mathbf{y}^{(L)} = \{y_1^L, \cdots, y_m^L\}$ is

$$P_{\mathbf{y}^{(L)}}: \quad \vec{\mathbf{r}} \bullet (\mathbf{y} - \mathbf{y}^{(L)}) = 0 \quad \Rightarrow \quad y_k - \left(f_k^A + t(f_k^B - f_k^A)\right) = 0$$

The intersection point of hyperplane $P_{\mathbf{y}^{(L)}}$ and the axis (on the axis) is

$$\begin{cases} \vec{\mathbf{r}} \bullet (\mathbf{y} - \mathbf{y}^{(L)}) = 0 \\ \mathbf{y} = \mathbf{0} + t\mathbf{r} \end{cases} \Rightarrow \begin{cases} y_k - \left(f_k^A + t^{(L)}(f_k^B - f_k^A)\right) = 0 \\ y_k = t \end{cases}$$

$$\Rightarrow \quad \mathbf{G} = (\overset{1}{0}, \cdots, \overset{k-1}{0}, \overset{k}{f_k^A + t^{(L)}(f_k^B - f_k^A)}, \overset{k+1}{0}, \cdots, \overset{m}{0}).$$

Hence the distance

$$\left|\overline{\mathbf{Gy}^{(L)}}\right| = \| \mathbf{y}^{(L)} - \mathbf{G} \| = \sqrt{(f_1^A)^2 + \cdots + (f_{k-1}^A)^2 + 0 + (f_{k+1}^A)^2 + \cdots + (f_m^A)^2}$$
$$= \left|\overline{\mathrm{AE}}\right| = \left|\overline{\mathrm{BF}}\right|.$$

Since $\left|\overline{\mathbf{Gy}^{(L)}}\right| = \left|\overline{\mathrm{AE}}\right| = \left|\overline{\mathrm{BF}}\right|$ holds for arbitrary $t^{(L)} \in R$, by Definition 6.2 and 6.3, $\overline{\mathrm{AB}}$ parallels with $k \in \{1, \cdots, m\}$ axis of the m dimension coordination system. □

In accordance with Theorem 6.1, c_i, in classification matrix $\mathbf{cF}^m$, describes the relationship of image point i with other image points:

- If $c_i \geq m-1$ then image point i may share with (at least) one other image point a same line which parallels with one axis of the coordinate system.

Put all such image points in set

$$\mathbf{N}^d = \{i : m-1 \leq c_i \leq m, i \in \{1, \cdots, n\}\} \subseteq \{1, \cdots, n\}. \tag{6.10}$$

Some image points in set $\mathbf{F}^d = \{\mathbf{f}_i : i \in \mathbf{N}^d\}$ may be on segments that parallel some axis of m dimension. By Theorem 6.1, if some image points (more than one point) are on the same segment then they have $m-1$ components that are the same. On this kind of segment, there exist two points: one is an inferior point and the other is a superior point of the image. Hence, we can distinguish these points on all segments in set $\mathbf{F}^d$. To find these image points on each segment, efficiently, we design a matrix named as $\mathbf{W}_{m\times(m-1)}$:

$$
\mathbf{W}_{m\times(m-1)} = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_j \\ \vdots \\ \mathbf{w}_m \end{bmatrix} = \begin{bmatrix} 2 & \cdots & \cdots & m \\ \vdots & & & \vdots \\ 1 & j-1 & j+1 & m \\ \vdots & & & \vdots \\ 1 & \cdots & \cdots & m-1 \end{bmatrix}_{m\times(m-1)} . \tag{6.11}
$$

We see that for arbitrary $i \in \mathbf{N}^d$, $\exists k = cf_{i,j}$ we have set

$$
\mathbf{IL}_{i,j} = \bigcap_{\bar{j}\in\mathbf{w}_j} \left\{ fm\_f_{\bar{k},\bar{j}} : \bar{k} = n_{k,\bar{j}}, n_{k,\bar{j}}+1, \cdots, n_{k,\bar{j}}+t_{k,\bar{j}}-1 \right\} \quad j \in \{1, \cdots, m\} , \tag{6.12}
$$

where $n_{k,\bar{j}}$ is an element in $\mathbf{Nb} = [n_{k,j}]_{s_j\times m}$, $t_{k,j}$ is an element in $\mathbf{TT} = [t_{k,j}]_{s_j\times m}$, and $fm\_f_{\bar{k},\bar{j}}$ is the location at $\mathbf{F}^m$ of element $fm_{\bar{k},\bar{j}}$ that is in $\mathbf{FM}^m = [fm_{i,j}]_{n\times m}$. If $\mathcal{N}(\mathbf{IL}_{i,j}) > 1$, i.e., then all image points in set $\{\mathbf{f}_k : k \in \mathbf{IL}_{i,j}\}$ can construct a segment which parallels the jth axis of the coordinate system. And if $\mathcal{N}(\mathbf{IL}_{i,j}) = 1$ then the image point i is the only image point on the line through it that parallels the jth axis of the coordinate system.

All the segments will be obtained by formula (6.12) in accordance with (6.10) and (6.11).

6.2.5 Determination of $\mathbf{F}^{\text{inf\_1},R}$ and $\mathbf{F}^{\text{sup\_1},R}$

During the determination of $\mathbf{F}^{\text{inf\_1},R}$ and $\mathbf{F}^{\text{sup\_1},R}$, we set two matrices named as **Min** and **Max** to prepare for a multiobjective solution to obtain the non-dominated set and the max_non-dominated set:

$$
\mathbf{Min} = \begin{bmatrix} mi_{1,1} & \cdots & mi_{1,j} & \cdots & mi_{1,m} \\ \vdots & & \vdots & & \vdots \\ mi_{i,1} & \cdots & mi_{i,j} & \cdots & mi_{i,m} \\ \vdots & & \vdots & & \vdots \\ mi_{n,1} & \cdots & mi_{n,j} & \cdots & mi_{n,m} \end{bmatrix}_{n\times m} ,
$$

$$
\mathbf{Max} = \begin{bmatrix} ma_{1,1} & \cdots & ma_{1,j} & \cdots & ma_{1,m} \\ \vdots & & & & \vdots \\ ma_{i,1} & \cdots & ma_{i,j} & \cdots & ma_{i,m} \\ \vdots & & & & \vdots \\ ma_{n,1} & \cdots & ma_{n,j} & \cdots & ma_{n,m} \end{bmatrix}_{n\times m} . \tag{6.13}
$$

Initially, set

$$\mathbf{Min} = \mathbf{Max} = \mathbf{FM\_F}^{m}, \tag{6.14}$$

i.e.,

$$mi_{i,j} = ma_{i,j} = fm\_f_{i,j}, \quad i = 1, \cdots, n \text{ and } j = 1, \cdots, m.$$

Formula (6.14) shows that an image point at location i, the number i will be dropped at the locations of its ordered components. The matrices will be updated:

$$\begin{cases} mi_{k,j} = 0 & \mathbf{f}_i \notin \mathbf{F}^{\inf\_1,R} \\ ma_{k,j} = 0 & \mathbf{f}_i \notin \mathbf{F}^{\sup\_1,R} \end{cases}, \; k = f\_fm_{i,j} \text{ and } j = 1, \cdots, m.$$

In the following, we will determine $\mathbf{F}^{\inf\_1,R}$ and $\mathbf{F}^{\sup\_1,R}$, meanwhile, updating matrices $\mathbf{Min}$ and $\mathbf{Max}$. In accordance to section 6.1.4, we have set $\mathbf{IL}_{i,j}$ for each $i \in \mathbf{N}^d$ and $j \in \{1, \cdots, m\}$. If $\mathbf{IL}_{i,j} = 1$ for some $i \in \mathbf{N}^d$ and $j \in \{1, \cdots, m\}$ then we keep the statuses in $\mathbf{cF}^m$, matrices $\mathbf{Min}$ and $\mathbf{Max}$ for these $i \in \mathbf{N}^d$. The variation of these statuses is only considered for those $\mathcal{N}(\mathbf{IL}_{i,j}) > 1$, $i \in \mathbf{N}^d$ and $j \in \{1, \cdots, m\}$.

For each $i \in \mathbf{N}^d$ and every $j \in \{1, \cdots, m\}$, if $\mathcal{N}(\mathbf{IL}_{i,j}) > 1$, i.e., there is a segment, such that we find the minimal image point and maximal image point on this segment. They are

$$f_{u_j,j} = \min_{i \in \mathbf{IL}_{i,j}} f_{i,j}, \quad u_j \in \mathbf{IL}_{i,j},$$
$$f_{v_j,j} = \max_{i \in \mathbf{IL}_{i,j}} f_{i,j}, \quad v_j \in \mathbf{IL}_{i,j}.$$

We denote $i_{\min,j} = u_j$ and $i_{\max,j} = v_j$, $j \in \{1, \cdots, m\}$. Obviously, $i_{\min,j} = i_{\max,j}$ holds if $\mathcal{N}(\mathbf{IL}_{i,j}) = 1$. We set

$$c_i = \begin{cases} 0 & \text{if } i = i_{\min,j} = i_{\max,j} \\ 1 & \text{if } i = i_{\min,j} \neq i_{\max,j} \\ 2 & \text{when } i = i_{\max,j} \neq i_{\min,j} \\ m+1 & \text{otherwise} \end{cases}, \quad i \in \mathbf{IL}_{i,j} \tag{6.15}$$

to change the marks in classification matrix $\mathbf{cF}^m$ at location i to progressively form characteristic matrix of image points $\mathbf{cF}^m$. Then we apply the mapping matrix $\mathbf{F\_FM}^m$ to vary some elements in matrices $\mathbf{Min}$ and $\mathbf{Max}$:

$$mi_{k,j} = ma_{k.j} = 0 \text{ for } j = 1, \cdots, m, \text{ when } c_i = m+1,$$

where $k = f\_fm_{i,j}$.

Finally, we have the characteristic matrix of image point $\mathbf{cF}^m$, matrices $\mathbf{Min}$ and $\mathbf{Max}$ from these two matrices, to get

$$\mathbf{F}^{\inf\_1,R} = \{\mathbf{f}_i : c_i = 0 \text{ or } c_i = 1, i = 1, \cdots, n\},$$
$$\mathbf{F}^{\sup\_1,R} = \{\mathbf{f}_i : c_i = 0 \text{ or } c_i = 2, i = 1, \cdots, n\},$$

and

$$mi_{k,j} = \begin{cases} 0 & if\ c_i = m+1 \text{ or } c_i = 2 \\ i & if\ c_i = 0 \text{ or } c_i = 1 \end{cases}, \quad j = 1, \cdots, m, \tag{6.16}$$

and

$$ma_{k,j} = \begin{cases} 0 & if\ c_i = m+1 \text{ or } c_i = 1 \\ i & if\ c_i = 0 \text{ or } c_i = 2 \end{cases}, \quad j = 1, \cdots, m, \tag{6.17}$$

where $k = f\_fm_{i,j}$.

6.3 Finding the Non-Dominated Set and Max Non-Dominated Set

Since $\mathbf{F}^m$ is a discrete set, $\mathbf{F}^{\inf\_1,R}$ and $\mathbf{F}^{\sup\_1,R}$ may not be $(\min\ and \max)$ non-dominated sets. From Chapter 5, the non-dominated set is given by $\mathbf{F}^{\inf\_1,R}$ and max non-dominated set is given by $\mathbf{F}^{\sup\_1,R}$. In section 6.2.5, formed matrices $\mathbf{Min}$ and $\mathbf{Max}$ store numbers that are numbered image points that are in $\mathbf{F}^{\inf\_1,R}$ and $\mathbf{F}^{\sup\_1,R}$, and are dropped on the locations of ordered components.

Theorem 6.2 Let $\mathbf{f}_i$, $i \in \{1, \cdots, n\}$ be an image point in $\mathbf{F}^m$, then $\mathbf{f}_i$ is a non-dominated point if and only if

$$\bigcap_{j=1}^{m}\left(\begin{array}{l}\{mi_{k,j} : k = 1, \cdots, f\_fm_{i,j}\} \\ \bigcup\{r = fm\_f_{q,j} : q = n_{p,j}, n_{p,j}+1, \cdots, n_{p,j}+t_{p,j}-1, p = cf_{i,j} \neq 0\}\end{array}\right)$$
$$= \{i = mi_{k,j}, k = f\_fm_{i,j}\} \text{ or } \{i\} \bigcup \{0\}$$

and $\mathbf{f}_i$ is a $\max-$non-dominated point if and only if

$$\bigcap_{j=1}^{m}\left(\begin{array}{l}\{ma_{k,j} : k = f\_fm_{i,j}, f\_fm_{i,j}+1, \cdots, n\} \\ \bigcup\{r = fm\_f_{q,j} : q = n_{p,j}, n_{p,j}+1, \cdots, n_{p,j}+t_{p,j}-1, p = cf_{i,j} \neq 0\}\end{array}\right)$$
$$= \{i\} \text{ or } \{i = ma_{k,j}, k = f\_fm_{i,j}\} \bigcup \{0\}.$$

Proof

We only prove the first part in this theorem.

Let $\mathbf{f}_i, i \in \{1, \cdots, n\}$ be a non-dominated point; then according to the definition, no other image point can dominate $\mathbf{f}_i$. That means no $\mathbf{f}_l$ exists such that $f_{l,j} \leq f_{i,j}$ for $l \in \{1, \cdots n\}$ and $l \neq i$. In matrix $\mathbf{Min}$, for $\mathbf{f}_i, i \in \{1, \cdots, n\}$, we have

$$mi_{k_j,j} = i, \quad k_j = f\_fm_{i,j}, \; j = 1, \cdots m.$$

Assume $\mathbf{f}_l$ dominates $\mathbf{f}_i$ then there are two cases as follows.

Case 1 $f_{l,j} \leq f_{i,j}$, for each $j \in \{1, \cdots m\}$ $\exists k'_j = f\_fm_{l,j}$ such that $1 \leq k'_j < k_j$.

Because $1 \leq k'_j < k_j$ for each $j \in \{1, \cdots m\}$ then $i, l \in \{mi_{k,j} : k = 1, \cdots, k_j\}$ for each $j \in \{1, \cdots m\}$, hence

$$i, l \in \bigcap_{j=1}^{m}\left(\begin{array}{l}\{mi_{k,j} : k = 1, \cdots, k_j\} \\ \bigcup\{r = fm\_f_{q,j} : q = n_{p,j}, n_{p,j}+1, \cdots, n_{p,j}+t_{p,j}-1, p = cf_{i,j} \neq 0\}\end{array}\right).$$

That is
$$\bigcap_{j=1}^{m}\left(\begin{array}{l}\{mi_{k,j} : k = 1, \cdots, k_j\} \\ \bigcup\{r = fm\_f_{q,j} : q = n_{p,j}, n_{p,j}+1, \cdots, n_{p,j}+t_{p,j}-1, p = cf_{i,j} \neq 0\}\end{array}\right)$$
$$\neq \{i\} \text{ or } \{i\} \bigcup \{0\}$$

Case 2 $f_{l,j} \leq f_{i,j}$ for some $j \in \{1, \cdots m\}$ $\exists k'_j = f\_fm_{l,j}$ such that $1 \leq k'_j < k_j$ and $f_{l,j} = f_{i,j}$ for remaining $j \in \{1, \cdots m\}$ $\exists k'_j = f\_fm_{l,j}$ such that $k'_j > k_j$.

We may divide set $\{1,\cdots m\}$ as two sets: J_1 and J_2, where

$$\forall j \in J_1,\ f_{l,j} \le f_{i,j} \text{ such that } 1 \le k'_j < k_j,$$
$$\forall j \in J_2,\ f_{l,j} = f_{i,j} \text{ such that } k'_j > k_j,$$
$$J_1 \bigcap J_2 = \Phi,$$
$$J_1 \bigcup J_2 = \{1,\cdots m\}.$$

By the proof in Case 1,

$$i,l \in \{mi_{k,j} : k = 1,\cdots, o_{i,j}\} \text{ for each } \forall j \in J_1. \text{ Hence}$$

$$i,l \in \bigcap_{j\in J_1}\begin{pmatrix}\{mi_{k,j} : k = 1,\cdots,k_j\}\\ \bigcup\{r = fm\_f_{q,j} : q = n_{p,j}, n_{p,j}+1,\cdots, n_{p,j}+t_{p,j}-1,\ p = cf_{i,j} \ne 0\}\end{pmatrix}$$
$$\ne \{i\} \text{ or } \{i\}\bigcup\{0\}$$

holds. Now consider $j \in J_2$. Because $f_{l,j} = f_{i,j}$, components $f_{l,j}$ and $f_{i,j}$ are in the same group $p = cf_{i,j}$:

$$i,l \in \{r = fm\_f_{q,j} : q = n_{p,j}, n_{p,j}+1,\cdots, n_{p,j}+t_{p,j}-1,\ p = cf_{i,j} \ne 0\}$$

for every $j \in J_2$ component. Hence

$$i,l \in \bigcap_{j\in J_2}\begin{pmatrix}\{mi_{k,j} : k = 1,\cdots, f\_fm_{i,j}\}\\ \bigcup\{r = fm\_f_{q,j} : q = n_{p,j}, n_{p,j}+1,\cdots, n_{p,j}+t_{p,j}-1,\ p = cf_{i,j} \ne 0\}\end{pmatrix}$$

holds. It is clear that

$$i,l \in \bigcap_{j=1}^{m}\begin{pmatrix}\{mi_{k,j} : k = 1,\cdots, f\_fm_{i,j}\}\\ \bigcup\{r = fm\_f_{q,j} : q = n_{p,j}, n_{p,j}+1,\cdots, n_{p,j}+t_{p,j}-1,\ p = cf_{i,j} \ne 0\}\end{pmatrix}$$
$$\ne \{i\} \text{ or } \{i\}\bigcup\{0\}$$

for Case 2.

It follows from the proofs of Case 1, Case 2 and the sufficient condition that there is no dominating point $\mathbf{f}_l$. □

Applying Theorem 6.2, we can find non-dominated points, max-non-dominated points from set $\mathbf{N}^d \subseteq \mathbf{cF}^m$ and update set $\mathbf{cF}^m$ (if image point k is a dominated point then we set $m+1$ at c_k). Finally, we obtain non-dominated set and max non-dominated set:

$$F_{\min}^m = \{i : c_i = 1, c_i \in \mathbf{cF}^m\},$$
$$F_{\max}^m = \{i : c_i = 2, c_i \in \mathbf{cF}^m\}.$$

6.4 Solving the Discrete Multicriteria Optimization Problem

The non-dominated and max non-dominated sets can be obtained by section 6.2. According to Theorem 5.6 and Theorem 5.7, it follows from the mapping characteristic matrix $\mathbf{r}$ and status matrix $\mathbf{xr}$, see (6.1), that the solutions of discrete multicriteria optimization problem are described as the following theorem.

Theorem 6.3 Let

$$\mathbf{X}_{\min} = \bigcup_{i\in \mathbf{F}_{\min}^m} \{xr_{i,1}, \cdots, xr_{i,r_i}\},$$
$$r_{\min} = \sum_{i\in \mathbf{F}_{\min}^m} r_i = \mathcal{N}(\mathbf{X}_{\min}),$$
$$n_{\min} = \mathcal{N}(\mathbf{F}_{\min}^m),$$

$$\mathbf{X}_{\max} = \bigcup_{i\in \mathbf{F}_{\max}^m} \{xr_{i,1}, \cdots, xr_{i,r_i}\},$$
$$r_{\max} = \sum_{i\in \mathbf{F}_{\max}^m} r_i = \mathcal{N}(\mathbf{X}_{\max}),$$
$$n_{\max} = \mathcal{N}(\mathbf{F}_{\max}^m).$$

Then

$$\mathbf{X}_{\min} = \begin{cases} \mathbf{X}_{wE} & r_{\min} > n_{\min} \\ \mathbf{X}_{E} & r_{\min} = n_{\min} > 1 \\ \mathbf{X}_{sE} & r_{\min} = n_{\min} = 1 \end{cases},$$

and

$$\mathbf{X}_{\max} = \begin{cases} \mathbf{X}_{M\_wE} & r_{\max} > n_{\max} \\ \mathbf{X}_{M\_E} & r_{\max} = n_{\max} > 1 \\ \mathbf{X}_{M\_sE} & r_{\max} = n_{\max} = 1 \end{cases}.$$

Referring to Chapter 5, Chapter 6 and all the notations therein, it is clear that Theorem 6.3 holds.

6.5 A Small Example: Finding the (min or max) Non-Dominated Point Set of the Discrete Image $\mathbf{F}^m (m = 4)$

The example shows the process of the direct method.

Let

$$\mathbf{F}^m = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{bmatrix} = \begin{bmatrix} 0.3 & 1.5 & 2.2 & 6.5 \\ 1.8 & 2.3 & 2.4 & 4.2 \\ 1.2 & 2.3 & 2.2 & 1.1 \\ 0.3 & 1.5 & 8.2 & 6.5 \\ 1.8 & 3.2 & 2.4 & 4.2 \\ 0.3 & 1.5 & 6.2 & 6.5 \end{bmatrix}_{6\times 4}$$

We obtain $\mathbf{FM}^m$, the component ordered matrix, and $\mathbf{F\_FM}^m$ and $\mathbf{FM\_F}^m$, the associated Permutation Matrices:

$$\mathbf{FM}^m = \begin{bmatrix} \mathbf{fm}_1 \\ \mathbf{fm}_2 \\ \mathbf{fm}_3 \\ \mathbf{fm}_4 \\ \mathbf{fm}_5 \\ \mathbf{fm}_6 \end{bmatrix} = \begin{bmatrix} 0.3 & 1.5 & 2.2 & 1.1 \\ 0.3 & 1.5 & 2.2 & 4.2 \\ 0.3 & 1.5 & 2.4 & 4.2 \\ 1.2 & 2.3 & 2.4 & 6.5 \\ 1.8 & 2.3 & 6.2 & 6.5 \\ 1.8 & 3.2 & 8.2 & 6.5 \end{bmatrix}_{6\times 4}$$

$$\mathbf{F\_FM}^m = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 5 & 4 & 3 & 2 \\ 4 & 5 & 2 & 1 \\ 2 & 2 & 6 & 5 \\ 6 & 6 & 4 & 3 \\ 3 & 3 & 5 & 6 \end{bmatrix}_{6\times 4} \qquad \mathbf{FM\_F}^m = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 4 & 3 & 2 \\ 6 & 6 & 2 & 5 \\ 3 & 2 & 5 & 1 \\ 2 & 3 & 6 & 4 \\ 5 & 5 & 4 & 6 \end{bmatrix}_{6\times 4}$$

By $\mathbf{FM}^m$ and $\mathbf{FM\_F}^m$, set the class for each component of every point:

$$\mathbf{CF}^m = \left[cf_{i,j}\right]_{6\times4} = \begin{matrix}\mathbf{f}_1\\ \mathbf{f}_2\\ \mathbf{f}_3\\ \mathbf{f}_4\\ \mathbf{f}_5\\ \mathbf{f}_6\end{matrix}\begin{bmatrix}1&1&1&2\\2&2&2&1\\0&2&1&0\\1&1&0&2\\2&0&2&1\\1&1&0&2\end{bmatrix}_{6\times4} \qquad \mathbf{cF}^m = \left[c_i\right]_{6\times1} = \begin{matrix}\mathbf{f}_1\\ \mathbf{f}_2\\ \mathbf{f}_3\\ \mathbf{f}_4\\ \mathbf{f}_5\\ \mathbf{f}_6\end{matrix}\begin{bmatrix}4\\4\\2\\3\\3\\3\end{bmatrix}_{6\times1}$$

By $\mathbf{F\_FM}^m$, find the starting number and the number of members in $\mathbf{F\_FM}^m$ for each class in every component:

$$\mathbf{Nb} = \begin{matrix}class1\\ class2\end{matrix}\begin{bmatrix}1&1&1&2\\5&4&3&4\end{bmatrix}_{2\times4} \quad \mathbf{TT} = \begin{matrix}class1\\ class2\end{matrix}\begin{bmatrix}3&3&2&2\\2&2&2&3\end{bmatrix}_{2\times4}$$

Initially, set

$$\mathbf{Min} = \mathbf{Max} = \mathbf{FM\_F}^m = \begin{bmatrix}1&1&1&3\\4&4&3&2\\6&6&2&5\\3&2&5&1\\2&3&6&4\\5&5&4&6\end{bmatrix}_{6\times4}$$

Method 1:

Now, we can operate the initialized $\mathbf{Min}$ and $\mathbf{Max}$ to find min and max non-dominated sets from $\mathbf{F\_FM}^m$, $\mathbf{FM\_F}^m$, $\mathbf{CF}^m$, $\mathbf{Nb}$, $\mathbf{TT}$ in the first way.

Min $\bigcup$

$$fm\_f_{1,1} = 1 \xrightarrow{\text{find the classes of the } given \text{ point}} [cf_{1,1}\ cf_{1,2}\ cf_{1,3}\ cf_{1,4}] = [1\,1\,1\,2]$$

$$\xrightarrow{\text{find the first position and the number of class members in } \mathbf{Nb} \text{ and } \mathbf{TT}} [1\,1\,1\,4],\ [3\,3\,2\,3]$$

$$\xrightarrow{\text{find the last position for every conponent in } \mathbf{Min} \text{ matrix}} [3\,3\,2\,6]$$

$$\longrightarrow \{1, 4, 6\} \cap \{1, 4, 6\} \cap \{1, 3\} \cap \{3, 2, 5, 1, 4, 6\} = \{1\}$$

$$fm\_f_{2,1} = 4 \xrightarrow{\text{find the classes of the } given \text{ point}} [cf_{4,1}\ cf_{4,2}\ cf_{4,3}\ cf_{4,4}] = [1\,1\,0\,2]$$

$$\xrightarrow{\text{find the first position and the number of class members in } \mathbf{Nb} \text{ and } \mathbf{TT}} [1\,1\,0\,4],\ [3\,3\,0\,3]$$

$$\xrightarrow{\text{find the last position for every conponent in } \mathbf{Min} \text{ maxtrix}} [3\ 3\ f\_fm_{4,3}(6)\ 6]$$

$$\longrightarrow \{1, 4, 6\} \cap \{1, 4, 6\} \cap \{1, 3, 2, 5, 6, 4\} \cap \{3, 2, 5, 1, 4, 6\} = \{1, 4, 6\}$$

Point $\mathbf{f}_4$ and point $\mathbf{f}_6$ are dominated by points $\mathbf{f}_1$, and hence, we apply by the information in $\mathbf{F\_FM}^m$:

$$[f\_fm_{4,1}\ f\_fm_{4,2}\ f\_fm_{4,3}\ f\_fm_{4,4}] = [2\,2\,6\,5] \text{ and}$$
$$[f\_fm_{6,1}\ f\_fm_{6,2}\ f\_fm_{6,3}\ f\_fm_{6,4}] = [3\,3\,5\,6]$$

to delete them in $\mathbf{Min}$:

$$\mathbf{Min} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 2 & 5 \\ 3 & 2 & 5 & 1 \\ 2 & 3 & 0 & 0 \\ 5 & 5 & 0 & 0 \end{bmatrix}_{6\times 4}$$

$$fm\_f_{4,1} = 3 \quad \xrightarrow{\text{find the classes of the } given \text{ point}} \quad [cf_{3,1}\ cf_{3,2}\ cf_{3,3}\ cf_{3,4}] = [0\,2\,1\,0]$$

$$\xrightarrow{\text{find the first positionand the number of the class members in } \mathbf{Nb} \text{ and } \mathbf{TT}} [0\,4\,1\,0],\ [0\,2\,2\,0]$$

$$\xrightarrow{\text{find the last position for every conponent in } \mathbf{Min} \text{ maxtrix}} [f\_fm_{3,1}(4)\ 5\ 2\ f\_fm_{3,4}(1)]$$

$$\longrightarrow \{1, 3\} \cap \{1, 2, 3\} \cap \{1, 3\} \cap \{3\} = \{3\}$$

$$fm\_f_{5,1} = 2 \quad \xrightarrow{\text{find the classes of the } given \text{ point}} \quad [cf_{2,1}\ cf_{2,2}\ cf_{2,3}\ cf_{2,4}] = [2\ 2\ 2\,1]$$

$$\xrightarrow{\text{find the first positionand the number of the class members in } \mathbf{Nb} \text{ and } \mathbf{TT}} [5\,4\,3\,2],\ [2\,2\,2\,2]$$

$$\xrightarrow{\text{find the last position for every conponent in } \mathbf{Min} \text{ maxtrix}} [6\,5\,4\,3]$$

$$\longrightarrow \{1, 3, 2, 5, 0\} \cap \{1, 0, 2, 3\} \cap \{1, 3, 2, 5\} \cap \{3, 2, 5\} = \{3, 2\}$$

Point $\mathbf{f}_2$ is dominated by point $\mathbf{f}_3$, and then we apply theinformation in $\mathbf{F\_FM}^m$:

$$[f\_fm_{2,1}\ f\_fm_{2,2}\ f\_fm_{2,3}\ f\_fm_{2,4}] = [5\ 4\ 3\ 2]$$

to delete number 2 in $\mathbf{Min}$:

$$\mathbf{Min} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 3 & 0 & 5 & 1 \\ 0 & 3 & 0 & 0 \\ 5 & 5 & 0 & 0 \end{bmatrix}_{6\times 4}$$

$$fm\_f_{6,1} = 5 \xrightarrow{\text{find the classes of the } given \text{ point}} [cf_{5,1}\ cf_{5,2}\ cf_{5,3}\ cf_{5,4}] = [2\ 0\ 2\ 1]$$

$$\xrightarrow{\text{find the first positionand the number of the class members in } \mathbf{Nb} \text{ and } \mathbf{TT}} [5\ 0\ 3\ 2],\ [2\ 0\ 2\ 2]$$

$$\xrightarrow{\text{find the last position for every conponent in } \mathbf{Min} \text{ maxtrix}} [6\ f\_fm_{5,2}(6)\ 4\ 3]$$

$$\longrightarrow \{1, 0, 3, 5\} \cap \{1, 0, 3, 5\} \cap \{1, 3, 0, 5\} \cap \{3, 0, 5\} = \{0, 3, 5\}$$

Point $\mathbf{f}_5$ is dominated by point $\mathbf{f}_3$, and then we apply the information in $\mathbf{F\_FM}^m$: $[f\_fm_{5,1}\ f\_fm_{5,2}\ f\_fm_{5,3}\ f\_fm_{5,4}] = [6\ 6\ 4\ 3]$ to delete number 5 in $\mathbf{Min}$:

$$\mathbf{Min} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

At the end, we have min non-dominated point set: $\{\mathbf{f}_1, \mathbf{f}_3\}$.

Max $\bigcap$

We start from $\max_{6,1}$.

$$fm\_f_{6,1}=5 \xrightarrow{\text{find the classes of the } \textit{given} \text{ point}} [cf_{5,1}\ cf_{5,2}\ cf_{5,3}\ cf_{5,4}]=[2\ 0\ 2\ 1]$$

$$\xrightarrow{\text{find the first position and the number of the class members in } \mathbf{Nb}} [5\ 0\ 3\ 2]$$

$$\xrightarrow{\text{find the last position for every conponent in } \mathbf{M}\textit{ax} \text{ maxtrix}} [5\ f\_fm_{5,2}(6)\ 3\ 2]$$

$$\longrightarrow \{2,5\}\cap\{5\}\cap\{2,5,6,4\}\cap\{2,5,1,4,6\}=\{5\}$$

$$fm\_f_{5,1}=2 \xrightarrow{\text{find the classes of the } \textit{given} \text{ point}} [cf_{2,1}\ cf_{2,2}\ cf_{2,3}\ cf_{2,4}]=[2\ 2\ 2\ 1]$$

$$\xrightarrow{\text{find the first positionand the number of the class members in } \mathbf{Nb}} [5\ 4\ 3\ 2]$$

$$\xrightarrow{\text{find the last position for every conponent in } \mathbf{M}\textit{ax} \text{ maxtrix}} [5\ 4\ 3\ 2]$$

$$\longrightarrow \{2,5\}\cap\{2,3,5\}\cap\{2,5,6,4\}\cap\{2,5,1,4,6\}=\{2,5\}$$

By $\mathbf{f}_5$ is non-dominated point, we set

$$\mathbf{Max}=\begin{bmatrix}1&1&1&3\\4&4&3&2\\6&6&2&5\\3&2&5&1\\2&3&6&4\\5&5&4&6\end{bmatrix}_{6\times4} \Rightarrow \begin{bmatrix}1&1&1&3\\4&4&3&0\\6&6&0&5\\3&0&5&1\\0&3&6&4\\5&5&4&6\end{bmatrix}_{6\times4}$$

$$fm\_f_{4,1}=3 \xrightarrow{\text{find the classes of the } \textit{given} \text{ point}} [cf_{3,1}\ cf_{3,2}\ cf_{3,3}\ cf_{3,4}]=[0\ 2\ 1\ 0]$$

$$\xrightarrow{\text{find the first positionand the number of the class members in } \mathbf{Nb}} [0\ 4\ 1\ 0]$$

$$\xrightarrow{\text{find the last position for every conponent in } \mathbf{M}\textit{ax} \text{ maxtrix}} [f\_fm_{3,1}(4)\ 4\ 1\ f\_fm_{3,4}(1)]$$

$$\longrightarrow \{3,0,5\}\cap\{0,3,5\}\cap\{1,3,0,5,6,4\}\cap\{3,0,5,1,4,6\}=\{3,5,0\}.$$

As the reason before, we have

$$\mathbf{Max} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 4 & 3 & 2 \\ 6 & 6 & 2 & 5 \\ 3 & 2 & 5 & 1 \\ 2 & 3 & 6 & 4 \\ 5 & 5 & 4 & 6 \end{bmatrix}_{6\times 4} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 4 & 3 & 0 \\ 6 & 6 & 0 & 5 \\ 3 & 0 & 5 & 1 \\ 0 & 3 & 6 & 4 \\ 5 & 5 & 4 & 6 \end{bmatrix}_{6\times 4} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 4 & 4 & 0 & 0 \\ 6 & 6 & 0 & 5 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 6 & 4 \\ 5 & 5 & 4 & 6 \end{bmatrix}_{6\times 4}$$

$$fm\_f_{3,1} = 6 \xrightarrow{\text{find the classes of the } \textit{given} \text{ point}} [cf_{6,1}\ cf_{6,2}\ cf_{6,3}\ cf_{6,4}] = [1 1 0 2]$$

$$\xrightarrow{\text{find the first position and the number of the class members in } \mathbf{Nb}} [1 1 0 4]$$

$$\xrightarrow{\text{find the last position for every conponent in } \mathbf{M}ax \text{ maxtrix}} [1 1\ f\_fm_{6,3}(5)\ 4]$$

$$\longrightarrow \{1, 4, 6, 0, 5\} \cap \{1, 4, 6, 0, 5\} \cap \{6, 4\} \cap \{1, 4, 6\} = \{4, 6\}.$$

$$fm\_f_{2,1} = 4 \xrightarrow{\text{find the classes of the } \textit{given} \text{ point}} [cf_{4,1}\ cf_{4,2}\ cf_{4,3}\ cf_{4,4}] = [1 1 0 2]$$

$$\xrightarrow{\text{find the first position and the number of the class members in } \mathbf{Nb}} [1 1 0 4]$$

$$\xrightarrow{\text{find the last position for every conponent in } \mathbf{M}ax \text{ maxtrix}} [1 1\ f\_fm_{4,3}(6)\ 4]$$

$$\longrightarrow \{1, 4, 6, 0, 5\} \cap \{1, 4, 6, 0, 5\} \cap \{4\} \cap \{1, 4, 6\} = \{4\}.$$

$$fm\_f_{1,1} = 1 \xrightarrow{\text{find the classes of the } \textit{given} \text{ point}} [cf_{1,1}\ cf_{1,2}\ cf_{1,3}\ cf_{1,4}] = [1 1 1 2]$$

$$\xrightarrow{\text{find the first position and the number of the class members in } \mathbf{Nb}} [1 1 1 4]$$

$$\xrightarrow{\text{find the last position for every conponent in } \mathbf{M}ax \text{ maxtrix}} [1 1 1 4]$$

$$\longrightarrow \{1, 4, 6, 0, 5\} \cap \{1, 4, 6, 0, 5\} \cap \{1, 4, 6, 0, 5\} \cap \{1, 4, 6\} = \{1, 4, 6\}.$$

We have max non-dominated point set $\{4, 5\}$

Method 2 (in this thesis):

We add two matrices:

$$\mathbf{cF}^m = \begin{matrix}\mathbf{f}_1\\ \mathbf{f}_2\\ \mathbf{f}_3\\ \mathbf{f}_4\\ \mathbf{f}_5\\ \mathbf{f}_6\end{matrix}\begin{bmatrix} c_1\\ c_2\\ c_3\\ c_4\\ c_5\\ c_6\end{bmatrix}_{6\times1} = \begin{bmatrix}4\\4\\2\\3\\3\\3\end{bmatrix}_{6\times1} \quad \text{and} \quad \mathbf{W}_{4\times3} = \left[w_{i,j}\right]_{4\times3} = \begin{bmatrix}2 & 3 & 4\\ 1 & 3 & 4\\ 1 & 2 & 4\\ 1 & 2 & 3\end{bmatrix}_{4\times(3-1)}.$$

<u>The first step: using line concept</u>

$\mathbf{f}_1$:

$c_1 = 4 \geq 3$, point $\mathbf{f}_1$ may be on the same lines which parallels axes with other point(s).

Parallels ol_1: $[w_{1,1}\ w_{1,2}\ w_{1,3}] = [2 \quad 3 \quad 4] \xrightarrow{\text{find the class of the number of point}}$

$[cf_{1,2}\ cf_{1,3}\ cf_{1,4}] = [1\,1\,2] \xrightarrow{\text{the } first \text{ position and the number of the classs member in } \mathbf{Nb} \text{ and } \mathbf{T}\text{T}}$

$[1\,1\,4], [3\,2\,3] \longrightarrow \{1, 4, 6\} \cap \{1, 3\} \cap \{1, 4, 6\} = \{1\}$

No points on the line which parallels ol_1 and passes point $\mathbf{f}_1$.

Parallels ol_2: $[w_{2,1}\ w_{2,2}\ w_{2,3}] = [1 \quad 3 \quad 4] \xrightarrow{\text{find the class of the number of point}}$

$[cf_{1,1}\ cf_{1,3}\ cf_{1,4}] = [1\,1\,2] \xrightarrow{\text{the } first \text{ position and the number of the classs member in } \mathbf{Nb} \text{ and } \mathbf{T}\text{T}}$

$[1\,1\,4], [3\,2\,3] \longrightarrow \{1, 4, 6\} \cap \{1, 3\} \cap \{1, 4, 6\} = \{1\}$

No points on the line which parallels ol_2 and passes point $\mathbf{f}_1$.

Parallels ol_3: $[w_{3,1}\ w_{3,2}\ w_{3,3}] = [1 \quad 2 \quad 4] \xrightarrow{\text{find the class of the number of point}}$

$[cf_{1,1}\ cf_{1,2}\ cf_{1,4}] = [1\,1\,2] \xrightarrow{\text{the } first \text{ position and the number of the classs member in } \mathbf{Nb} \text{ and } \mathbf{T}\text{T}}$

$[1\,1\,4], [3\,3\,3] \longrightarrow \{1, 4, 6\} \cap \{1, 4, 6\} \cap \{1, 4, 6\} = \{1, 4, 6\}$

Points $\mathbf{f}_1$, $\mathbf{f}_4$ and $\mathbf{f}_6$ are on the same line which parallels ol_3.

$\min\{f_{1,3},\ f_{4,3},\ f_{6,3}\} = f_{1,3}$ and $\max\{f_{1,3},\ f_{4,3},\ f_{6,3}\} = f_{4,3}$

Operate **Min** and **Max** by position information $\mathbf{F\_FM}^m$

$$\mathbf{Min} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 4 & 3 & 2 \\ 6 & 6 & 2 & 5 \\ 3 & 2 & 5 & 1 \\ 2 & 3 & 6 & 4 \\ 5 & 5 & 4 & 6 \end{bmatrix}_{6\times 4} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 2 & 5 \\ 3 & 2 & 5 & 1 \\ 2 & 3 & 0 & 0 \\ 5 & 5 & 0 & 0 \end{bmatrix}_{6\times 4}$$

$$\mathbf{Max} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 4 & 3 & 2 \\ 6 & 6 & 2 & 5 \\ 3 & 2 & 5 & 1 \\ 2 & 3 & 6 & 4 \\ 5 & 5 & 4 & 6 \end{bmatrix}_{6\times 4} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 3 \\ 4 & 4 & 3 & 2 \\ 0 & 0 & 2 & 5 \\ 3 & 2 & 5 & 0 \\ 2 & 3 & 0 & 4 \\ 5 & 5 & 4 & 0 \end{bmatrix}_{6\times 4}$$

Use the same way for $\mathbf{f}_2$ and $[w_{2,1}\ w_{2,2}\ w_{2,3}] = [1 \quad 3 \quad 4]$, we obtain

$$\mathbf{Min} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 4 & 3 & 2 \\ 6 & 6 & 2 & 5 \\ 3 & 2 & 5 & 1 \\ 2 & 3 & 6 & 4 \\ 5 & 5 & 4 & 6 \end{bmatrix}_{6\times 4} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 2 & 5 \\ 3 & 2 & 5 & 1 \\ 2 & 3 & 0 & 0 \\ 5 & 5 & 0 & 0 \end{bmatrix}_{6\times 4} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 2 & 0 \\ 3 & 2 & 0 & 1 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 4}$$

$$\mathbf{Max} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 4 & 3 & 2 \\ 6 & 6 & 2 & 5 \\ 3 & 2 & 5 & 1 \\ 2 & 3 & 6 & 4 \\ 5 & 5 & 4 & 6 \end{bmatrix}_{6\times 4} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 3 \\ 4 & 4 & 3 & 2 \\ 0 & 0 & 2 & 5 \\ 3 & 2 & 5 & 0 \\ 2 & 3 & 0 & 4 \\ 5 & 5 & 4 & 0 \end{bmatrix}_{6\times 4} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 3 \\ 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 3 & 0 & 5 & 0 \\ 0 & 3 & 0 & 4 \\ 5 & 5 & 4 & 0 \end{bmatrix}_{6\times 4}$$

The Second step (using Method 1)

Apply Method 1 on the matrices that are obtained:

$$fm\_f_{5,1} = 2 \xrightarrow{\text{find the classes of the } given \text{ point}} [cf_{2,1}\ cf_{2,2}\ cf_{2,3}\ cf_{2,4}] = [2\ 2\ 2\ 1]$$

$$\xrightarrow{\text{find the first position and the number of the class members in } \mathbf{Nb} \text{ and } \mathbf{TT}} [5\ 4\ 3\ 2],\ [2\ 2\ 2\ 2]$$

$$\xrightarrow{\text{find the last position for every conponent in } \mathbf{M}in \text{ maxtrix}} [6\ 5\ 4\ 3]$$

$$\longrightarrow \{1, 0, 3, 2\} \cap \{1, 0, 2, 3\} \cap \{0, 1, 3, 2\} \cap \{3, 2, 0\} = \{0, 3, 2\}$$

Point $\mathbf{f}_2$ is not a dominated point, and then we apply the information in $\mathbf{F\_FM}^m$:
$[f\_fm_{2,1}\ f\_fm_{2,2}\ f\_fm_{2,3}\ f\_fm_{2,4}] = [5\ 4\ 3\ 2]$ to delete number 2 in $\mathbf{Min}$:

$$\mathbf{Min} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 2 & 0 \\ 3 & 2 & 0 & 1 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 4} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 4}$$

The min non-dominated point set: $\{\mathbf{f}_1, \mathbf{f}_3\}$.

$$\mathbf{Max} = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 4 & 4 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 3 & 0 & 5 & 0 \\ 0 & 3 & 0 & 4 \\ 5 & 5 & 4 & 0 \end{bmatrix}_{6\times 4} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 4 \\ 5 & 5 & 4 & 0 \end{bmatrix}_{6\times 4}$$

The max non-dominated point set: $\{\mathbf{f}_4, \mathbf{f}_5\}$.

By the way, final matrix $\mathbf{cF}^m = [c_i]_{6\times 1} = \begin{matrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \end{matrix} \begin{bmatrix} 1 \\ 5 \\ 1 \\ 2 \\ 2 \\ 5 \end{bmatrix}_{6\times 1}$

Chapter 7 The Direct Method Algorithm of Solving the Discrete Multiobjective Optimization Problem

Let $\mathbf{X}=\{\mathbf{x}_1,\cdots,\mathbf{x}_h\}$ be feasible set, $\mathbf{F}_{n\times m}$ be its mapping matrix, and $\mathbf{r}_{n\times 1}$ be mapping characteristic matrix and $\mathbf{xr}_{n\times\overline{m}}$ be status matrix be given and formed.

Step 1 Form ordered matrix $\mathbf{FM}_{n\times m}$ permutation matrix $\mathbf{FM\_F}_{n\times m}$ and permutation matrix $\mathbf{F\_FM}_{n\times m}$

1−1 $[\mathbf{FM},\mathbf{FM\_F}]\Leftarrow\mathbf{sort}(\mathbf{F},n)$

1−2 $[\sim,\mathbf{F\_FM}]\Leftarrow\mathbf{sort}(\mathbf{FM\_F},n)$

Step 2 Form component classification matrix $\mathbf{CF}_{n\times m}$, additional matrix $\mathbf{Nb}=[n_{k,j}]_{s_j\times m}$ and additional matrix $\mathbf{TT}=[t_{k,j}]_{s_j\times m}$ according to matrix $\mathbf{FM}_{n\times m}$

2−1 $\mathbf{CF}_{n\times m}\Leftarrow\mathbf{0}_{n\times m},\mathbf{SS}_{1\times m}\Leftarrow\mathbf{0}_{1\times m}$

2−2 Do **for** $j=1,\cdots,m$

In accordance with j column of sorted matrix $\mathbf{FM}_{n\times m}$

(1) Apply the principles of classification (6.6) in section 6.1.2 to determine the class for each image point at column j (fill the value in $\mathbf{CF}_{n\times m}$)

(2) Determine the location in $\mathbf{FM}_j$ of the first image point in each class (>1) and numbers in each class (fill the value in column j in $\mathbf{Nb}$ and $\mathbf{TT}$)

(3) Find the numbers of classes in column j of $\mathbf{F}_{n\times m}$ or $\mathbf{FM}_{n\times m}$ (fill the value in column j in $\mathbf{SS}_{1\times m}$)

Enddo.

Step 3 Form image point classification matrix $\mathbf{cF}_{n\times 1}$

3−1 $\mathbf{cF}_{n\times 1}\Leftarrow\mathbf{0}_{n\times 1}$

3−2 Do **for** $i=1,\cdots,n$

In accordance with the rule of (6.7) in section 6.1.2 to fill in $\mathbf{cF}_{n\times 1}$ for image point i

Enddo.

Step 4 Form matrix $\mathbf{W}_{m\times(m-1)}$

4−1 Follow formula (6.11) to form matrix $\mathbf{W}_{m\times(m-1)}$

Step 5 Fill in matrices $\mathbf{Min}$ and $\mathbf{Max}$

$5-1$ $\mathbf{Max}_{n\times m}, \mathbf{Min}_{n\times m} \Leftarrow \mathbf{FM\_F}_{n\times m}$

$5-2$ Do **for** $i = 1, \cdots, n$

Find the characteristic of i image point from $\mathbf{cF}_{n\times 1}$ matrix

If the value of the characteristic of i image point $\geq m-1$

(1) Follow matrix $\mathbf{W}_{m\times(m-1)}$ to find all points that are on the same segments with i image point according to information ($cf_{i,j}$ $j = 1, \cdots, m$) of each component classification of i image point in matrix $\mathbf{CF}_{n\times m}$ and additional matrices $\mathbf{Nb}, \mathbf{TT}$

(2) Find max and min points on each segment

(3) Apply permutation matrix $\mathbf{F\_FM}_{n\times m}$ to update matrices $\mathbf{Max}_{n\times m}$, $\mathbf{Min}_{n\times m}$ and $\mathbf{cF}_{n\times 1}$ in accordance with the rules (6.16), (6.17) and (6.15)

Endif

Enddo.

Step 6 Solve matrices $\mathbf{Min}_{n\times m}$ and $\mathbf{Max}_{n\times m}$ to find the non-dominated sets

$6-1$ Solve matrix $\mathbf{Min}_{n\times m}$ to determine whether the image is a non-dominated point

Do **for** $i = n, \cdots, 1$

(1) Find the value of the first column at i row

(2) If the value > 0, (i.e., the image point)

(i) Following Theorem (6.2) to find whether the image point is a non-dominated point

(ii) If the image point is a dominated point then fill 0 at the location $f\_fm_{i,j}$, $j = 1, \cdots, m$ in each column in $\mathbf{Min}_{n\times m}$ and $m+1$ at location i in matrix $\mathbf{cF}_{n\times 1}$

Endif

Enddo

$6-2$ Solve matrices $\mathbf{Max}_{n\times m}$ to determine whether the image is a non-dominated point

Do **for** $i = 1, \cdots, n$

(1) Find the value of the first column at i row

(2) If the value > 0, (i.e., the image point)

(i) Following Theorem (6.2) to find whether the image point is a non-dominated point

(ii) If the image point is a dominated point then fill 0 at the location $f\_fm_{i,j}$, $j = 1, \cdots, m$ in each column in $\mathbf{Max}_{n \times m}$ and $m + 1$ at location i in matrix $\mathbf{cF}_{n \times 1}$

Endif

Enddo

Chapter 8 Several Examples of the Discrete Multiobjective Optimization Problem

We design the software in accordance with the algorithm in Chapter 7 to solve several examples of discrete multiobjective problems. The results, which include non-dominated set and max-non-dominated set and the running time, will be listed. The software is designed in MATLAB 6.5 and is performed on a computer:

System: Microsoft Window XP,
Computer: Intel (R),
Pentium (R) 4 CPU 3.00GHz,
2.99 GHz. 504MB.

8.1 Examples of Non-Dominated Sets that can be Perceived by Inspection

8.1.1 Construct Image Points

Let $ol_1 \cdots l_m$ be coordinate system in $m \in N^+$ dimension space and the center of a super sphere with radius $r \in N^+$ be located at point $(\overbrace{r, \cdots, r}^{m})$. All designed points are on the areas of planes $ol_1 l_j$, $j = 2, \cdots, m$ which are touched by the surface of the super sphere: $\sum_{j=1}^{m} (x_j - r)^2 = r^2$, where $(x_1, \cdots, x_m)$ is a varied point at the surface. The area's boundary can be described as

$$\begin{cases} ol_1 l_j \text{ plane}, \ j = 2, \cdots, m, \\ \text{super sphere: } \sum_{j=1}^{m} (x_j - r)^2 = r^2. \end{cases}$$

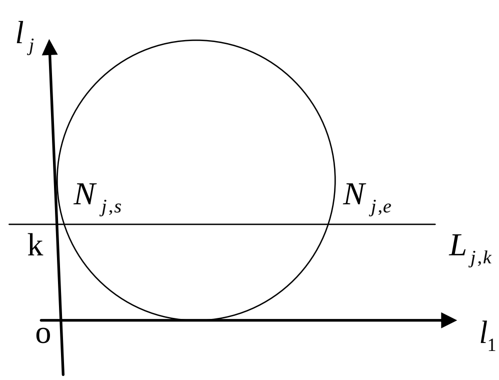

Figure 5: the two intersection points of line $L_{j,k}$ and an area of disk

Let $N_{j,s}$ and $N_{j,e}$ be the two intersection points of line $L_{j,k}$ and an area of disk in ol_1l_j plane:

$$\begin{cases} \text{line } L_{j,k}: \begin{cases} x_1 = x,\ x \in R \\ x_j = k,\ k \in \{1, \cdots, r-1\} & j \in \{2, \cdots, m\} \\ x_s = 0 & s \neq 1,\ j \end{cases} \\ \text{super sphere: } \sum_{j=1}^{m} (x_j - r)^2 = r^2 \end{cases}$$

We pick up some points on the segments $N_{j,s}N_{j,e}$, $j = 2, \cdots, m$:

$$A = \left\{ \begin{array}{l} (i, r, \cdots, \overset{j-1}{r}, \overset{j}{2r-k}, \overset{j+1}{r}, \cdots, \overset{m}{r}), (i, r, \cdots, \overset{j-1}{r}, \overset{j}{k}, \overset{j+1}{r}, \cdots, \overset{m}{r}): \\ \qquad i = \overline{N}_{j,s}, \overline{N}_{j,s} + 1, \cdots, \overline{N}_{j,e}, \overline{N}_{j,s} = r - n1 \\ \qquad \overline{N}_{j,e} = r + n1,\ n1 = fix\left(\sqrt{k(2r-k)}\right), \\ \qquad k = 1, \cdots, r-1,\ j = 2, \cdots, m \end{array} \right\},$$

where $n1 \leq \sqrt{k(2r-k)} < n1 + 1,\ n1 \in N^+ \cup \{0\}$. In addition, some points are on l_1 axis ,

$$B = \left\{ (i, r, \cdots, \overset{m}{r}) : i = 0, 1, \cdots, 2r \right\}.$$

All image points are formed by $A \cup B$.

8.1.2 Construct Set $\mathbf{F}^m$

Let n be the number of image points in set $A \cup B$. Change the n points order state in set $A \cup B$ by random permutation to obtain a random permutation of the integers 1~n:

$$A \cup B \overset{random\ permutation}{\Longrightarrow} \mathbf{F}^m,$$

where

$$A \bigcup B = \mathbf{F}^m = \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_i \\ \vdots \\ \mathbf{f}_n \end{bmatrix} = \begin{bmatrix} f_{1,1} & \cdots & f_{1,j} & \cdots & f_{1,m} \\ \vdots & & \vdots & & \vdots \\ f_{i,1} & \cdots & f_{i,j} & \cdots & f_{i,m} \\ \vdots & & \vdots & & \vdots \\ f_{n,1} & \cdots & f_{n,j} & \cdots & f_{n,m} \end{bmatrix}_{n \times m} .$$

8.1.3 Examples and Results

Example 1 $r = 5, m = 3$.

We obtain the total number of image points = 147, the number of min points = 5, and the number of max points = 5 and elapsed time = 0.0930 seconds. The solutions are

Number	The Coordinates of the (min) non-dominated points
115	0 5 5
63	1 5 2
72	1 2 5
20	2 1 5
139	2 5 1

Number	The Coordinates of the (max) non-dominated points
59	8 9 5
132	8 5 9
58	9 5 8
144	9 8 5
86	10 5 5

Figure 6 shows the results on the planes:

ol_1l_2 plane: j1=1 j2=2,

ol_1l_3 plane: j1=1 j2=3.

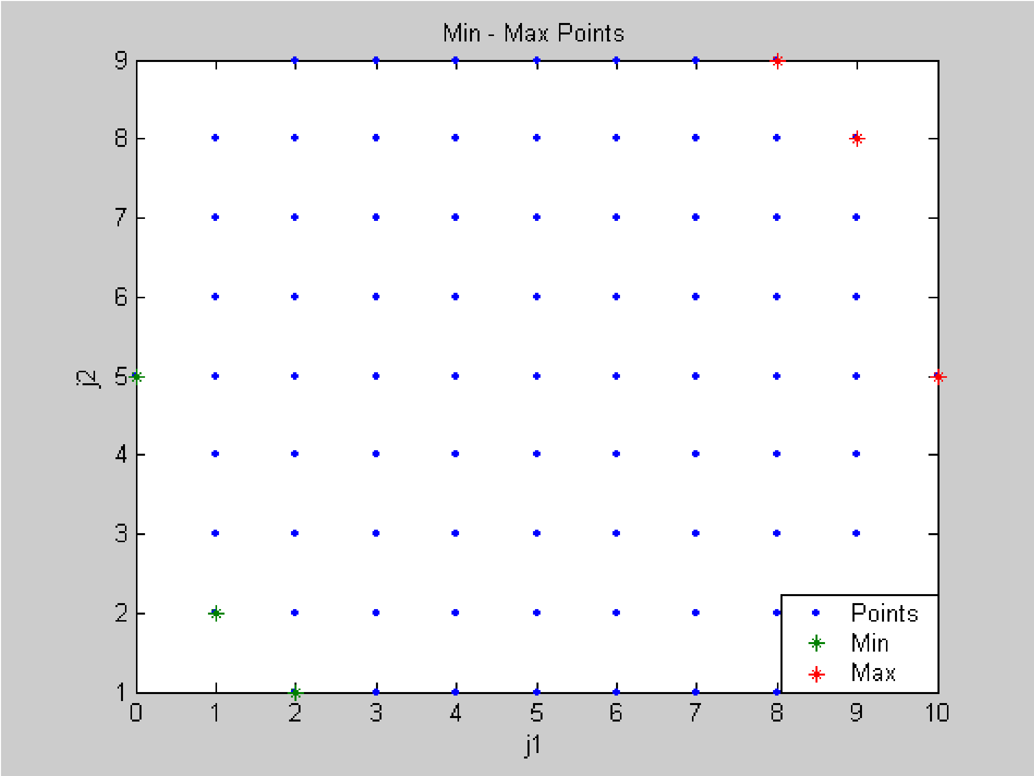

Figure 6: image points, min points and max points in Example 1

Example 2 $r = 8, m = 4$.

The total number of image points = 551, the number of min points = 13, and the number of max points = 13 and elapsed time = 0.2190 seconds. The solutions are

Number	The Coordinates of the (min) non-dominated points			
337	0	8	8	8
144	1	5	8	8
262	1	8	5	8
532	1	8	8	5
35	2	8	3	8
505	2	8	8	3
543	2	3	8	8
24	3	8	8	2
124	3	2	8	8
402	3	8	2	8
229	5	8	8	1
503	5	8	1	8
539	5	1	8	8

Number	The Coordinates of the (max) non-dominated points
159	11 8 8 15
393	11 8 15 8
491	11 15 8 8
60	13 14 8 8
250	13 8 8 14
410	13 8 14 8
311	14 13 8 8
323	14 8 13 8
377	14 8 8 13
66	15 11 8 8
105	15 8 8 11
483	15 8 11 8
453	16 8 8 8

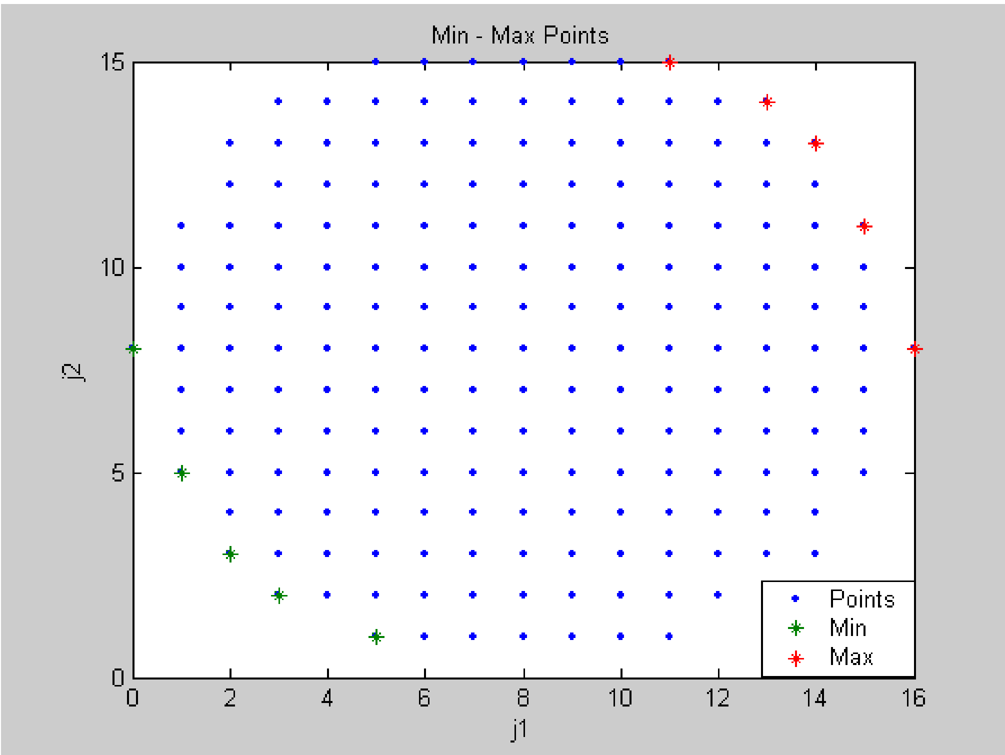

Figure 7: image points, min points and max points in Example 2

Figure 7 shows the results on the planes:

ol_1l_2 plane: j1=1 j2=2,

ol_1l_3 plane: j1=1 j2=3,

ol_1l_4 plane: j1=1 j2=4,

Example 3 $r = 20, m = 8$.

The total created number of image points = 8539, the min points = 78, and the max points = 78 and running time = 43.1710 seconds. The solutions are

Number	The Coordinates of the (min) non-dominated points							
3196	0	20	20	20	20	20	20	20
843	1	20	20	14	20	20	20	20
3022	1	20	14	20	20	20	20	20
7225	1	14	20	20	20	20	20	20
7423	1	20	20	20	20	20	14	20
7632	1	20	20	20	20	20	20	14
7899	1	20	20	20	14	20	20	20
7915	1	20	20	20	20	14	20	20
1589	2	20	12	20	20	20	20	20
3661	2	20	20	20	20	20	12	20
5896	2	20	20	20	12	20	20	20
6551	2	12	20	20	20	20	20	20
6759	2	20	20	20	20	12	20	20
6991	2	20	20	20	20	20	20	12
7016	2	20	20	12	20	20	20	20
521	3	20	20	10	20	20	20	20
2113	3	10	20	20	20	20	20	20
3095	3	20	10	20	20	20	20	20
4185	3	20	20	20	20	10	20	20
5153	3	20	20	20	20	20	10	20
7702	3	20	20	20	10	20	20	20
7843	3	20	20	20	20	20	20	10
643	4	20	20	20	20	20	8	20
2055	4	20	20	8	20	20	20	20
2813	4	20	8	20	20	20	20	20
3191	4	20	20	20	20	20	20	8
3947	4	8	20	20	20	20	20	20
7960	4	20	20	20	8	20	20	20
8437	4	20	20	20	20	8	20	20
2010	5	20	20	7	20	20	20	20
2415	5	20	7	20	20	20	20	20
3018	5	7	20	20	20	20	20	20
5155	5	20	20	20	20	7	20	20
6850	5	20	20	20	20	20	20	7
7506	5	20	20	20	7	20	20	20
7569	5	20	20	20	20	20	7	20
2921	6	20	20	20	6	20	20	20

Number	The Coordinates of the (min) non-dominated points							
3764	6	20	6	20	20	20	20	20
5359	6	20	20	6	20	20	20	20
7158	6	20	20	20	20	20	20	6
7739	6	20	20	20	20	6	20	20
8173	6	20	20	20	20	20	6	20
8442	6	6	20	20	20	20	20	20
1962	7	5	20	20	20	20	20	20
2439	7	20	20	20	5	20	20	20
4952	7	20	20	20	20	20	5	20
5102	7	20	20	5	20	20	20	20
6568	7	20	20	20	20	5	20	20
8459	7	20	20	20	20	20	20	5
8468	7	20	5	20	20	20	20	20
200	8	20	4	20	20	20	20	20
910	8	4	20	20	20	20	20	20
2277	8	20	20	20	20	20	20	4
3009	8	20	20	20	4	20	20	20
3494	8	20	20	20	20	20	4	20
3631	8	20	20	20	20	4	20	20
8267	8	20	20	4	20	20	20	20
328	10	20	3	20	20	20	20	20
1572	10	20	20	20	3	20	20	20
1632	10	20	20	20	20	20	20	3
2589	10	20	20	20	20	3	20	20
3501	10	20	20	20	20	20	3	20
6497	10	3	20	20	20	20	20	20
8241	10	20	20	3	20	20	20	20
886	12	20	2	20	20	20	20	20
1684	12	20	20	20	20	20	20	2
2548	12	2	20	20	20	20	20	20
2870	12	20	20	20	2	20	20	20
3437	12	20	20	20	20	20	2	20
7134	12	20	20	20	20	2	20	20
8052	12	20	20	2	20	20	20	20
1141	14	20	20	20	20	20	20	1
3412	14	20	20	20	20	20	1	20
3542	14	20	20	20	1	20	20	20
3639	14	20	1	20	20	20	20	20
4476	14	20	20	1	20	20	20	20
5878	14	1	20	20	20	20	20	20
7829	14	20	20	20	20	1	20	20

Number	The Coordinates of the (max) non-dominated points							
1562	26	20	39	20	20	20	20	20
2874	26	20	20	39	20	20	20	20
4540	26	20	20	20	20	20	20	39
5779	26	20	20	20	20	20	39	20
6277	26	39	20	20	20	20	20	20
7817	26	20	20	20	39	20	20	20
8124	26	20	20	20	20	39	20	20
33	28	20	38	20	20	20	20	20
695	28	20	20	38	20	20	20	20
1546	28	20	20	20	20	20	38	20
3553	28	20	20	20	20	38	20	20
5235	28	20	20	20	38	20	20	20
6960	28	20	20	20	20	20	20	38
7429	28	38	20	20	20	20	20	20
375	30	20	20	20	20	20	37	20
543	30	20	37	20	20	20	20	20
3259	30	20	20	20	37	20	20	20
4173	30	20	20	37	20	20	20	20
5109	30	20	20	20	20	20	20	37
6263	30	20	20	20	20	37	20	20
7752	30	37	20	20	20	20	20	20
79	32	20	20	36	20	20	20	20
669	32	20	20	20	20	20	20	36
706	32	36	20	20	20	20	20	20
985	32	20	36	20	20	20	20	20
1657	32	20	20	20	36	20	20	20
1859	32	20	20	20	20	20	36	20
8511	32	20	20	20	20	36	20	20
3926	33	20	20	20	20	35	20	20
4263	33	20	20	35	20	20	20	20
4280	33	20	20	20	35	20	20	20
5384	33	20	20	20	20	20	20	35
5706	33	20	35	20	20	20	20	20
8182	33	20	20	20	20	20	35	20
8482	33	35	20	20	20	20	20	20
1698	34	34	20	20	20	20	20	20
2912	34	20	20	34	20	20	20	20
3797	34	20	20	20	20	34	20	20
4301	34	20	20	20	20	20	20	34
5429	34	20	20	20	34	20	20	20
5822	34	20	34	20	20	20	20	20
7793	34	20	20	20	20	20	34	20
345	35	20	20	20	20	20	20	33
2925	35	20	20	20	33	20	20	20

Number	The Coordinates of the (max) non-dominated points							
3870	35	20	20	33	20	20	20	20
6432	35	20	33	20	20	20	20	20
7034	35	33	20	20	20	20	20	20
7583	35	20	20	20	20	20	33	20
7963	35	20	20	20	20	33	20	20
3094	36	20	20	20	32	20	20	20
5824	36	20	32	20	20	20	20	20
6315	36	20	20	20	20	20	20	32
6331	36	20	20	20	20	32	20	20
6376	36	20	20	32	20	20	20	20
7189	36	32	20	20	20	20	20	20
8390	36	20	20	20	20	20	32	20
317	37	30	20	20	20	20	20	20
1576	37	20	20	20	30	20	20	20
2751	37	20	30	20	20	20	20	20
3026	37	20	20	30	20	20	20	20
3226	37	20	20	20	20	20	30	20
3499	37	20	20	20	20	30	20	20
4199	37	20	20	20	20	20	20	30
785	38	28	20	20	20	20	20	20
1789	38	20	20	20	28	20	20	20
2195	38	20	20	20	20	20	28	20
5543	38	20	20	28	20	20	20	20
6517	38	20	20	20	20	20	20	28
7036	38	20	28	20	20	20	20	20
7437	38	20	20	20	20	28	20	20
1448	39	26	20	20	20	20	20	20
2067	39	20	20	20	20	20	20	26
2907	39	20	26	20	20	20	20	20
3958	39	20	20	20	20	20	26	20
5060	39	20	20	20	26	20	20	20
7263	39	20	20	26	20	20	20	20
8320	39	20	20	20	20	26	20	20
5962	40	20	20	20	20	20	20	20

Figure 8 shows the results on the planes:

ol_1l_2 plane: j1=1 j2=2,
ol_1l_3 plane: j1=1 j2=3,
ol_1l_4 plane: j1=1 j2=4,
ol_1l_5 plane: j1=1 j2=5,
ol_1l_6 plane: j1=1 j2=6,
ol_1l_7 plane: j1=1 j2=7,

ol_1l_8 plane: j1=1 j2=8.

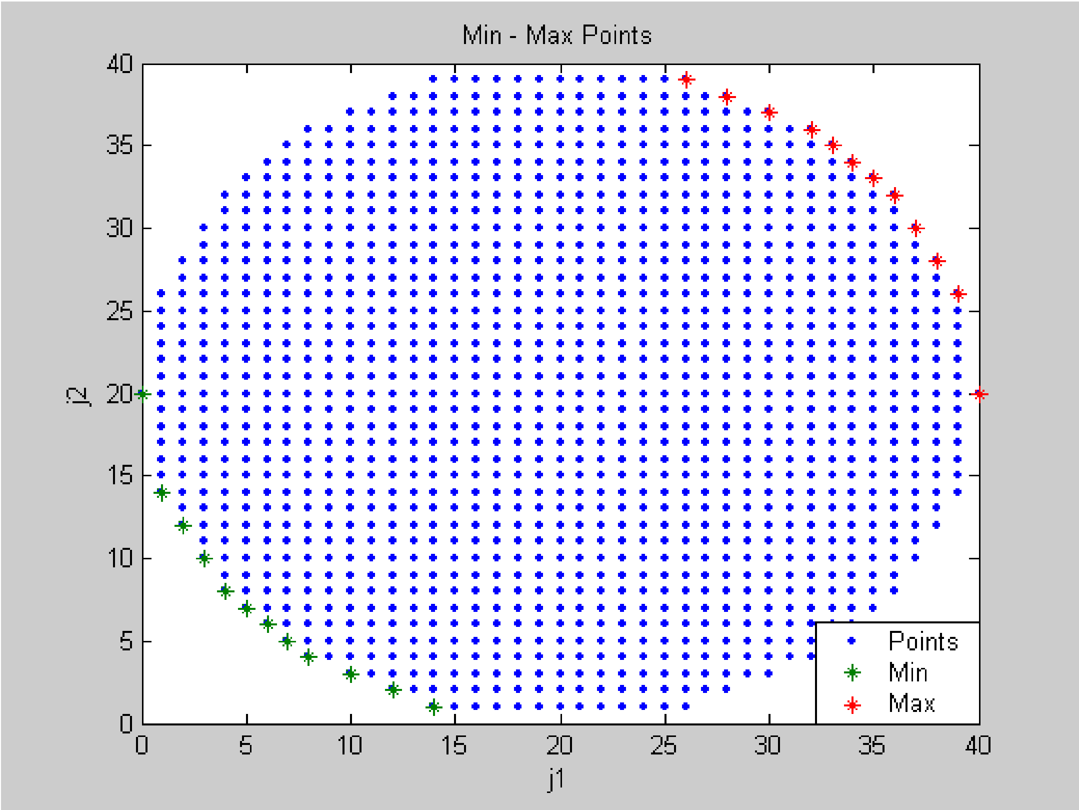

Figure 8: image points, min points and max points in Example 3

Example 4 $r = 50, m = 10$

The total number of image points is 69779; the the number of the min points obtained is 262 and the max points is 262. The running time is much larger than the previous three examples, that is 1934.815 seconds (32 minutes 14.815 seconds). The detail solutions are listed in Appendix II. Here, we only provide the Figure which is obtained by running the software.

ol_1l_2 plane: j1=1 j2=2
ol_1l_3 plane: j1=1 j2=3
ol_1l_4 plane: j1=1 j2=4
ol_1l_5 plane: j1=1 j2=5
ol_1l_6 plane: j1=1 j2=6
ol_1l_7 plane: j1=1 j2=7
ol_1l_8 plane: j1=1 j2=8
ol_1l_9 plane: j1=1 j2=9

ol_1l_{10} plane: j1=1 j2=10

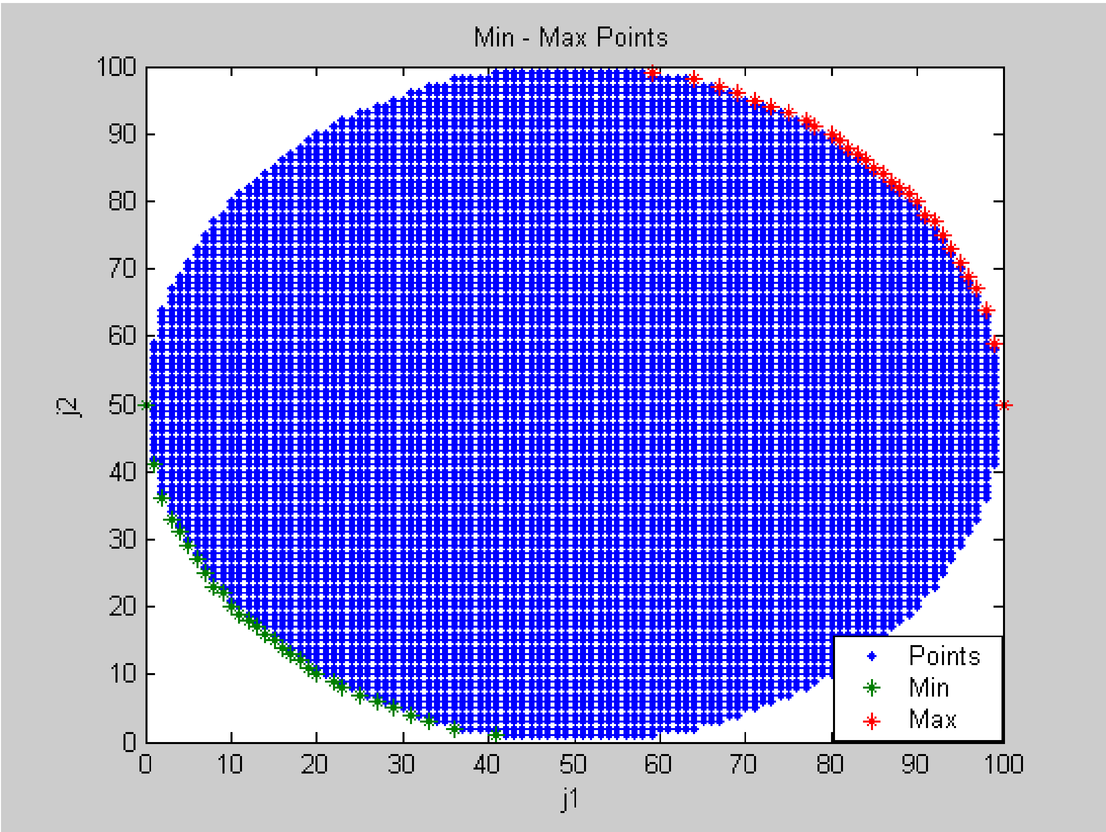

Figure 9: image points, min points and max points in Example 4

If using HP Compaq 6005 Pro Microtower to run example 4, then the time is 1029.321451 second (17 minutes 9.321451 seconds).

We easily obtain the solutions by calculation for the previous four examples and verify all the solutions are consistent with our solutions. Actually, the designed software fits any kind of discrete multiobjective problem.

8.2 Examples of Non-Dominated Sets that cannot be Directly Perceived by Inspection

8.2.1 Construct Image Points

$$\mathbf{F} = rand([n, m]) * rr,$$

where n is the number of image points, m is the dimension of each image point and $rr > 0$ is a real positive number.

8.2.2 Examples and Results

Example 5 $n = 147$, $m = 3$ and $rr = 10$.

Elapsed time is 0.083852 seconds. For detailed results, see Appendix III (a).

Example 6 $n = 551$, $m = 4$ and $rr = 16$.

For detailed results, see Appendix III (b).

Example 7 $n = 8539$, $m = 8$ and $rr = 40$.

Elapsed time is 175.449175 seconds. For detailed results, see Appendix III (c).

8.3 Special Example of Non-Dominated Set can be Directly Perceived by Inspection

8.3.1 Construct Image Points

$$
\begin{aligned}
&\mathbf{F}x = rand(n) * rr, \\
&\mathbf{F}(i,1) = \mathbf{F}x(i), \qquad i = 1, 2, \cdots, n, \\
&\mathbf{F}(i,j) = -\mathbf{F}x(i) + rr, \quad i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, m,
\end{aligned}
$$

where $n = 70000$, $m = 10$ and $rr = 200$.

8.3.2 Results

Elapsed time is 518 mintues. For partial results, see Appendix III (d).

Chapter 9 Conclusion and Further Research

9.1 Conclusion

In this thesis, it has been clear that: the Multicriteria Optimization Problem is, in geometry, finding a requested partial boundary of an image, which is mapped by a feasible set and inverting the partial boundary. The requested partial boundary is filtered by various filters (a kind of non-dominated) designed by an operator. The author of the thesis applies synthesized knowledge of analysis, algebra and geometry to strictly prove the facts and develop an algorithm for MOP and M_MOP. Based on solutions to the multiobjective optimization problem being the main subject of solving MOP (M_MOP) and most MOP (M_MOP) being discrete (or discrete and continuous), the author designed a software selection partially listed in Index I in MATLAB for solving discrete multiobjective problems. The results of running several examples are listed in Chapter 8 and Index II. This research opens up giant applications of MOP and M_MOP in all fields such as management, engineering, medical science and so on, and also establishes a solid foundation for further research in the MOP field.

There is a lot of further research to be done in this field, including theory and applications. We list some of them as the follows.

9.2 Further Research

Data Classification

The experimental data, statistical data and simulated data are mostly huge and high dimension need in classification. We can use MOP (M_MOP) to obtain the initial set as the first class by using a type of filter, then we can take away these points and obtain a set of new points as a second class. Continuing the work at the terminal, we will obtain a base which contains these classes in using the same filter. Using the obtained base, we can then construct a group. If there are u multicriteria optimization classes in the base then we can construct a group which contains 2^u different classes. The data classes depend on the selected filter (the requirement in the application). The multicriteria optimization classification has to be researched and completed in this field.

Data classification should include resource classification such as human resources, company resources, natural resources, school resources, and so on. For example, the (discipline) multicriteria optimization classification determines the class of a person in his/her best work field(s). The student and employee multicriteria optimization classification are also very useful.

Date classification also involves dynamic multicriteria optimization classification because of dynamic data.

In private companies, the classification can be used rationally for all kinds of resources.

Management Application

Management systems make some decisions in accordance with the obtained information (an image in criteria space).

For example, the government of a country may have a very clear picture of the human resources, company resource, natural resources, school resources, and so on, which governments could use when they need to optimize decisions, such as opening or closing a school, opening or closing some majors in universities, setting new research projects, investments, and courses setting in university, immigration types, and so on. The optimal decision set will be found by solving the multicriteria optimization. The government will make appropriate decisions for the country development.

The multicriteria optimization can be applied in insurance management and (banks) investment management. An insurance owner can manage his company well and reduce his costs (rent and employees) and customers see very clearly how and why we are in this class and paying. In investing, we can determine risk level we are in.

A military organization that selects a team from staff to fulfill a special task is also a multicriteria optimization problem. It is a team optimization according to the special task and the military qualities of every member of the team. Each member of the team may be in different classes in the military.

The Multicriteria Optimization Control

We have mentioned a typical example of a multicriteria optimization control problem in section 2.1, an automatic control system design of a type of flying vehicle.

During the flying of an airplane, such as landing or taking off, we obtain the class risk of status of the airplane and the optimal operations set in any flying point, we will give the best suggestions to the pilot to obtain the best flying control.

In fact, we have lots of work to do in this field. Here, we list only a few of them. We would leave it to the reader find others.

References

[1] Antonio Flores-Tlacuahuac, Pilar Morales, and Martín Rivera-Toledo, "Multiobjective Nonlinear Model Predictive Control of a Class of Chemical Reactors," *Industrial & Engineering Chemistry Research*, vol. 51(17), pp. 5891-5899, 2011

[2] A. L. Custódio, J. F. A. Madeira, A. I. F. Vaz, and L. N. Vicente, "Direct Multisearch for Multiobjective Optimization," *SIAM Journal on Optimization*, vol. 21(3), pp. 1109-1140, 2011

[3] Benson H. P., "Existence of Efficient Solutions for Vector Maximization Problems," *Journal of Optimization Theory and Applications*, 26(4), pp. 569-580, 1978

[4] Charles Audet, Gilles Savard, Walid Zghal, "A mesh adaptive direct search algorithm for multiobjective optimization," *European Journal of Operational Research*, vol. 204(3), pp. 545-556, 2010

[5] Crina Grosan, Ajith Abraham, "Approximating Pareto frontier using a hybrid line search approach," *Information Sciences*, vol. 180(14), pp. 2674-2695, 2010

[6] Dingzhou Cao, Alper Murat, Ratna Babu Chinnam, "Efficient exact optimization of multi-objective redundancy allocation problems in series-parallel systems," *Reliability Engineering & System Safety*, vol. 111, pp. 154-163, 2013

[7] Emam Tarek, "Method of centers algorithm for multi-objective programming problems," *Acta Mathematica Scientia*, vol. 29(5), pp. 1128-1142, 2009

[8] Ehrgott, M. and Ryan, D., "Constructing robust crew schedules with bicriteria optimization," *Journal of Multi-Criteria Decision Analysis*, 11(3), 139–150, 2002

[9] F. Logist, J. Van Impe, "Multi-objective dynamic optimisation of cyclic chemical reactors with distributed parameters," *Chemical Engineering Science*, vol. 80, pp. 429-434, 2012

[10] Gabriele Eichfelder, "Multiobjective bilevel optimization," *Mathematical Programming*, vol. 123(2), pp. 419-449, 2010

[11] Göpfert, A and Nehse, R., "Vektoroptimierung (in German)," *B.G. Teubner, Leipzig*, 1990

[12] Hella Tokos, Zorka Novak Pintarič, Yongrong Yang, "Bi-objective optimization of a water network via benchmarking," *Journal of Cleaner Production*, vol. 39, pp. 168-179, 2013

[13] Heng Xu, Yinglin Wang, Xin Xu, "Multiobjective Particle Swarm Optimization based on Dimensional Update," *Engineering optimization*, vol. 22(3), 2013

[14] H. Zidani, E. Pagnacco, R. Sampaio, R. Ellaia & J. E. Souza de Cursi, "Multi-objective optimization by a new hybridized method: applications to random mechanical systems," *Journal of Engineering optimization*, vol. 45(8), pp. 917-939, 2013

[15] Kaisa Meittinen, "Nonlinear Multiobjective Optimization," *Kluwer Academic Publishers*, 2012

[16] Jörg Fliege, Huifu Xu, "Stochastic Multiobjective Optimization: Sample Average Approximation and Applications," *Journal of Optimization Theory and Applications*, vol. 151(1), pp. 135-162, 2011

[17] José Oscar H. Sendín, Antonio A. Alonso, Julio R. Banga, “Efficient and robust multi-objective optimization of food processing,” *Journal of Food Engineering*, vol. 98(3), pp. 317-324, 2010
[18] Lorena Pedraza-Segura and Hector Toribio-Cuaya, “Multiobjective Optimization Approach for Cellulosic Biomass Pretreatment,” *Industrial & Engineering Chemistry Research*, 52(15), pp. 5357-5364, 2013
[19] Maik Ringkamp, Sina Ober-Blöbaum, Michael Dellnitz & Oliver Schütze, “Handling high-dimensional problems with multi-objective continuation methods via successive approximation of the tangent space,” *Engineering Optimization* vol. 44(9), pp. 1117-1146, 2012
[20] Massimiliano Gobbi, Paolo Guarneri, Leonardo Scala, Lorenzo Scotti, “A local approximation based multi-objective optimization algorithm with applications,” *Optimization and Engineering,* vol. 15, pp. 619-641, 2014
[21] Matthew J. Daskilewicz, Brian J. German, Timothy T. Takahashi, Shane Donovan, Arvin Shajanian, “Effects of disciplinary uncertainty on multi-objective optimization in aircraft conceptual design,” *Structural and Multidisciplinary Optimization*, vol. 44(6), pp. 831-846, 2011
[22] Matthias Ehrgott, “Multicriteria Optimization,” *Springer*, 2005
[23] M. Secanell, J. Wishart, P. Dobson, “Computational design and optimization of fuel cells and fuel cell systems: A review,” *Journal of Power Sources*, vol. 196(8), pp. 3690-3704, 2011
[24] Pradyumn Kumar Shukla, Kalyanmoy Deb, “ On finding multiple Pareto-optimal solutions using classical and evolutionary generating methods,” *European Journal of Operational Research*, vol. 181, issue 3, pp. 1630-1652, 2007
[25] Qu, S, Goh, M and Liang, B, “Trust region methods for solving multi-objective optimization,” *Journal of Optimization Methods and Software*, vol. 28(4), pp. 796-811, 2013
[26] Renato de S. Motta, Silvana, M. B. Afonso, Paulo R. M. Lyra, “A modified NBI and NC method for the solution of N-multiobjective optimization problems,” *Structural and Multidisciplinary Optimization*, vol. 46(2), pp. 239-259, 2012
[27] R. S. Burachik, C. Y. Kaya, M. M. Rizvi, “A New Scalarization Technique to Approximate Pareto Fronts of Problems with Disconnected Feasible Sets,” *Journal of Optimization Theory and Applications*, Vol. 162(2), 428-446, 2014
[28] R. Timothy Marler, Jasbir S. Arora, “The weighted sum method for multi-objective optimization: new insights,” *Structural and Multidisciplinary Optimization*, vol. 41(6), pp. 853-862, 2010
[29] T. Ganesan, I. Elamvazuthi, Ku Zilati, Ku Shaari, P. Vasant, “Swarm intelligence and gravitational search algorithm for multi-objective optimization of synthesis gas production,” *Applied Energy,* vol. 103, pp. 374-378, 2013
[30] T. Ganesan, P. Vasant, I. Elamvazuthi, “Normal-boundary intersection based parametric multi-objective optimization of green sand mould system,” *Journal of Manufacturing Systems*, vol. 32(1), pp. 197-205, 2013
[31] Walter Rudin, “Principles of Mathematical Analysis,” *International Series in Pure and Applied Mathematics*, *Third Edition, McGraw-Hill, Inc.*,1976

Appendix I

Software for Solving the Discrete Multiobjective Problem

```
clc;
clear all;

fprintf('**** Please Input Initial Information Using Keyboard as the Following ****\n\n');
r=input('The Radius of the Super sphere = ');
dm=input('The Dimension of the Super sphere = ');
nr=fix(r+0.001);
m=fix(dm+0.01);
nu=nr-1;
nu2=nu*2;

Ns=zeros(nu,1);
Ne=zeros(nu,1);

sum=0;
for k=1:nu;
   n1=fix(sqrt(k*(2*nr-k)));
   Ns(k,1)=nr-n1;
   Ne(k,1)=nr+n1;
   sum=sum+2*n1+1;
end

n=2*sum*(m-1)+2*nr+1;
MA=randperm(n);
F=nr*ones(n,m);

iii=0;
for i=1:2*nr+1;
   iii=iii+1;
   i1=MA(iii);
   F(i1,1)=i-1;
end
for j=2:m;
   for k=1:nu;
      for i=Ns(k,1):Ne(k,1);
         iii=iii+1;
         i1=MA(iii);
         F(i1,1)=i;
         F(i1,j)=k;
         iii=iii+1;
         i1=MA(iii);
         F(i1,1)=i;
```

```
         F(i1,j)=nr+nu-k+1;
      end

   end
end
clear('MA');
clear('Ns');
clear('Ne');

tic;
[FM, FM_F]=sort(F,1);
[~, F_FM]=sort(FM_F,1);


CF=zeros(n,m);
SS=zeros(1,m);

****
****

Min=zeros(n,m);
Max=zeros(n,m);
IL0=zeros(n,1);

toc;

fprintf('**** Please v******************** 1111\n\n');
pause

tic;

Min=FM_F;
Max=FM_F;

for i=1:n;
   if (cF(i,1) > m-2);
      for ij=1:m;
         for i1=1:n;
            IL0(i1,1)=i1;
         end

         for j=1:m-1;
            h=WW(ij,j);
            kc=CF(i,h);
            VV=zeros(n,1);
```

```
  if (kc == 0);

     ****
    ****

  end
  for i2=1:n;
     na=abs(IL0(i2,1)-VV(i2,1));
     if ( na > 0);
        IL0(i2,1)=0;
     end
  end
end

IL1=zeros(n,1);
kke=0;
for i2=1:n;
  if (IL0(i2,1) > 0);
     kke=kke+1;
     IL1(kke,1)=IL0(i2,1);
  end
end

maxi=0;
mini=800;
for it=1:kke;
  ii=IL1(it,1);
  if (F(ii,ij) > maxi);
     maxi=F(ii,ij);
     i_max=it;
  end

  if (F(ii,ij) < mini);
     mini=F(ii,ij);
     i_min=it;
  end
end

for it=1:kke;
  ii=IL1(it);
  if (it == i_max);
     cF(ii,1)=m-1;
  elseif (it == i_min);
     cF(ii,1)=-1;
     for jj=1:m;
        kk=F_FM(ii,jj);
```

```
                Max(kk,jj)=0;
            end
        else
            cF(ii,1)=0;
            for jj=1:m;
                kk=F_FM(ii,jj);
                Min(kk,jj)=0;
                Max(kk,jj)=0;
            end
        end
      end
    end
  end
end

toc;

fprintf('**** Please v******************** 2222\n\n');
pause

tic;
for i=1:n;
    iis=Min(i,1);
    if (iis > 0);
        for i1=1:n;
            IL0(i1,1)=i1;
        end

        ****
        ****

        n_min=nnz(IL0);
        if (n_min > 1);
            for j=1:m;
                ie=F_FM(iis,j);
                Min(ie,j)=0;
            end;
        end
    end
end

for i=n:-1:1;
    iis=Max(i,1);
    if (iis > 0);
        for s=1:n;
            IL0(s,1)=s;
```

```
        end

        ****
       ****

        n_max=nnz(IL0);
        if (n_max > 1);
           for j=1:m;
              ie=F_FM(iis,j);
              Max(ie,j)=0;
           end;
        end
     end
  end

  clear('IL0');
  clear('IL1');

  ms=0;
  me=0;
  for i=1:n;
     iis=Min(i,1);
     if (iis > 0);
        ms=ms+1;
        nods(ms,1)=iis;
     end
     iie=Max(i,1);
     if (iie > 0);
        me=me+1;
        node(me,1)=iie;
     end
  end

  toc;

  fprintf('**** Please v******************** 4444\n\n');
  pause

  P_s=zeros(ms,m);
  P_e=zeros(me,m);
  fprintf ('  Number  The Coordinates of Min Point  \n ');
  for i=1:ms;
     ii=nods(i,1);
     fprintf (' %4d  ',ii);
     for j=1:m;
        P_s(i,j)=F(ii,j);
```

```
        fprintf ('%4d ',F(ii,j));
    end
    fprintf('\n');
end
fprintf('\n\n');
fprintf ('  Number  The Coordinates of Max Point  \n ');
for i=1:me;
    ii=node(i,1);
    fprintf (' %4d  ',ii);
    for j=1:m;
        P_e(i,j)=F(ii,j);
        fprintf ('%4d ',F(ii,j));
    end
    fprintf('\n');
end
fprintf('\n\n');

IG0=zeros(m,1);
IG=zeros(m-2,1);
for j3=1:m;
    IG0(j3,1)=j3;
end
for j1=1;
    for j2=2:m;
        fprintf('plane j1oj2:  j1=%3d j2=%3d ',j1,j2);
        fprintf('\n');

        ik=0;
        for j3=1:m;
            na=abs(IG0(j3,1)-j1);
            nb=abs(IG0(j3,1)-j2);
            nab=na*nb;
            if (nab > 0);
                ik=ik+1;
                IG(ik,1)=IG0(j3,1);
            end
        end

        ie=0;

        for i=1:n;
            ih=0;
            for jk=1:ik;
                k1=IG(jk,1);
                ih=ih+nr-F(i,k1);
            end
```

```
        if (ih == 0);
           ie=ie+1;
           F1(ie,1)=F(i,j1);
           F1(ie,2)=F(i,j2);
        end
     end

     ie1=0;

     for i1=1:ms;
        ih1=0;
        for jk=1:ik;
           k1=IG(jk,1);
           ih1=ih1+nr-P_s(i1,k1);
        end
        if (ih1 == 0);
           ie1=ie1+1;
           P_s1(ie1,1)=P_s(i1,j1);
           P_s1(ie1,2)=P_s(i1,j2);
        end
     end

     ie2=0;

     for i2=1:me;
        ih2=0;
        for jk=1:ik;
           k1=IG(jk,1);
           ih2=ih2+nr-P_e(i2,k1);
        end
        if (ih2 == 0);
           ie2=ie2+1;
           P_e1(ie2,1)=P_e(i2,j1);
           P_e1(ie2,2)=P_e(i2,j2);
        end
     end

     plot(F1(:,1),F1(:,2),'.',P_s1(:,1),P_s1(:,2),'*',P_e1(:,1),P_e1(:,2),'*')
     xlabel(' j1 ')
     ylabel(' j2 ')
     title('Min - Max Points')
     legend('Points','Min','Max')
     pause
  end
end
fprintf('\n\n')
```

Appendix II

A Detailed Results for Example 4

Number	The Coordinates of the (min) non-dominated points									
36295	0	50	50	50	50	50	50	50	50	50
9969	1	50	50	50	50	41	50	50	50	50
14554	1	50	50	50	50	50	41	50	50	50
15537	1	50	41	50	50	50	50	50	50	50
24632	1	50	50	50	50	50	50	50	41	50
25649	1	50	50	50	41	50	50	50	50	50
44937	1	50	50	50	50	50	50	41	50	50
49477	1	50	50	50	50	50	50	50	50	41
51050	1	50	50	41	50	50	50	50	50	50
55612	1	41	50	50	50	50	50	50	50	50
2618	2	50	50	50	50	36	50	50	50	50
18718	2	50	50	36	50	50	50	50	50	50
22809	2	50	50	50	50	50	50	36	50	50
26571	2	50	50	50	50	50	36	50	50	50
34071	2	50	50	50	50	50	50	50	50	36
38658	2	50	36	50	50	50	50	50	50	50
40806	2	36	50	50	50	50	50	50	50	50
47517	2	50	50	50	50	50	50	50	36	50
64695	2	50	50	50	36	50	50	50	50	50
2941	3	50	50	50	50	50	50	50	33	50
9058	3	50	33	50	50	50	50	50	50	50
32286	3	33	50	50	50	50	50	50	50	50
38747	3	50	50	50	50	33	50	50	50	50
47252	3	50	50	33	50	50	50	50	50	50
50573	3	50	50	50	33	50	50	50	50	50
53036	3	50	50	50	50	50	50	33	50	50
54725	3	50	50	50	50	50	50	50	50	33
62344	3	50	50	50	50	50	33	50	50	50
23859	4	50	50	50	50	50	50	50	50	31
28357	4	31	50	50	50	50	50	50	50	50
36943	4	50	50	31	50	50	50	50	50	50
42162	4	50	50	50	50	50	50	31	50	50
42430	4	50	50	50	31	50	50	50	50	50
54073	4	50	31	50	50	50	50	50	50	50
57835	4	50	50	50	50	31	50	50	50	50
58855	4	50	50	50	50	50	50	50	31	50
59846	4	50	50	50	50	50	31	50	50	50
10567	5	50	50	50	29	50	50	50	50	50
12214	5	29	50	50	50	50	50	50	50	50
19912	5	50	50	50	50	50	50	50	50	29

32049	5	50	50	50	50	50	50	50	29	50
33309	5	50	50	50	50	50	29	50	50	50
36992	5	50	50	50	50	50	50	29	50	50
46242	5	50	50	50	50	29	50	50	50	50
56386	5	50	50	29	50	50	50	50	50	50
61040	5	50	29	50	50	50	50	50	50	50
327	6	50	50	50	50	50	27	50	50	50
14173	6	27	50	50	50	50	50	50	50	50
24560	6	50	50	27	50	50	50	50	50	50
24631	6	50	50	50	50	50	50	50	50	27
33389	6	50	50	50	50	27	50	50	50	50
34395	6	50	50	50	50	50	50	27	50	50
35584	6	50	50	50	27	50	50	50	50	50
49521	6	50	50	50	50	50	50	50	27	50
51685	6	50	27	50	50	50	50	50	50	50
6001	7	50	50	50	50	25	50	50	50	50
15128	7	50	25	50	50	50	50	50	50	50
17847	7	50	50	50	25	50	50	50	50	50
35608	7	50	50	50	50	50	50	50	25	50
39352	7	25	50	50	50	50	50	50	50	50
46393	7	50	50	25	50	50	50	50	50	50
50548	7	50	50	50	50	50	50	50	50	25
55581	7	50	50	50	50	50	25	50	50	50
66914	7	50	50	50	50	50	50	25	50	50
1050	8	50	50	50	23	50	50	50	50	50
26849	8	50	50	50	50	50	23	50	50	50
28365	8	50	50	50	50	23	50	50	50	50
52067	8	50	50	23	50	50	50	50	50	50
52519	8	23	50	50	50	50	50	50	50	50
60351	8	50	50	50	50	50	50	50	50	23
61046	8	50	50	50	50	50	50	23	50	50
61866	8	50	50	50	50	50	50	50	23	50
65119	8	50	23	50	50	50	50	50	50	50
9820	9	50	50	50	50	50	22	50	50	50
14155	9	50	22	50	50	50	50	50	50	50
30030	9	50	50	50	22	50	50	50	50	50
44459	9	50	50	50	50	50	50	50	22	50
56059	9	22	50	50	50	50	50	50	50	50
56487	9	50	50	22	50	50	50	50	50	50
59532	9	50	50	50	50	22	50	50	50	50
60103	9	50	50	50	50	50	50	50	50	22
65978	9	50	50	50	50	50	50	22	50	50
5038	10	50	50	50	50	50	20	50	50	50
12820	10	50	50	50	20	50	50	50	50	50
14553	10	20	50	50	50	50	50	50	50	50
24018	10	50	50	50	50	20	50	50	50	50

30708	10	50	50	50	50	50	50	20	50	50
35466	10	50	50	50	50	50	50	50	20	50
43519	10	50	50	20	50	50	50	50	50	50
47490	10	50	50	50	50	50	50	50	50	20
51553	10	50	20	50	50	50	50	50	50	50
1490	11	50	19	50	50	50	50	50	50	50
10542	11	50	50	50	50	50	50	19	50	50
21808	11	50	50	50	50	50	19	50	50	50
32055	11	19	50	50	50	50	50	50	50	50
37985	11	50	50	50	50	50	50	50	50	19
47331	11	50	50	50	50	50	50	50	19	50
62163	11	50	50	50	19	50	50	50	50	50
64241	11	50	50	50	50	19	50	50	50	50
66700	11	50	50	19	50	50	50	50	50	50
9884	12	50	50	50	50	18	50	50	50	50
39434	12	50	50	50	50	50	18	50	50	50
46686	12	50	50	50	50	50	50	50	18	50
49613	12	50	18	50	50	50	50	50	50	50
52003	12	50	50	50	50	50	50	50	50	18
54148	12	50	50	18	50	50	50	50	50	50
56568	12	18	50	50	50	50	50	50	50	50
58917	12	50	50	50	18	50	50	50	50	50
68745	12	50	50	50	50	50	50	18	50	50
19227	13	50	17	50	50	50	50	50	50	50
24390	13	50	50	50	17	50	50	50	50	50
24477	13	50	50	17	50	50	50	50	50	50
26487	13	50	50	50	50	50	17	50	50	50
32229	13	50	50	50	50	17	50	50	50	50
33238	13	50	50	50	50	50	50	17	50	50
37109	13	50	50	50	50	50	50	50	50	17
49605	13	50	50	50	50	50	50	50	17	50
58932	13	17	50	50	50	50	50	50	50	50
17232	14	50	16	50	50	50	50	50	50	50
30123	14	50	50	50	50	50	16	50	50	50
38631	14	50	50	50	50	16	50	50	50	50
46681	14	50	50	50	16	50	50	50	50	50
48374	14	50	50	50	50	50	50	16	50	50
48640	14	50	50	16	50	50	50	50	50	50
49041	14	50	50	50	50	50	50	50	16	50
60150	14	50	50	50	50	50	50	50	50	16
65844	14	16	50	50	50	50	50	50	50	50
2183	15	50	15	50	50	50	50	50	50	50
6314	15	50	50	50	50	50	15	50	50	50
15345	15	15	50	50	50	50	50	50	50	50
21054	15	50	50	50	50	50	50	50	50	15
22473	15	50	50	50	15	50	50	50	50	50

25021	15	50	50	15	50	50	50	50	50	50
43368	15	50	50	50	50	15	50	50	50	50
47138	15	50	50	50	50	50	50	50	15	50
64737	15	50	50	50	50	50	50	15	50	50
3063	16	50	14	50	50	50	50	50	50	50
5635	16	50	50	50	50	50	50	14	50	50
25630	16	50	50	50	50	50	50	50	50	14
48642	16	50	50	50	50	50	14	50	50	50
50757	16	50	50	14	50	50	50	50	50	50
52154	16	50	50	50	14	50	50	50	50	50
53097	16	50	50	50	50	50	50	50	14	50
68228	16	50	50	50	50	14	50	50	50	50
68656	16	14	50	50	50	50	50	50	50	50
8946	17	50	50	50	50	50	13	50	50	50
12655	17	50	50	13	50	50	50	50	50	50
18160	17	50	13	50	50	50	50	50	50	50
19878	17	50	50	50	13	50	50	50	50	50
43103	17	50	50	50	50	50	50	13	50	50
55540	17	13	50	50	50	50	50	50	50	50
56554	17	50	50	50	50	50	50	50	13	50
62692	17	50	50	50	50	50	50	50	50	13
67333	17	50	50	50	50	13	50	50	50	50
486	18	50	50	50	50	50	50	12	50	50
10018	18	50	50	12	50	50	50	50	50	50
19639	18	12	50	50	50	50	50	50	50	50
21055	18	50	50	50	50	50	50	50	50	12
23733	18	50	12	50	50	50	50	50	50	50
25529	18	50	50	50	50	50	12	50	50	50
29270	18	50	50	50	12	50	50	50	50	50
30944	18	50	50	50	50	12	50	50	50	50
56367	18	50	50	50	50	50	50	50	12	50
15078	19	11	50	50	50	50	50	50	50	50
17123	19	50	50	50	11	50	50	50	50	50
19576	19	50	50	50	50	11	50	50	50	50
24383	19	50	50	50	50	50	50	11	50	50
30357	19	50	50	50	50	50	50	50	50	11
41974	19	50	50	11	50	50	50	50	50	50
61167	19	50	11	50	50	50	50	50	50	50
61625	19	50	50	50	50	50	50	50	11	50
66563	19	50	50	50	50	50	11	50	50	50
1978	20	50	50	10	50	50	50	50	50	50
2593	20	50	50	50	50	10	50	50	50	50
5152	20	10	50	50	50	50	50	50	50	50
14341	20	50	50	50	50	50	50	50	10	50
21910	20	50	50	50	50	50	50	10	50	50
40915	20	50	50	50	10	50	50	50	50	50

46079	20	50	10	50	50	50	50	50	50	50
64002	20	50	50	50	50	50	10	50	50	50
64151	20	50	50	50	50	50	50	50	50	10
901	22	50	50	50	9	50	50	50	50	50
1878	22	50	50	50	50	50	50	50	50	9
7292	22	50	50	9	50	50	50	50	50	50
17270	22	50	9	50	50	50	50	50	50	50
28741	22	50	50	50	50	9	50	50	50	50
34230	22	50	50	50	50	50	50	50	9	50
47425	22	9	50	50	50	50	50	50	50	50
48033	22	50	50	50	50	50	9	50	50	50
68767	22	50	50	50	50	50	50	9	50	50
2419	23	50	50	50	50	50	8	50	50	50
11240	23	50	50	50	50	50	50	8	50	50
15328	23	8	50	50	50	50	50	50	50	50
16104	23	50	50	50	50	50	50	50	50	8
31201	23	50	50	50	50	8	50	50	50	50
36106	23	50	50	50	50	50	50	50	8	50
43964	23	50	50	50	8	50	50	50	50	50
48745	23	50	50	8	50	50	50	50	50	50
60703	23	50	8	50	50	50	50	50	50	50
815	25	50	7	50	50	50	50	50	50	50
8356	25	50	50	50	7	50	50	50	50	50
49097	25	50	50	50	50	50	7	50	50	50
58993	25	50	50	7	50	50	50	50	50	50
59779	25	50	50	50	50	50	50	50	7	50
60002	25	50	50	50	50	50	50	7	50	50
60417	25	7	50	50	50	50	50	50	50	50
66511	25	50	50	50	50	7	50	50	50	50
69324	25	50	50	50	50	50	50	50	50	7
304	27	50	50	6	50	50	50	50	50	50
3008	27	50	50	50	50	50	6	50	50	50
13615	27	50	50	50	50	6	50	50	50	50
35537	27	50	50	50	50	50	50	50	6	50
45739	27	50	50	50	6	50	50	50	50	50
52288	27	50	50	50	50	50	50	6	50	50
58688	27	50	50	50	50	50	50	50	50	6
60579	27	50	6	50	50	50	50	50	50	50
64439	27	6	50	50	50	50	50	50	50	50
3877	29	50	50	5	50	50	50	50	50	50
8064	29	50	50	50	50	50	50	50	50	5
10522	29	50	50	50	50	5	50	50	50	50
11606	29	5	50	50	50	50	50	50	50	50
23996	29	50	50	50	50	50	5	50	50	50
32734	29	50	50	50	5	50	50	50	50	50
55289	29	50	50	50	50	50	50	5	50	50

55462	29	50	5	50	50	50	50	50	50	50
60049	29	50	50	50	50	50	50	50	5	50
3794	31	50	4	50	50	50	50	50	50	50
5571	31	50	50	50	50	50	50	50	50	4
8900	31	50	50	50	50	50	50	4	50	50
15067	31	50	50	50	4	50	50	50	50	50
16875	31	4	50	50	50	50	50	50	50	50
32392	31	50	50	4	50	50	50	50	50	50
36514	31	50	50	50	50	4	50	50	50	50
59658	31	50	50	50	50	50	4	50	50	50
59781	31	50	50	50	50	50	50	50	4	50
2064	33	50	50	50	50	50	3	50	50	50
3806	33	50	50	3	50	50	50	50	50	50
13074	33	50	50	50	50	50	50	3	50	50
18366	33	3	50	50	50	50	50	50	50	50
33478	33	50	50	50	3	50	50	50	50	50
47559	33	50	50	50	50	3	50	50	50	50
49620	33	50	50	50	50	50	50	50	3	50
59738	33	50	3	50	50	50	50	50	50	50
64603	33	50	50	50	50	50	50	50	50	3
11668	36	50	50	50	50	50	50	50	50	2
19253	36	50	50	50	50	50	50	2	50	50
24599	36	50	50	50	50	2	50	50	50	50
26966	36	50	50	50	50	50	2	50	50	50
36233	36	50	50	2	50	50	50	50	50	50
47924	36	50	2	50	50	50	50	50	50	50
50167	36	2	50	50	50	50	50	50	50	50
62819	36	50	50	50	2	50	50	50	50	50
68281	36	50	50	50	50	50	50	50	2	50
8860	41	1	50	50	50	50	50	50	50	50
11875	41	50	50	50	50	50	50	1	50	50
11904	41	50	50	50	50	50	50	50	50	1
14284	41	50	50	50	50	50	50	50	1	50
25136	41	50	1	50	50	50	50	50	50	50
25528	41	50	50	50	50	1	50	50	50	50
34283	41	50	50	1	50	50	50	50	50	50
53474	41	50	50	50	50	50	1	50	50	50
56332	41	50	50	50	1	50	50	50	50	50

Number	The Coordinates of the (max) non-dominated points									
2229	59	50	50	50	50	99	50	50	50	50
4673	59	50	99	50	50	50	50	50	50	50
10960	59	50	50	50	50	50	50	50	50	99
15134	59	99	50	50	50	50	50	50	50	50
15674	59	50	50	50	50	50	50	99	50	50

38845	59	50	50	50	50	50	50	50	99	50
49174	59	50	50	99	50	50	50	50	50	50
58176	59	50	50	50	50	50	99	50	50	50
64856	59	50	50	50	99	50	50	50	50	50
4567	64	50	50	50	50	50	50	50	98	50
28835	64	98	50	50	50	50	50	50	50	50
31025	64	50	98	50	50	50	50	50	50	50
37849	64	50	50	50	50	50	98	50	50	50
40832	64	50	50	50	50	98	50	50	50	50
43414	64	50	50	50	50	50	50	98	50	50
48606	64	50	50	98	50	50	50	50	50	50
51684	64	50	50	50	98	50	50	50	50	50
67654	64	50	50	50	50	50	50	50	50	98
23309	67	50	50	50	50	50	50	97	50	50
40408	67	50	50	97	50	50	50	50	50	50
46419	67	50	97	50	50	50	50	50	50	50
55975	67	50	50	50	97	50	50	50	50	50
57198	67	50	50	50	50	50	50	50	50	97
58593	67	50	50	50	50	97	50	50	50	50
63420	67	50	50	50	50	50	50	50	97	50
63612	67	97	50	50	50	50	50	50	50	50
65884	67	50	50	50	50	50	97	50	50	50
7892	69	50	50	50	96	50	50	50	50	50
24024	69	50	50	50	50	50	50	50	96	50
41995	69	50	50	50	50	50	50	50	50	96
47258	69	50	96	50	50	50	50	50	50	50
55501	69	50	50	50	50	50	50	96	50	50
55667	69	50	50	50	50	96	50	50	50	50
63700	69	50	50	50	50	50	96	50	50	50
63814	69	96	50	50	50	50	50	50	50	50
66109	69	50	50	96	50	50	50	50	50	50
3830	71	50	50	50	50	95	50	50	50	50
7420	71	50	50	95	50	50	50	50	50	50
20438	71	50	95	50	50	50	50	50	50	50
42998	71	50	50	50	50	50	50	50	50	95
45850	71	50	50	50	50	50	50	50	95	50
49510	71	50	50	50	50	50	95	50	50	50
51125	71	50	50	50	95	50	50	50	50	50
51191	71	50	50	50	50	50	50	95	50	50
68664	71	95	50	50	50	50	50	50	50	50
16325	73	50	50	50	50	50	50	94	50	50
16639	73	50	94	50	50	50	50	50	50	50
18412	73	50	50	50	50	50	94	50	50	50
31524	73	94	50	50	50	50	50	50	50	50
45946	73	50	50	50	50	50	50	50	50	94
45960	73	50	50	50	94	50	50	50	50	50

52841	73	50	50	94	50	50	50	50	50	50
53476	73	50	50	50	50	50	50	50	94	50
67429	73	50	50	50	50	94	50	50	50	50
188	75	50	50	93	50	50	50	50	50	50
13314	75	93	50	50	50	50	50	50	50	50
29807	75	50	50	50	50	50	93	50	50	50
37318	75	50	93	50	50	50	50	50	50	50
39528	75	50	50	50	50	50	50	50	50	93
42890	75	50	50	50	50	50	50	93	50	50
48158	75	50	50	50	50	93	50	50	50	50
50694	75	50	50	50	93	50	50	50	50	50
68905	75	50	50	50	50	50	50	50	93	50
6138	77	50	50	50	50	92	50	50	50	50
11228	77	50	50	50	50	50	50	50	92	50
19915	77	50	50	50	92	50	50	50	50	50
22613	77	50	50	50	50	50	50	92	50	50
30521	77	50	50	92	50	50	50	50	50	50
35347	77	50	92	50	50	50	50	50	50	50
38538	77	50	50	50	50	50	50	50	50	92
52577	77	92	50	50	50	50	50	50	50	50
64031	77	50	50	50	50	50	92	50	50	50
8643	78	50	50	50	50	50	50	50	91	50
12665	78	50	50	50	50	91	50	50	50	50
22011	78	50	50	50	50	50	50	50	50	91
27876	78	50	50	50	50	50	91	50	50	50
28121	78	50	50	91	50	50	50	50	50	50
29811	78	50	91	50	50	50	50	50	50	50
58137	78	91	50	50	50	50	50	50	50	50
66289	78	50	50	50	50	50	50	91	50	50
66568	78	50	50	50	91	50	50	50	50	50
439	80	50	50	50	50	50	50	50	50	90
5046	80	50	50	50	50	50	50	90	50	50
9834	80	50	50	50	90	50	50	50	50	50
27527	80	50	50	50	50	50	50	50	90	50
37188	80	50	50	50	50	50	90	50	50	50
44423	80	50	50	50	50	90	50	50	50	50
54271	80	90	50	50	50	50	50	50	50	50
54540	80	50	50	90	50	50	50	50	50	50
55023	80	50	90	50	50	50	50	50	50	50
5206	81	50	50	50	50	50	50	89	50	50
15643	81	50	50	89	50	50	50	50	50	50
17005	81	50	89	50	50	50	50	50	50	50
24934	81	89	50	50	50	50	50	50	50	50
43770	81	50	50	50	50	50	50	50	89	50
45913	81	50	50	50	50	89	50	50	50	50
52969	81	50	50	50	50	50	50	50	50	89

61471	81	50	50	50	50	50	89	50	50	50
64448	81	50	50	50	89	50	50	50	50	50
1712	82	50	50	50	50	88	50	50	50	50
2258	82	88	50	50	50	50	50	50	50	50
4261	82	50	50	50	88	50	50	50	50	50
17575	82	50	50	50	50	50	88	50	50	50
20705	82	50	50	88	50	50	50	50	50	50
30751	82	50	50	50	50	50	50	50	88	50
41525	82	50	50	50	50	50	50	50	50	88
54280	82	50	50	50	50	50	50	88	50	50
58080	82	50	88	50	50	50	50	50	50	50
7624	83	50	87	50	50	50	50	50	50	50
16891	83	50	50	50	50	50	50	50	50	87
21202	83	50	50	50	50	50	87	50	50	50
26530	83	87	50	50	50	50	50	50	50	50
31295	83	50	50	50	50	50	50	50	87	50
33585	83	50	50	50	87	50	50	50	50	50
44936	83	50	50	50	50	50	50	87	50	50
52775	83	50	50	87	50	50	50	50	50	50
66629	83	50	50	50	50	87	50	50	50	50
16278	84	50	50	50	50	50	86	50	50	50
24026	84	50	86	50	50	50	50	50	50	50
26467	84	50	50	50	50	50	50	50	86	50
43063	84	50	50	50	50	86	50	50	50	50
43206	84	50	50	50	50	50	50	86	50	50
45918	84	50	50	50	86	50	50	50	50	50
51961	84	86	50	50	50	50	50	50	50	50
52442	84	50	50	86	50	50	50	50	50	50
67256	84	50	50	50	50	50	50	50	50	86
5037	85	50	85	50	50	50	50	50	50	50
15573	85	50	50	50	50	50	50	50	50	85
28153	85	50	50	50	50	50	85	50	50	50
43021	85	85	50	50	50	50	50	50	50	50
58197	85	50	50	50	50	50	50	85	50	50
58258	85	50	50	50	50	50	50	50	85	50
61565	85	50	50	85	50	50	50	50	50	50
63891	85	50	50	50	85	50	50	50	50	50
68788	85	50	50	50	50	85	50	50	50	50
3065	86	50	50	50	50	50	84	50	50	50
7740	86	50	50	50	50	50	50	84	50	50
10507	86	50	50	50	50	50	50	50	50	84
14333	86	50	84	50	50	50	50	50	50	50
34801	86	50	50	50	84	50	50	50	50	50
54621	86	50	50	84	50	50	50	50	50	50
63422	86	50	50	50	50	50	50	50	84	50
67165	86	84	50	50	50	50	50	50	50	50

68761	86	50	50	50	50	84	50	50	50	50
6330	87	50	50	50	50	50	83	50	50	50
7690	87	50	50	50	50	50	50	50	83	50
11823	87	50	50	50	50	50	50	50	50	83
19534	87	50	83	50	50	50	50	50	50	50
53241	87	50	50	50	50	50	50	83	50	50
56316	87	50	50	50	83	50	50	50	50	50
56544	87	50	50	50	50	83	50	50	50	50
59713	87	50	50	83	50	50	50	50	50	50
69560	87	83	50	50	50	50	50	50	50	50
24317	88	82	50	50	50	50	50	50	50	50
28888	88	50	82	50	50	50	50	50	50	50
41970	88	50	50	50	82	50	50	50	50	50
48395	88	50	50	50	50	50	50	50	50	82
53217	88	50	50	50	50	50	82	50	50	50
59294	88	50	50	82	50	50	50	50	50	50
61760	88	50	50	50	50	50	50	50	82	50
66308	88	50	50	50	50	50	50	82	50	50
66969	88	50	50	50	50	82	50	50	50	50
7737	89	81	50	50	50	50	50	50	50	50
17958	89	50	50	50	50	81	50	50	50	50
30524	89	50	50	50	50	50	50	50	81	50
39483	89	50	81	50	50	50	50	50	50	50
46444	89	50	50	50	50	50	50	81	50	50
51648	89	50	50	81	50	50	50	50	50	50
54755	89	50	50	50	50	50	81	50	50	50
59228	89	50	50	50	50	50	50	50	50	81
60587	89	50	50	50	81	50	50	50	50	50
9536	90	50	50	50	50	50	50	50	80	50
15830	90	50	50	50	50	80	50	50	50	50
17218	90	50	50	50	50	50	80	50	50	50
29816	90	50	50	50	50	50	50	80	50	50
31169	90	80	50	50	50	50	50	50	50	50
44323	90	50	50	50	80	50	50	50	50	50
54203	90	50	50	50	50	50	50	50	50	80
60066	90	50	50	80	50	50	50	50	50	50
67442	90	50	80	50	50	50	50	50	50	50
3618	91	50	50	50	50	50	50	50	50	78
14782	91	50	50	50	50	50	50	78	50	50
15313	91	78	50	50	50	50	50	50	50	50
16317	91	50	50	78	50	50	50	50	50	50
21013	91	50	50	50	50	50	78	50	50	50
28074	91	50	50	50	50	78	50	50	50	50
28787	91	50	50	50	50	50	50	50	78	50
40339	91	50	78	50	50	50	50	50	50	50
44620	91	50	50	50	78	50	50	50	50	50

9077	92	50	77	50	50	50	50	50	50	50
23851	92	50	50	50	77	50	50	50	50	50
27908	92	50	50	50	50	50	50	50	77	50
29579	92	50	50	50	50	50	77	50	50	50
33707	92	50	50	50	50	77	50	50	50	50
36401	92	77	50	50	50	50	50	50	50	50
41260	92	50	50	50	50	50	50	50	50	77
51567	92	50	50	50	50	50	50	77	50	50
68365	92	50	50	77	50	50	50	50	50	50
9010	93	50	50	50	50	50	75	50	50	50
26729	93	50	75	50	50	50	50	50	50	50
44032	93	50	50	50	50	75	50	50	50	50
47417	93	75	50	50	50	50	50	50	50	50
49090	93	50	50	75	50	50	50	50	50	50
49168	93	50	50	50	50	50	50	50	75	50
50905	93	50	50	50	50	50	50	50	50	75
62543	93	50	50	50	50	50	50	75	50	50
63107	93	50	50	50	75	50	50	50	50	50
7154	94	50	50	50	50	73	50	50	50	50
20952	94	50	50	50	73	50	50	50	50	50
24931	94	50	50	50	50	50	50	50	50	73
27172	94	50	50	73	50	50	50	50	50	50
35995	94	50	50	50	50	50	50	73	50	50
48118	94	50	73	50	50	50	50	50	50	50
49918	94	73	50	50	50	50	50	50	50	50
56859	94	50	50	50	50	50	50	50	73	50
56935	94	50	50	50	50	50	73	50	50	50
2676	95	71	50	50	50	50	50	50	50	50
4998	95	50	50	50	50	50	50	50	71	50
9725	95	50	50	50	50	50	50	71	50	50
17301	95	50	50	50	50	50	71	50	50	50
18941	95	50	71	50	50	50	50	50	50	50
27096	95	50	50	71	50	50	50	50	50	50
41422	95	50	50	50	50	50	50	50	50	71
64251	95	50	50	50	71	50	50	50	50	50
65296	95	50	50	50	50	71	50	50	50	50
791	96	50	50	50	50	50	69	50	50	50
2488	96	50	50	50	50	69	50	50	50	50
3489	96	50	50	50	50	50	50	50	69	50
27896	96	50	50	50	50	50	50	69	50	50
31248	96	50	50	50	50	50	50	50	50	69
40682	96	50	50	69	50	50	50	50	50	50
53072	96	69	50	50	50	50	50	50	50	50
59461	96	50	69	50	50	50	50	50	50	50
59848	96	50	50	50	69	50	50	50	50	50
8551	97	50	50	50	50	50	67	50	50	50

22679	97	50	50	50	50	67	50	50	50	50
25817	97	67	50	50	50	50	50	50	50	50
33275	97	50	50	50	50	50	50	67	50	50
35246	97	50	50	50	50	50	50	50	67	50
47644	97	50	50	50	50	50	50	50	50	67
52792	97	50	67	50	50	50	50	50	50	50
53570	97	50	50	50	67	50	50	50	50	50
66734	97	50	50	67	50	50	50	50	50	50
4822	98	50	64	50	50	50	50	50	50	50
13982	98	50	50	50	50	50	64	50	50	50
16765	98	50	50	64	50	50	50	50	50	50
34899	98	50	50	50	50	50	50	50	50	64
38887	98	50	50	50	50	64	50	50	50	50
49845	98	50	50	50	50	50	50	64	50	50
59305	98	50	50	50	50	50	50	50	64	50
64868	98	64	50	50	50	50	50	50	50	50
67120	98	50	50	50	64	50	50	50	50	50
6059	99	50	50	59	50	50	50	50	50	50
17910	99	50	50	50	50	50	50	59	50	50
23246	99	50	50	50	50	50	50	50	59	50
33146	99	50	50	50	50	59	50	50	50	50
35001	99	50	50	50	50	50	50	50	50	59
41889	99	59	50	50	50	50	50	50	50	50
50377	99	50	50	50	59	50	50	50	50	50
51631	99	50	50	50	50	50	59	50	50	50
64215	99	50	59	50	50	50	50	50	50	50
52981	100	50	50	50	50	50	50	50	50	50

Appendix III

(a) A Detailed Results for Example 5

Number	The Coordinates of Points of F		
1	2.2404	7.8996	1.8419
2	6.6783	3.1852	5.9721
3	8.4439	5.3406	2.9994
4	3.4446	0.8995	1.3412
5	7.8052	1.1171	2.1260
6	6.7533	1.3629	8.9494
7	0.0672	6.7865	0.7145
8	6.0217	4.9518	2.4249
9	3.8677	1.8971	0.5375
10	9.1599	4.9501	4.4172
11	0.0115	1.4761	0.1328
12	4.6245	0.5497	8.9719
13	4.2435	8.5071	1.9666
14	4.6092	5.6056	0.9337
15	7.7016	9.2961	3.0737
16	3.2247	6.9667	4.5606
17	7.8474	5.8279	1.0167
18	4.7136	8.1540	9.9539
19	0.3576	8.7901	3.3209
20	1.7587	9.8891	2.9735
21	7.2176	0.0052	0.6205
22	4.7349	8.6544	2.9824
23	1.5272	6.1257	0.4635
24	3.4112	9.8995	5.0543
25	6.0739	5.2768	7.6143
26	1.9175	4.7952	6.3107
27	7.3843	8.0135	0.8989
28	2.4285	2.2784	0.8086
29	9.1742	4.9809	7.7724
30	2.6906	9.0085	9.0513
31	7.6550	5.7466	5.3377
32	1.8866	8.4518	1.0915
33	2.8750	7.3864	8.2581
34	0.9111	5.8599	3.3810
35	5.7621	2.4673	2.9397
36	6.8336	6.6642	7.4631
37	5.4659	0.8348	0.1034
38	4.2573	6.2596	0.4845
39	6.4444	6.6094	6.6792
40	6.4762	7.2975	6.0347

41	6.7902	8.9075	5.2610
42	6.3579	9.8230	7.2971
43	9.4517	7.6903	7.0725
44	2.0893	5.8145	7.8138
45	7.0928	9.2831	2.8798
46	2.3623	5.8009	6.9253
47	1.1940	0.1698	5.5667
48	6.0730	1.2086	3.9652
49	4.5014	8.6271	0.6159
50	4.5873	4.8430	7.8018
51	6.6194	8.4486	3.3758
52	7.7029	2.0941	6.0787
53	3.5022	5.5229	7.4125
54	6.6201	6.2988	1.0481
55	4.1616	0.3199	1.2789
56	8.4193	6.1471	5.4954
57	8.3292	3.6241	4.8523
58	2.5644	0.4953	8.9048
59	6.1346	4.8957	7.9896
60	5.8225	1.9251	7.3434
61	5.4074	1.2308	0.5133
62	8.6994	2.0549	0.7289
63	2.6478	1.4651	0.8853
64	3.1807	1.8907	7.9835
65	1.1921	0.4265	9.4301
66	9.3983	6.3520	6.8372
67	6.4555	2.8187	1.3208
68	4.7946	5.3860	7.2272
69	6.3932	6.9516	1.1035
70	5.4472	4.9912	1.1749
71	6.4731	5.3580	6.4072
72	5.4389	4.4518	3.2881
73	7.2105	1.2393	6.5381
74	5.2250	4.9036	7.4913
75	9.9370	8.5300	5.8319
76	2.1868	8.7393	7.4003
77	1.0580	2.7029	2.3483
78	1.0970	2.0846	7.3496
79	0.6359	5.6498	9.7060
80	4.0458	6.4031	8.6693
81	4.4837	4.1703	0.8623
82	3.6582	2.0598	3.6644
83	7.6350	9.4793	3.6920
84	6.2790	0.8207	6.8503
85	7.7198	1.0571	5.9794
86	9.3285	1.4204	7.8936

87	9.7274	1.6646	3.6765
88	1.9203	6.2096	2.0603
89	1.3887	5.7371	0.8667
90	6.9627	0.5208	7.7193
91	0.9382	9.3120	2.0567
92	5.2540	7.2866	3.8827
93	5.3034	7.3784	5.5178
94	8.6114	0.6340	2.2895
95	4.8485	8.6044	6.4194
96	3.9346	9.3441	4.8448
97	6.7143	9.8440	1.5185
98	7.4126	8.5894	7.8193
99	5.2005	7.8556	1.0061
100	3.4771	5.1338	2.9407
101	1.5000	1.7760	2.3737
102	5.8609	3.9859	5.3087
103	2.6215	1.3393	0.9150
104	0.4445	0.3089	4.0532
105	7.5493	9.3914	1.0485
106	2.4279	3.0131	1.1228
107	4.4240	2.9553	7.8443
108	6.8780	3.3294	2.9157
109	3.5923	4.6707	6.0353
110	7.3634	6.4820	9.6442
111	3.9471	0.2523	4.3248
112	6.8342	8.4221	6.9475
113	7.0405	5.5903	7.5810
114	4.4231	8.5410	4.3264
115	0.1958	3.4788	6.5550
116	3.3086	4.4603	1.0976
117	4.2431	0.5424	9.3376
118	2.7027	1.7711	1.8746
119	1.9705	6.6281	2.6618
120	8.2172	3.3083	7.9783
121	4.2992	8.9849	4.8760
122	8.8777	1.1816	7.6896
123	3.9118	9.8842	3.9601
124	7.6911	5.3998	2.7294
125	3.9679	7.0692	0.3723
126	8.0851	9.9949	6.7329
127	7.5508	2.8785	4.2956
128	3.7740	4.1452	4.5174
129	2.1602	4.6484	6.0986
130	7.9041	7.6396	0.5940
131	9.4930	8.1820	3.1581
132	3.2757	1.0022	7.7272

133	6.7126	1.7812	6.9643
134	4.3864	3.5963	1.2533
135	8.3350	0.5670	1.3015
136	7.6885	5.2189	0.9235
137	1.6725	3.3585	0.0782
138	8.6198	1.7567	4.2311
139	9.8987	2.0895	6.5557
140	5.1442	9.0515	7.2292
141	8.8428	6.7539	5.3121
142	5.8803	4.6847	1.0882
143	1.5475	9.1213	6.3177
144	1.9986	1.0401	1.2650
145	4.0695	7.4555	1.3430
146	7.4871	7.3627	0.9859
147	8.2558	5.6186	1.4203

Total numbers of points of set F = 147

Number	The Coordinates of the Minimal Points		
11	0.0115	1.4761	0.1328
104	0.4445	0.3089	4.0532
47	1.1940	0.1698	5.5667
137	1.6725	3.3585	0.0782
144	1.9986	1.0401	1.2650
103	2.6215	1.3393	0.9150
63	2.6478	1.4651	0.8853
4	3.4446	0.8995	1.3412
111	3.9471	0.2523	4.3248
55	4.1616	0.3199	1.2789
61	5.4074	1.2308	0.5133
37	5.4659	0.8348	0.1034
21	7.2176	0.0052	0.6205

Total number of the Minimal Points = 13

Number	The Coordinates of the Maximal Points		
30	2.6906	9.0085	9.0513
18	4.7136	8.1540	9.9539
42	6.3579	9.8230	7.2971
110	7.3634	6.4820	9.6442
98	7.4126	8.5894	7.8193
126	8.0851	9.9949	6.7329
120	8.2172	3.3083	7.9783
29	9.1742	4.9809	7.7724

86	9.3285	1.4204	7.8936
43	9.4517	7.6903	7.0725
139	9.8987	2.0895	6.5557
75	9.9370	8.5300	5.8319

Total number of the Maximal Points = 12

Appendix III

(b) A Detailed Results for Example 6

Number	The Coordinates of Points of F			
1	12.2014	15.9916	6.9948	0.8398
2	14.1198	15.6957	6.0774	12.8188
3	4.5592	2.0326	15.6745	10.8571
4	10.7716	3.7158	6.3839	15.1361
5	10.6285	0.3781	7.0430	1.4649
6	1.9650	9.7189	2.5089	14.5350
7	6.5171	1.7729	5.2165	8.1592
8	4.4046	6.5194	5.0250	9.8385
9	11.4667	14.1452	14.3120	5.0571
10	4.5342	8.7701	3.9524	1.2398
11	14.3392	5.9040	4.9709	13.6098
12	13.2253	3.3335	6.5419	2.3124
13	6.2404	7.0551	11.3282	5.9278
14	7.9664	15.2991	2.2982	9.9583
15	11.1169	1.9844	13.9412	15.9608
16	13.3499	7.5322	1.3305	8.2775
17	9.7541	13.7103	7.3878	15.8482
18	9.1958	0.6942	0.4862	3.6246
19	5.2167	11.0660	12.0512	6.3681
20	7.3028	15.6638	11.2007	11.1451
21	11.4207	4.5323	3.4322	1.0343
22	14.1505	2.1405	10.8785	11.9626
23	11.5337	10.9645	8.9167	6.7264
24	0.2978	14.5513	13.6109	12.9811
25	10.7964	9.7739	8.9370	6.0737
26	7.0161	14.3997	14.4284	5.1051
27	7.0051	3.0949	6.7123	15.7768
28	1.8726	12.0708	5.7301	11.4909
29	13.0349	5.5402	7.8238	6.6109
30	5.1977	6.6980	4.0954	1.5781
31	3.9396	2.4915	14.8667	11.7529
32	5.4834	13.1040	7.4681	10.1969
33	6.0111	9.9988	4.0641	1.1815
34	8.7449	11.8170	6.8995	1.9281
35	8.9907	12.8818	11.2405	15.7055
36	6.3332	1.0756	6.4373	7.9488

37	6.3701	15.2126	2.9094	0.3586
38	8.2459	7.9612	13.7000	0.8613
39	10.5205	12.0823	9.3472	2.2540
40	15.2146	11.8785	5.9773	14.2956
41	11.5576	13.2981	3.5471	7.4531
42	6.4013	2.5040	3.5039	8.9737
43	13.3099	7.3169	8.3557	7.9113
44	2.1494	9.8896	6.9348	1.0846
45	0.9675	14.9149	11.8609	14.3623
46	1.3480	13.3614	1.1272	4.6170
47	2.6224	14.3268	13.5573	4.3047
48	5.1875	9.3203	10.8781	9.5071
49	4.8276	9.3239	2.1864	7.6141
50	0.1869	13.6788	13.7344	5.8930
51	8.6385	0.5579	3.1973	10.4898
52	1.5260	14.1667	9.7174	15.0112
53	2.3442	6.5237	8.6887	9.9268
54	10.0983	0.5821	2.5972	4.5254
55	13.7491	11.9384	0.0904	3.2829
56	15.5875	2.4773	12.3438	7.0261
57	9.1334	2.3025	12.2366	0.4360
58	15.9496	9.6953	6.7371	14.0189
59	8.8567	4.0717	0.9090	9.7615
60	8.2473	5.1865	9.3720	3.2575
61	5.2909	6.4287	2.7865	8.3187
62	6.8800	6.5020	11.6578	0.8612
63	7.8689	6.1791	8.5487	13.7950
64	1.1366	9.7568	4.0490	7.0870
65	14.2038	2.6703	14.6729	8.7681
66	1.0341	3.0095	12.1311	9.0698
67	6.9790	1.5141	14.1925	10.8863
68	13.2261	5.1710	1.1008	5.9421
69	6.3126	12.3136	2.9365	1.2517
70	9.8156	3.7459	11.7932	7.3016
71	13.0983	11.8458	11.1474	0.7655
72	14.1798	11.0851	12.4319	11.8121
73	14.8978	13.1853	8.0305	0.6080
74	3.0526	13.2476	6.8079	15.2679
75	4.1373	4.6939	9.7798	11.8780
76	14.3659	4.9499	13.6924	14.9992
77	9.4938	8.3685	10.7328	8.2138
78	8.0614	5.2048	8.3775	3.8545
79	9.8050	13.3095	4.7810	4.1594

80	13.1108	12.9647	11.2635	12.1436
81	8.5102	8.9120	6.1058	15.8935
82	3.2332	4.2074	9.0830	5.7073
83	7.2623	10.8891	14.2058	12.0457
84	6.8466	3.7385	13.4872	1.7608
85	15.4568	7.3028	14.3808	9.5527
86	9.9209	6.1531	15.0240	6.8895
87	11.1262	8.6176	13.0470	11.6915
88	11.5226	15.8673	0.0217	4.1788
89	5.5503	12.0835	0.0495	1.5169
90	8.2718	15.6873	1.3995	7.2154
91	8.9071	3.7565	4.1716	10.2412
92	2.5039	8.4569	0.3648	2.1126
93	8.9929	0.8230	6.7854	7.2452
94	11.1169	12.1100	5.4570	10.4352
95	6.8233	9.6317	8.6617	13.2320
96	13.3803	13.7147	14.8187	4.9292
97	11.7022	15.8124	4.7760	6.4378
98	5.7605	14.8718	5.4094	14.1477
99	7.2674	6.5522	13.7517	11.2093
100	6.1822	0.0055	5.4477	3.8700
101	12.4089	8.6541	2.2099	12.1573
102	11.7483	3.3237	8.1248	4.6548
103	6.8844	3.5085	13.7065	4.4390
104	11.1000	5.2129	6.1490	0.0977
105	15.1234	1.5352	11.1311	5.9954
106	12.5477	11.9605	10.0465	6.9909
107	11.2891	11.9761	7.2062	4.8688
108	1.7493	8.6928	7.5779	4.6538
109	6.2389	5.4101	15.1953	3.8803
110	9.4545	13.3173	1.3360	14.9869
111	7.3501	8.8412	4.4773	13.7630
112	0.8054	15.3207	7.1521	6.3557
113	3.6590	14.2853	9.4011	7.6707
114	13.3470	5.7041	14.0421	9.0399
115	0.2503	8.7424	7.5056	7.8339
116	13.8194	5.5469	6.9987	4.3170
117	1.2491	9.9648	11.9390	15.8358
118	10.7047	12.7460	7.4866	2.9388
119	8.0034	11.9340	13.7732	13.7865
120	3.4879	2.0086	7.4642	0.5221
121	9.1459	13.1583	7.9697	5.3113
122	1.9550	0.4024	7.7989	11.9800

123	10.7387	6.6309	3.6715	10.3099
124	9.5934	11.7025	1.3688	2.7078
125	0.8956	12.5020	1.0781	15.2353
126	0.9015	5.8766	14.2143	8.6923
127	2.4400	11.9179	3.7307	4.0226
128	0.3139	14.2763	13.7855	9.2572
129	6.9628	3.8817	11.3878	14.6476
130	13.3155	2.0736	13.9650	14.3295
131	9.8782	3.6011	15.0080	7.7201
132	8.3221	5.6002	2.2350	7.0838
133	13.8219	4.5934	6.3024	4.9881
134	1.5632	14.8398	15.6890	0.8850
135	14.5288	0.8210	10.3167	12.0607
136	1.7283	9.4827	14.3426	2.1112
137	8.2719	2.6064	7.7157	5.6948
138	2.2905	13.4145	0.2255	6.3339
139	8.9499	2.6810	9.9661	14.1683
140	0.0733	8.0352	3.6975	0.3398
141	12.2669	15.9893	8.4389	13.5053
142	13.5793	5.6865	11.5999	4.6091
143	14.6691	0.7532	9.7187	4.0055
144	15.7915	3.4186	9.4139	7.8141
145	8.0821	6.3654	6.9350	11.6646
146	4.3427	5.3387	3.9068	3.2419
147	1.6120	3.6736	6.8634	3.4604
148	8.1256	14.9779	0.1628	15.6214
149	9.3697	10.9310	9.7411	9.4918
150	12.2062	15.3938	15.3276	4.8705
151	1.3274	7.0076	1.5271	15.4832
152	10.5855	15.0454	0.5694	14.3355
153	8.2717	0.0933	14.1798	3.0404
154	2.7368	9.7649	3.9511	0.0288
155	15.0169	12.8172	0.1426	11.3882
156	9.4477	3.7277	13.0387	13.8836
157	7.0502	14.9195	2.2480	1.8929
158	15.0707	12.2122	14.0779	0.6244
159	10.4946	13.2232	1.5260	9.5711
160	7.2311	9.1754	5.6410	9.6690
161	13.4352	12.6813	9.4947	8.2629
162	8.5220	5.2647	9.3629	0.1201
163	8.8622	3.5754	10.6829	11.0229
164	10.8810	4.9982	10.3684	15.1363
165	5.8750	9.3524	6.9339	13.9766

166	3.8286	13.2786	2.2361	1.8124
167	9.2628	4.6474	12.0309	5.6731
168	13.8702	6.4409	3.8686	3.8710
169	6.5084	13.7929	10.4073	8.9653
170	1.8018	9.8358	13.7180	9.8036
171	7.1015	15.8590	1.3499	4.8129
172	4.8030	3.2592	15.5534	12.7703
173	6.4222	13.2353	0.5034	12.7303
174	13.3338	10.8138	13.3665	12.4975
175	6.4581	3.9832	13.3714	5.6176
176	6.2428	7.6126	0.7977	0.8688
177	5.7672	6.3852	8.7342	11.3393
178	2.2441	9.5910	15.0907	15.8869
179	4.1621	12.8084	5.1436	2.5996
180	1.3890	1.6811	12.9035	1.8171
181	6.8704	13.1431	9.6224	14.6060
182	4.1165	13.4574	12.6339	7.7065
183	4.7609	5.6721	12.7870	13.6289
184	6.7977	6.8811	0.7930	12.9586
185	1.9073	9.1558	4.5312	2.9882
186	7.9211	11.2132	10.4553	3.9552
187	11.3025	11.8795	7.8345	0.8670
188	3.8972	12.1261	15.5656	9.7434
189	12.5611	6.2261	11.9758	12.4357
190	1.1854	6.8688	9.0855	8.1770
191	6.3021	15.3015	4.7834	0.4440
192	0.0543	9.1675	4.0978	15.8462
193	3.5308	13.5956	14.1850	8.0150
194	0.0208	4.4215	7.1488	5.3120
195	3.0269	9.9572	13.0558	2.7821
196	2.2797	9.4138	1.5734	10.0102
197	4.2892	15.4155	13.7535	9.2022
198	2.7983	1.3744	0.4421	12.0158
199	2.2184	8.0080	14.3865	2.4563
200	9.5822	8.3454	14.3990	5.7086
201	14.4169	1.4427	8.3857	2.3032
202	15.0301	14.4747	1.9232	13.6097
203	3.5390	14.1502	2.8447	5.4059
204	7.7227	7.0238	11.2977	4.4031
205	6.0162	12.5076	13.3018	0.0962
206	8.3805	2.3754	0.5573	12.8306
207	4.2380	9.9171	12.1254	7.9585
208	1.0937	4.1700	15.3138	8.6055

209	6.9812	7.1305	5.4859	13.9346
210	2.7816	13.5040	10.2119	11.5655
211	0.4177	3.1393	5.4881	10.6894
212	15.2749	4.8616	3.4635	2.8612
213	6.8895	7.7327	12.5792	8.8079
214	15.3849	5.4050	11.5694	15.3580
215	12.1986	12.7758	4.4614	9.5363
216	0.1176	15.7998	9.3189	12.9371
217	10.8806	2.5448	6.7361	15.7525
218	11.2952	3.7901	1.4731	14.1748
219	10.3221	11.2358	0.3844	3.4214
220	8.8370	6.0075	7.8583	0.5541
221	3.4897	15.5793	4.4523	7.2180
222	12.3579	15.5569	5.4361	0.2207
223	3.6485	10.2992	4.5976	7.5794
224	5.9338	13.7616	2.7345	15.2192
225	14.2549	6.4301	6.3882	3.9832
226	13.7020	10.1109	11.1624	6.1828
227	6.4389	15.7638	3.2588	6.9029
228	5.0883	8.9516	10.6612	13.2942
229	9.7382	14.9375	7.0891	13.1943
230	14.5631	11.5255	6.9327	7.2480
231	14.5456	7.7446	2.8038	6.0890
232	9.4655	10.2245	3.0912	14.8139
233	5.3211	14.2022	9.8627	11.8535
234	13.6490	3.1798	4.3042	11.8021
235	7.0784	6.3259	8.9548	15.1507
236	14.4697	15.8748	15.1165	8.1616
237	0.5309	6.4376	11.4315	12.6701
238	8.5188	10.5417	10.8675	7.2348
239	11.4640	14.4216	15.3501	13.5872
240	2.8688	15.9261	12.4053	6.2469
241	5.3845	10.4506	9.7236	11.8140
242	3.0034	1.7350	15.1680	15.6230
243	5.1508	0.5778	0.9543	8.3728
244	6.4617	9.8895	4.2994	6.8786
245	8.7771	9.0743	15.7869	3.3145
246	0.7798	15.3914	12.3553	5.1744
247	8.8437	11.9377	7.6057	1.7739
248	4.3970	10.6003	10.8944	6.0034
249	3.8640	8.3730	6.6710	5.2785
250	3.8903	4.1583	6.0824	5.4737
251	2.4666	15.3919	3.4123	13.0739

252	15.3027	8.6433	6.1270	8.5070
253	14.9706	0.4843	0.4747	8.3380
254	13.0994	11.1410	7.5571	12.3890
255	11.6522	8.3155	5.3340	1.9242
256	2.8130	0.9445	15.6135	10.0072
257	5.7659	14.2406	8.8871	5.5464
258	3.0206	5.2832	13.5409	5.3539
259	0.0192	3.6752	6.5290	9.1939
260	5.0627	1.8232	7.3923	13.8230
261	11.1939	4.9748	13.2209	3.1770
262	10.0041	3.6549	15.8593	10.7592
263	8.6890	10.4320	8.3832	14.4293
264	7.0246	1.0586	14.8070	3.1865
265	4.5988	4.4069	11.8243	4.7725
266	8.0265	4.5091	9.0789	7.9443
267	12.1847	14.0811	15.5004	14.2385
268	12.1985	7.1093	13.1920	8.0226
269	9.2169	12.0946	15.3537	4.4319
270	11.9626	9.6527	10.3415	8.5433
271	10.3286	12.5323	6.0732	9.1880
272	1.9715	1.8229	7.6252	6.6049
273	8.0704	15.6570	14.5902	0.2362
274	5.5562	13.5775	0.2377	11.2442
275	1.4744	0.8103	2.5071	8.1080
276	2.3656	7.4592	7.5451	6.1003
277	3.1707	5.2105	8.6879	1.0389
278	10.7563	10.0833	0.9551	5.7373
279	6.9042	3.6848	10.5285	3.7480
280	11.1105	9.2782	14.2342	3.2561
281	4.1086	9.6505	1.7541	13.0205
282	0.1561	9.5981	7.0044	6.2950
283	8.5165	7.1748	4.4837	0.8572
284	4.4703	0.5668	15.7639	6.0008
285	15.1397	8.2210	9.7401	12.3999
286	14.5031	6.5237	4.0600	2.6448
287	6.2830	1.7287	2.1218	14.5955
288	0.3977	7.3580	8.7201	5.1073
289	10.7430	7.2141	13.2449	5.2765
290	13.3947	8.8182	13.3921	3.2678
291	15.5440	12.8865	13.3336	12.2754
292	0.9109	11.2136	3.2594	1.1196
293	7.2052	13.9558	8.7107	15.2005
294	9.3195	0.8351	13.9991	2.5314

295	10.9862	3.5149	1.9360	4.5831
296	11.5109	7.3543	13.7016	10.9941
297	10.4007	15.3365	14.3964	2.2584
298	11.6306	12.6407	3.4858	8.1934
299	5.9816	7.2300	1.2317	11.5412
300	9.3053	5.3349	7.5874	14.8615
301	1.8579	0.9455	13.3604	11.7137
302	0.9225	11.8545	7.5103	11.9976
303	15.6762	8.1087	6.6203	6.5171
304	4.5572	3.1988	8.0439	3.8319
305	9.5196	6.8351	2.0070	8.3336
306	15.3946	2.6990	2.1166	3.5052
307	2.9725	12.0271	13.9276	13.4782
308	3.0886	5.8936	9.6472	10.6069
309	5.4663	15.0691	4.2448	13.0598
310	14.9264	0.2748	13.8368	12.7020
311	6.2507	13.2649	0.9298	7.5057
312	4.3715	10.0255	7.3241	4.9524
313	2.4312	8.6199	11.5554	11.0013
314	6.3537	10.4081	5.4240	15.7896
315	5.9956	11.6261	6.4195	12.3189
316	2.0978	1.5118	8.4317	13.2733
317	6.9607	14.0412	14.3078	11.2974
318	1.4642	0.2298	12.4538	9.5254
319	9.8340	4.7088	1.1099	12.0460
320	0.1757	2.8786	4.4606	7.9476
321	9.1722	14.8207	6.0699	13.8421
322	12.6357	1.0909	13.8348	1.0884
323	3.7659	9.2975	6.7194	15.4967
324	7.1683	10.1944	3.8380	1.5801
325	9.1097	10.4203	9.5625	8.7516
326	0.9824	13.8340	7.6705	6.4475
327	7.9406	0.8952	14.3768	1.7126
328	10.2770	13.0697	14.9553	11.5867
329	3.5403	8.4628	13.0862	9.8189
330	13.3929	11.1096	11.3425	12.5275
331	15.5372	3.3985	11.8915	9.0659
332	13.5420	8.6925	14.3954	12.9811
333	8.0960	11.2403	1.0438	9.2284
334	4.4620	15.3030	5.3746	15.1045
335	11.9459	7.1127	0.0694	13.9432
336	3.7909	1.3664	13.2495	8.1216
337	15.3175	0.9174	8.1190	12.6212

338	9.9242	10.0712	5.8586	7.5685
339	9.6042	12.7389	3.6262	13.2608
340	2.7617	11.0591	8.5573	5.1597
341	1.4455	5.5249	4.6318	15.6183
342	4.0842	15.1491	1.0939	4.4514
343	13.7371	8.3230	1.3595	1.1653
344	14.5771	15.2610	1.0934	12.0196
345	11.1941	1.1775	6.5571	13.2990
346	11.6029	3.3125	1.9741	14.7574
347	3.6782	12.4004	7.0883	5.2324
348	9.2169	14.6270	14.3830	12.8651
349	12.9700	12.5208	5.6582	8.6120
350	6.4615	4.7285	1.9228	7.4127
351	15.8150	2.4295	9.1058	13.1320
352	1.4400	13.5666	14.0005	15.2305
353	5.1351	12.5577	5.5772	1.2204
354	8.1825	4.3333	0.6707	11.3387
355	0.9697	3.6450	2.2774	3.7588
356	11.6110	5.1364	1.2255	6.3823
357	8.9049	13.2730	11.8484	4.2900
358	8.4698	13.1549	7.3044	13.3202
359	13.2797	9.1309	10.6920	15.9260
360	13.7401	9.1493	11.1879	10.3960
361	12.6245	4.5763	9.1417	11.2632
362	5.0853	11.1861	10.0591	14.9169
363	7.2353	12.7401	14.0442	11.0024
364	12.0356	7.0654	10.5976	9.0937
365	1.7578	7.1394	14.0066	6.0936
366	1.7559	7.4506	7.4803	10.1533
367	4.3181	4.4646	2.2614	5.8117
368	8.3942	10.8060	1.0902	6.5219
369	15.5624	14.4586	11.4279	5.8992
370	11.3665	14.5364	4.9278	7.4944
371	4.9898	11.9552	10.7386	8.0546
372	4.6633	4.1682	10.4387	14.5686
373	13.6057	11.0342	8.4968	3.3029
374	14.5864	2.1093	11.4417	5.4177
375	10.2284	1.9760	8.0770	9.1860
376	4.0859	3.0544	7.8080	7.7909
377	1.4187	2.3317	7.9654	4.1955
378	13.4121	9.3607	14.9756	9.2735
379	9.3555	1.1738	6.2285	14.0532
380	15.1697	13.1572	1.8743	0.9752

381	0.9765	11.5664	3.8468	7.0540
382	9.3543	14.8137	10.9585	1.3481
383	4.5617	7.8822	13.4281	9.0118
384	13.2437	10.4781	15.5223	8.6290
385	3.0558	14.2420	3.4427	12.2889
386	7.0805	8.6164	12.1655	3.7294
387	6.2946	4.5153	9.3456	9.3978
388	13.2252	15.6153	6.4472	7.3436
389	10.8299	0.5828	8.1606	13.7757
390	3.3216	5.2199	7.9303	10.5734
391	5.0897	15.5682	10.4219	5.6621
392	2.1410	5.8405	11.8993	5.5550
393	10.7434	4.9464	4.8313	4.0595
394	9.1359	1.9346	1.4338	15.2405
395	2.7163	14.6523	13.2154	4.7712
396	2.3625	2.1677	6.2334	2.5345
397	7.6173	5.3139	12.4049	5.7808
398	14.5296	14.3597	2.8699	11.8661
399	8.8348	7.9944	1.7498	11.2944
400	0.5270	9.8446	14.4825	11.2143
401	0.8618	9.3301	14.0216	0.0996
402	12.8810	11.1721	15.9967	5.9895
403	7.2220	0.4693	13.8281	14.4239
404	6.1223	8.4461	0.5900	5.0935
405	12.6343	0.5132	8.7149	9.5533
406	5.8286	13.2343	15.9619	4.7647
407	8.5176	5.4398	8.1761	2.0002
408	11.3865	13.5474	13.9762	6.2137
409	13.9436	3.9371	1.1235	13.0830
410	5.2590	9.3039	15.8006	15.6988
411	10.4019	15.0028	14.7634	13.7918
412	15.5974	0.7646	9.0282	1.3411
413	1.2155	0.8636	6.9034	5.4034
414	9.3923	0.3299	5.4055	3.7781
415	6.6222	10.9037	11.5316	5.0849
416	4.9462	9.5781	0.2185	15.7512
417	4.2213	1.8245	5.9850	8.7720
418	12.1403	12.7399	14.7630	11.9880
419	15.9235	9.8856	8.7437	13.4696
420	2.9851	1.1234	7.5822	2.6702
421	12.4983	1.1085	7.9446	14.4496
422	3.1328	2.1761	4.9435	1.6820
423	15.8777	12.6223	15.2134	11.9215

424	12.8362	1.4784	15.7119	11.6700
425	6.7876	3.8059	8.2170	11.4795
426	11.6618	3.8984	15.8814	2.1349
427	7.9737	1.6772	7.2935	7.1326
428	12.9438	13.7336	6.8167	8.1406
429	5.7041	11.1712	3.4114	8.4878
430	1.1719	11.7399	3.0920	13.7555
431	9.4559	10.4085	13.3241	10.8436
432	14.5630	8.2603	11.6262	12.8934
433	3.1003	5.2222	8.4759	8.4999
434	6.9179	10.5884	13.2653	15.2943
435	11.9866	1.8810	8.1900	1.0668
436	0.6270	2.3651	8.8314	8.6643
437	15.1412	0.3162	3.4126	4.5066
438	12.2188	15.4287	9.4051	7.6944
439	8.9411	15.5260	2.2842	10.9578
440	2.9415	1.9818	0.8356	3.3321
441	7.9672	7.4786	10.9329	9.7306
442	8.2855	10.5071	9.7370	5.2188
443	15.9079	4.6430	3.5145	14.0936
444	13.6776	12.0726	6.5005	2.1343
445	15.3985	8.9299	10.0785	1.6385
446	10.8631	6.8447	8.8849	15.3459
447	6.4560	4.2751	2.0413	2.4464
448	14.9597	12.0598	2.7072	2.4406
449	7.6718	14.3740	0.0160	2.4888
450	3.7087	11.6551	6.6906	1.4331
451	6.3406	6.5093	7.8158	7.2708
452	11.2812	15.0131	2.5579	10.7023
453	8.9369	4.0868	10.6694	13.3008
454	12.1061	8.5306	0.2867	12.6438
455	15.9277	15.2761	1.9148	11.4034
456	15.3989	4.2840	15.2340	7.5616
457	8.5611	4.0014	15.6137	11.3374
458	15.4219	14.8428	0.4946	15.3290
459	1.8500	1.0973	7.9021	8.0924
460	0.8232	4.7904	13.8036	4.8808
461	4.8696	9.4653	3.8860	12.6370
462	9.2831	3.2528	13.3483	3.7822
463	8.4954	10.1741	13.0175	3.7488
464	14.4193	12.7739	10.0636	7.4352
465	8.6488	8.0272	0.0358	9.9102
466	6.9117	10.4130	6.0756	9.8453

467	8.6827	12.7353	14.4705	1.9620
468	11.3986	3.7340	10.8856	1.9807
469	0.2668	9.6134	6.0612	4.5513
470	12.8147	1.7994	10.1113	11.7717
471	2.2801	8.2523	3.8923	6.5809
472	7.6556	13.4055	9.1426	13.2637
473	4.1094	14.7326	15.7077	14.9618
474	5.9055	7.9716	13.5949	6.3851
475	10.5882	4.4418	4.5351	0.8354
476	2.7137	10.4403	10.9193	9.1390
477	4.4605	14.6768	5.7300	11.9627
478	3.1715	8.1574	15.7908	5.1239
479	3.1211	15.5871	1.3438	7.8869
480	5.2294	3.1565	4.0049	3.5464
481	14.0854	1.7790	12.9815	15.0284
482	7.5376	4.7577	1.3508	7.7169
483	6.4635	6.3427	8.5001	8.6399
484	2.8677	6.7321	12.8100	3.5369
485	15.5028	4.9836	11.8212	1.5351
486	6.5193	11.1015	2.2666	0.9626
487	13.5118	1.4699	7.0063	13.1121
488	9.8452	6.4334	5.6061	12.3437
489	6.0258	4.7229	7.6560	3.1311
490	14.0349	4.9039	9.3985	14.3219
491	12.5576	1.6890	2.3329	10.9488
492	7.4393	9.5012	14.4853	10.5095
493	13.0236	4.5236	10.2431	15.8461
494	14.3751	2.4835	2.6070	0.5391
495	6.8678	0.0105	9.0546	6.7881
496	5.3493	4.5375	14.9059	7.8397
497	9.5464	8.8130	12.5296	9.3361
498	14.4319	13.9344	10.9710	1.3323
499	11.2331	0.6761	7.4595	10.5625
500	6.0393	14.4756	4.1651	0.8369
501	11.7593	2.0956	9.1083	8.9093
502	15.2656	13.3397	3.9803	11.3924
503	8.6850	12.8075	5.1088	7.8066
504	8.6417	14.6861	14.5728	9.8816
505	4.9778	2.1969	14.1635	3.4204
506	1.1398	8.0757	12.7134	10.3305
507	2.9117	6.4793	14.8130	6.0903
508	1.4878	2.7772	2.8614	1.6594
509	7.4158	9.2029	8.2807	6.0402

510	0.1493	9.6995	10.0321	4.2058
511	14.6404	3.4311	14.6109	3.8606
512	10.2839	8.3189	10.6235	9.9668
513	0.0227	15.8270	6.2271	8.3668
514	0.4862	7.8386	11.8401	6.6118
515	3.3355	11.1180	13.0822	3.4847
516	7.2795	6.5827	9.6055	13.7369
517	2.0363	0.5564	1.3600	13.7761
518	0.1384	4.6853	14.7577	4.5430
519	11.6333	12.8231	0.8576	9.8463
520	5.6659	5.5440	8.4324	12.4718
521	12.4871	1.3331	1.9017	15.2776
522	6.9865	8.1777	6.0823	14.7137
523	6.9849	5.8693	13.0053	6.1571
524	0.7874	11.8317	3.9055	2.6023
525	0.7941	8.3958	14.1508	12.7480
526	1.4576	12.8723	11.4023	1.8211
527	9.5046	13.0706	6.0504	2.5412
528	3.8573	3.0315	3.9827	5.6932
529	13.4619	1.9791	4.0457	13.5641
530	13.7154	13.1359	12.2759	9.3245
531	15.4178	10.2064	0.7978	9.3788
532	7.8224	0.2579	10.9646	14.8134
533	3.5250	14.3353	9.9244	9.2013
534	3.6193	8.2460	11.9470	0.1596
535	8.5886	8.7124	15.6361	12.9500
536	12.1938	9.7031	6.1426	9.7409
537	5.5611	12.1670	4.1633	7.6786
538	7.3797	13.6855	14.0395	4.2950
539	10.2292	6.1259	12.8975	4.1295
540	14.6774	1.3544	7.3779	7.6962
541	2.5852	11.7420	1.4554	3.6375
542	11.4502	5.3118	9.0283	0.7776
543	9.2438	13.4360	2.9981	2.7079
544	6.9328	5.9476	8.5070	4.1351
545	14.1479	13.2514	5.6805	3.1666
546	6.2888	2.8243	5.0365	9.6911
547	2.8636	2.0723	11.6279	13.1793
548	10.1333	14.0781	8.2524	12.9698
549	9.9840	0.7053	12.6503	12.8358
550	5.2471	10.9875	3.2719	11.3298
551	12.8474	11.7404	10.8497	13.7500

Total numbers of points of set F = 551

Number The Coordinates of the Minimal Points

259	0.0192	3.6752	6.5290	9.1939
194	0.0208	4.4215	7.1488	5.3120
513	0.0227	15.8270	6.2271	8.3668
192	0.0543	9.1675	4.0978	15.8462
140	0.0733	8.0352	3.6975	0.3398
518	0.1384	4.6853	14.7577	4.5430
320	0.1757	2.8786	4.4606	7.9476
288	0.3977	7.3580	8.7201	5.1073
436	0.6270	2.3651	8.8314	8.6643
460	0.8232	4.7904	13.8036	4.8808
401	0.8618	9.3301	14.0216	0.0996
125	0.8956	12.5020	1.0781	15.2353
292	0.9109	11.2136	3.2594	1.1196
355	0.9697	3.6450	2.2774	3.7588
413	1.2155	0.8636	6.9034	5.4034
151	1.3274	7.0076	1.5271	15.4832
46	1.3480	13.3614	1.1272	4.6170
180	1.3890	1.6811	12.9035	1.8171
377	1.4187	2.3317	7.9654	4.1955
318	1.4642	0.2298	12.4538	9.5254
275	1.4744	0.8103	2.5071	8.1080
508	1.4878	2.7772	2.8614	1.6594
122	1.9550	0.4024	7.7989	11.9800
517	2.0363	0.5564	1.3600	13.7761
196	2.2797	9.4138	1.5734	10.0102
138	2.2905	13.4145	0.2255	6.3339
396	2.3625	2.1677	6.2334	2.5345
92	2.5039	8.4569	0.3648	2.1126
154	2.7368	9.7649	3.9511	0.0288
198	2.7983	1.3744	0.4421	12.0158
440	2.9415	1.9818	0.8356	3.3321
420	2.9851	1.1234	7.5822	2.6702
422	3.1328	2.1761	4.9435	1.6820
277	3.1707	5.2105	8.6879	1.0389
120	3.4879	2.0086	7.4642	0.5221
534	3.6193	8.2460	11.9470	0.1596
166	3.8286	13.2786	2.2361	1.8124
284	4.4703	0.5668	15.7639	6.0008
416	4.9462	9.5781	0.2185	15.7512

243	5.1508	0.5778	0.9543	8.3728
30	5.1977	6.6980	4.0954	1.5781
89	5.5503	12.0835	0.0495	1.5169
404	6.1223	8.4461	0.5900	5.0935
100	6.1822	0.0055	5.4477	3.8700
176	6.2428	7.6126	0.7977	0.8688
37	6.3701	15.2126	2.9094	0.3586
447	6.4560	4.2751	2.0413	2.4464
264	7.0246	1.0586	14.8070	3.1865
449	7.6718	14.3740	0.0160	2.4888
327	7.9406	0.8952	14.3768	1.7126
354	8.1825	4.3333	0.6707	11.3387
153	8.2717	0.0933	14.1798	3.0404
283	8.5165	7.1748	4.4837	0.8572
162	8.5220	5.2647	9.3629	0.1201
51	8.6385	0.5579	3.1973	10.4898
465	8.6488	8.0272	0.0358	9.9102
57	9.1334	2.3025	12.2366	0.4360
18	9.1958	0.6942	0.4862	3.6246
294	9.3195	0.8351	13.9991	2.5314
414	9.3923	0.3299	5.4055	3.7781
54	10.0983	0.5821	2.5972	4.5254
475	10.5882	4.4418	4.5351	0.8354
5	10.6285	0.3781	7.0430	1.4649
104	11.1000	5.2129	6.1490	0.0977
21	11.4207	4.5323	3.4322	1.0343
335	11.9459	7.1127	0.0694	13.9432
435	11.9866	1.8810	8.1900	1.0668
322	12.6357	1.0909	13.8348	1.0884
55	13.7491	11.9384	0.0904	3.2829
494	14.3751	2.4835	2.6070	0.5391
253	14.9706	0.4843	0.4747	8.3380
437	15.1412	0.3162	3.4126	4.5066
412	15.5974	0.7646	9.0282	1.3411

Total number of the Minimal Points = 73

Number	The Coordinates of the Maximal Points			
216	0.1176	15.7998	9.3189	12.9371
45	0.9675	14.9149	11.8609	14.3623
117	1.2491	9.9648	11.9390	15.8358
352	1.4400	13.5666	14.0005	15.2305

52	1.5260	14.1667	9.7174	15.0112
134	1.5632	14.8398	15.6890	0.8850
178	2.2441	9.5910	15.0907	15.8869
240	2.8688	15.9261	12.4053	6.2469
473	4.1094	14.7326	15.7077	14.9618
197	4.2892	15.4155	13.7535	9.2022
334	4.4620	15.3030	5.3746	15.1045
410	5.2590	9.3039	15.8006	15.6988
98	5.7605	14.8718	5.4094	14.1477
406	5.8286	13.2343	15.9619	4.7647
224	5.9338	13.7616	2.7345	15.2192
181	6.8704	13.1431	9.6224	14.6060
434	6.9179	10.5884	13.2653	15.2943
293	7.2052	13.9558	8.7107	15.2005
20	7.3028	15.6638	11.2007	11.1451
148	8.1256	14.9779	0.1628	15.6214
535	8.5886	8.7124	15.6361	12.9500
35	8.9907	12.8818	11.2405	15.7055
321	9.1722	14.8207	6.0699	13.8421
17	9.7541	13.7103	7.3878	15.8482
262	10.0041	3.6549	15.8593	10.7592
411	10.4019	15.0028	14.7634	13.7918
152	10.5855	15.0454	0.5694	14.3355
15	11.1169	1.9844	13.9412	15.9608
239	11.4640	14.4216	15.3501	13.5872
267	12.1847	14.0811	15.5004	14.2385
1	12.2014	15.9916	6.9948	0.8398
150	12.2062	15.3938	15.3276	4.8705
141	12.2669	15.9893	8.4389	13.5053
424	12.8362	1.4784	15.7119	11.6700
551	12.8474	11.7404	10.8497	13.7500
402	12.8810	11.1721	15.9967	5.9895
254	13.0994	11.1410	7.5571	12.3890
80	13.1108	12.9647	11.2635	12.1436
384	13.2437	10.4781	15.5223	8.6290
359	13.2797	9.1309	10.6920	15.9260
130	13.3155	2.0736	13.9650	14.3295
174	13.3338	10.8138	13.3665	12.4975
330	13.3929	11.1096	11.3425	12.5275
332	13.5420	8.6925	14.3954	12.9811
530	13.7154	13.1359	12.2759	9.3245
481	14.0854	1.7790	12.9815	15.0284
2	14.1198	15.6957	6.0774	12.8188

76	14.3659	4.9499	13.6924	14.9992
236	14.4697	15.8748	15.1165	8.1616
398	14.5296	14.3597	2.8699	11.8661
432	14.5630	8.2603	11.6262	12.8934
344	14.5771	15.2610	1.0934	12.0196
310	14.9264	0.2748	13.8368	12.7020
202	15.0301	14.4747	1.9232	13.6097
285	15.1397	8.2210	9.7401	12.3999
40	15.2146	11.8785	5.9773	14.2956
502	15.2656	13.3397	3.9803	11.3924
214	15.3849	5.4050	11.5694	15.3580
456	15.3989	4.2840	15.2340	7.5616
458	15.4219	14.8428	0.4946	15.3290
291	15.5440	12.8865	13.3336	12.2754
369	15.5624	14.4586	11.4279	5.8992
351	15.8150	2.4295	9.1058	13.1320
423	15.8777	12.6223	15.2134	11.9215
443	15.9079	4.6430	3.5145	14.0936
419	15.9235	9.8856	8.7437	13.4696
455	15.9277	15.2761	1.9148	11.4034
58	15.9496	9.6953	6.7371	14.0189

Total number of the Maximal Points = 68

Appendix III

(c) Partial Results for Example 7

Number	The Coordinates of Points of F							
….								
…..								
8000	9.3711	20.4806	31.2683	36.2505	21.8931	2.8150	15.5893	27.0889
8001	19.8941	9.9671	20.3927	20.8402	21.5654	11.5905	38.8753	11.4882
8002	31.6233	15.0114	10.2780	34.1817	19.7075	25.2043	5.4582	30.5340
8003	15.2093	23.5876	23.9932	12.9926	5.9525	32.2403	19.4281	26.4420
8004	8.7433	39.8073	22.5823	34.5301	12.9564	9.2417	22.6861	2.4943
8005	28.9066	1.5341	14.1785	20.8601	1.5650	37.5691	23.6790	37.9125
8006	30.9133	10.6274	6.4425	36.5776	2.7954	6.8085	15.7368	38.7498
8007	27.7313	12.0876	20.9337	11.1507	1.3902	7.6312	34.0809	9.5545
8008	24.7870	8.2457	6.1936	37.0493	22.5402	17.7420	0.1638	10.5942
8009	30.4921	25.3480	36.2021	34.4395	21.3015	14.6580	25.4070	24.9615
8010	20.4592	28.6233	28.0205	12.5543	23.4698	24.9169	27.8198	9.4216
8011	13.1762	21.4201	37.3562	4.9984	9.8613	2.8468	35.8587	30.7474
8012	0.0538	2.9846	21.0495	26.7662	30.1150	22.6396	25.5417	5.6594
8013	23.2258	6.1344	10.9897	31.3470	30.9606	0.5576	27.7449	39.2551
8014	6.8735	27.3463	9.2236	15.4330	21.8183	9.1417	33.9053	19.0993
8015	7.7963	13.3512	12.4635	15.4825	33.5500	17.0319	21.9586	17.7369
8016	6.0129	33.6794	14.6267	34.0669	8.4261	31.7138	22.8377	9.7563
8017	20.2168	37.9197	39.7819	17.9630	17.0702	14.3481	24.2532	30.4710
8018	0.6955	30.0674	6.1189	33.1433	4.8299	8.4053	26.9041	27.7692
8019	18.8483	1.0876	33.2958	33.4236	27.6679	24.0920	13.5314	20.4643
8020	1.8431	29.7640	10.9500	35.7038	35.5946	17.3509	8.5196	16.8155
8021	17.6906	13.4702	36.5078	30.0577	6.1616	30.2270	13.7933	2.6755
8022	6.5424	32.0541	18.8681	26.9850	17.7594	20.2132	3.2563	36.8423
8023	22.8840	6.6761	39.4446	4.8112	39.6291	33.4996	5.0044	12.5140
8024	32.7328	25.4675	3.4911	1.1809	22.3673	1.0656	34.6232	39.8304
8025	3.4137	5.7297	24.1482	24.5432	20.9117	31.1805	32.9636	25.2712
8026	9.3134	32.8045	1.5333	29.2799	31.1596	13.2388	2.6472	12.9475
8027	9.8418	7.1392	11.7542	38.0995	35.4467	19.2612	22.8303	6.4322
8028	14.0774	7.1751	19.4129	25.0553	4.3405	28.5423	16.3855	32.2834
8029	16.3634	6.0670	6.7464	18.7534	25.4364	32.2995	24.4000	35.2863
8030	4.5697	17.3849	24.3572	19.4451	13.3543	36.7120	31.6571	23.6006
8031	16.2886	35.2000	20.1860	34.6523	8.6886	0.2508	7.8761	7.5827
8032	27.0372	19.9808	31.4260	32.7089	0.1019	30.5178	12.1498	9.5141
8033	21.7210	22.7205	5.6370	24.3354	0.2643	23.3135	9.3890	21.6472
8034	14.2852	3.5023	13.7699	22.7285	20.6900	13.0747	34.2160	35.6194
8035	26.4637	17.7582	5.8130	20.2781	14.0214	12.6099	36.5932	24.3385
8036	27.1481	28.1828	19.5369	34.5998	10.3545	27.1378	11.7918	6.8606
8037	5.2624	39.2722	18.7940	26.8869	33.0970	5.9985	17.5400	17.7542
8038	17.5848	14.3243	33.6218	11.5433	10.9368	13.6734	19.9275	33.2694

8039	37.4878	4.6127	21.3040	36.2340	30.7359	38.9348	22.0075	33.4056
8040	22.8090	10.4050	32.6933	2.3783	3.6654	14.3708	29.8416	27.4269
8041	23.1056	39.8112	30.1812	21.1399	28.2042	34.8560	16.2586	34.3096
8042	30.5841	33.5000	1.5611	20.3297	34.2339	23.7740	9.4206	25.0335
8043	9.8781	30.0027	4.1747	19.5187	16.6028	25.4945	27.0768	14.6585
8044	27.7542	30.5820	11.6043	33.9891	39.4050	12.4026	29.4244	11.3610
8045	11.7945	39.0281	21.7881	35.3515	22.6768	5.5695	3.0073	36.1167
8046	22.8280	35.3432	4.8612	37.2588	15.8164	25.2662	32.5464	28.9423
8047	25.3922	25.6365	24.2266	6.0196	27.3511	31.9842	21.3949	8.2180
8048	15.9556	36.1835	26.4029	1.2505	7.9667	19.0493	7.8775	8.1148
8049	9.9553	32.4288	14.4238	7.8370	7.2848	16.2467	6.4249	5.7148
8050	37.4133	24.2416	10.6620	25.4789	32.4693	8.2628	14.1966	16.0460
8051	17.9561	34.2941	35.5567	30.0335	27.7874	8.9081	37.0405	3.2988
8052	21.2714	6.2760	15.0048	12.3534	3.2722	9.7449	19.0955	28.6608
8053	25.3935	12.9365	26.9749	17.0269	26.5193	30.2483	39.4064	25.7703
8054	26.7248	20.2589	12.9526	25.0947	20.4318	34.9014	17.4232	30.1710
8055	17.5605	15.7307	32.3688	24.9513	6.7344	7.3142	0.5252	7.8324
8056	6.9196	11.2283	3.5670	23.5261	2.0783	35.4839	13.3432	10.5786
8057	27.7754	35.5388	13.6298	12.2989	15.9094	12.8632	10.8838	2.5689
8058	36.9970	12.2373	20.0326	13.0109	2.8940	4.4529	17.1418	21.6345
8059	14.8242	33.0825	5.2716	37.7759	23.3907	11.0410	27.7754	21.2592
8060	25.5515	7.3246	38.6237	34.7150	3.3198	11.8375	28.4118	11.5876
8061	35.5872	34.0421	16.3432	25.3255	27.6727	28.8256	1.7927	20.6866
8062	19.3177	11.8428	38.8494	22.2604	4.3534	4.1200	5.6379	33.2133
8063	25.1927	1.2466	25.4836	14.7907	3.1559	13.1724	29.1853	34.2216
8064	18.6007	14.8395	18.4248	21.2475	37.8401	34.4389	3.5188	37.1894
8065	4.3669	15.4070	10.0624	27.8172	19.4131	20.9188	32.4423	34.3647
8066	4.5247	34.2608	7.1185	29.4182	1.1380	14.8430	27.5308	15.9151
8067	4.3157	36.9016	23.5609	25.7088	25.0055	10.0506	1.3643	6.2685
8068	17.8292	10.2702	6.3807	30.8288	9.8506	24.0947	16.7095	37.7185
8069	37.5538	23.1317	16.0677	8.9839	15.3730	26.3322	31.6463	38.6717
8070	25.3526	13.0910	25.5271	17.2732	11.9365	1.1909	12.6954	10.7724
8071	37.1868	18.9836	28.0283	11.2222	5.7369	5.7342	3.2638	38.1421
8072	2.5151	29.7237	17.9168	32.9961	5.2408	39.9848	8.7564	38.2093
8073	12.2776	26.8328	11.0092	22.7062	2.6916	5.1265	5.8425	30.1198
8074	37.9906	2.5556	2.6824	31.1086	32.1008	19.9241	29.4998	31.2655
8075	0.4146	36.0729	20.7655	16.1012	18.2480	23.4173	27.7375	22.9110
8076	6.7131	28.8370	22.2375	27.4229	24.0470	4.8377	30.8869	0.6231
8077	30.2109	17.7318	32.3489	33.2079	13.3374	0.9526	34.4366	17.3979
8078	33.3529	6.8941	24.3338	32.4369	8.6312	37.9819	11.3694	25.2521
8079	24.2344	16.6376	20.2908	33.7268	22.4710	25.2517	23.3150	32.9663
8080	20.7167	0.0286	21.6837	13.9349	35.0177	26.0843	11.3634	8.0250
8081	37.7143	21.8617	18.1478	37.5516	18.3982	27.9561	19.5306	16.4760
8082	29.6483	22.5721	10.8512	2.9057	33.1311	13.3346	1.8403	38.4058
8083	28.0432	2.6027	33.1673	7.6597	24.5121	29.8734	24.9276	2.6246
8084	29.7887	15.4678	18.9135	21.9723	28.2686	25.9535	36.3541	16.4969

8085	34.9530	20.1986	25.7425	18.9227	4.9022	37.6127	29.1681	31.1879
8086	38.5648	34.7429	23.8684	3.0387	11.9839	0.6756	30.0198	22.0274
8087	39.2767	29.9405	5.4380	1.1068	27.2956	16.1426	36.5509	27.3720
8088	36.7453	29.8286	9.9446	2.0661	30.1126	37.3698	12.6056	21.9997
8089	11.3392	33.9175	13.1532	15.7717	21.8271	36.8365	31.4452	34.3702
8090	2.3450	9.4840	2.1952	38.8912	28.8993	29.5896	14.8559	12.7543
8091	32.8398	13.5012	28.8228	14.3173	18.5170	7.6326	21.4290	8.4676
8092	7.1809	38.5652	31.6570	21.1921	1.7491	38.9227	19.9100	16.0440
8093	1.3973	5.7405	39.1297	30.1751	2.0190	28.6247	15.1516	32.5714
8094	30.8531	4.1414	3.9675	14.4477	26.4802	12.3864	33.9731	25.5733
8095	21.6210	36.3458	27.9644	8.6255	34.5433	31.5359	10.4078	15.0063
8096	13.2616	4.6074	5.4326	0.5840	19.6579	12.4846	6.1021	13.6363
8097	23.1287	29.7748	12.6243	26.8703	17.0708	31.5922	20.2005	9.0049
8098	34.1803	28.9866	16.5156	7.3363	31.6922	35.7228	32.1113	29.4404
8099	25.6950	4.8888	1.0290	21.4515	15.6265	0.0409	1.1057	0.7666
8100	16.8041	39.3962	24.7614	10.5178	19.0354	4.6329	17.2789	4.4443
8101	21.8365	8.3258	20.0355	13.0804	6.6547	8.5425	9.3880	10.3534
8102	7.2919	9.9897	13.7915	39.1223	22.2725	13.6294	31.9991	24.4467
8103	9.4510	19.4496	0.3119	7.9780	2.0272	2.3866	30.1415	2.5702
8104	8.2165	11.7098	27.5176	32.0997	27.2527	31.5505	37.7733	2.6843
8105	10.3875	28.6599	14.0011	18.2989	37.6849	3.2485	30.1547	17.5307
8106	37.7053	22.9677	39.2830	22.8984	26.4930	6.9508	32.2016	39.8665
8107	25.6628	28.4410	32.6620	7.4331	4.6095	30.4581	4.7164	17.8450
8108	33.1625	4.0325	17.4627	4.7750	6.1830	7.3352	28.7031	7.5637
8109	15.5505	12.5550	22.9060	10.4282	11.7453	5.7160	2.1417	4.4672
8110	6.7365	37.1315	38.8531	24.7200	37.0711	30.3149	37.5958	10.9449
8111	29.6047	35.7172	37.4023	19.8804	32.3218	7.1640	7.7033	0.2445
8112	34.9118	23.9482	21.5231	39.5282	23.0087	37.5178	14.6431	10.0872
8113	14.2210	19.8659	36.7124	5.7227	22.5438	8.3302	20.5799	3.7109
8114	7.1531	33.0281	30.4481	14.1598	22.1170	11.4049	34.8463	31.3296
8115	39.8569	32.6312	39.1474	21.1557	37.5934	39.1486	1.3452	38.2634
8116	38.3651	5.6481	16.7675	27.4026	26.3399	19.0102	20.0949	27.6134
8117	6.2398	24.1866	7.3888	20.6330	12.9256	10.7805	9.8622	4.4124
8118	37.1574	31.6229	35.4921	3.6298	1.4487	1.6349	30.9182	35.7147
8119	16.9830	29.9434	38.9448	38.3557	38.8842	30.1921	25.9491	38.4902
8120	8.9234	0.3929	34.1561	17.8516	18.5598	9.8205	5.1182	15.8581
8121	5.4558	7.7361	1.9252	13.0836	23.6417	36.1894	11.1286	34.1415
8122	10.4070	15.3917	8.0977	1.4508	28.8256	18.4157	36.2161	22.4813
8123	21.7786	17.5503	33.2208	20.5329	0.4685	28.8855	37.1534	4.6834
8124	3.2883	35.6918	12.9347	32.2124	33.2109	2.3535	20.9917	21.5194
8125	23.9251	37.2114	35.5304	37.4394	25.0971	0.4717	1.0579	33.5129
8126	1.1674	24.0946	5.2821	19.3899	39.8737	16.0103	8.4340	21.9635
8127	18.1964	29.1590	38.1539	18.7940	2.2235	39.8052	18.5270	17.7819
8128	13.2999	35.7746	9.2423	33.7723	12.8905	37.4132	32.8185	37.0155
8129	37.7469	14.4736	2.1591	4.8499	39.4032	28.9654	13.2759	39.5177
8130	12.1241	8.1476	23.7510	6.0505	31.1612	37.6371	32.1732	18.0844

8131	17.1290	21.2112	36.2260	26.5828	28.8327	11.1000	8.4516	16.7555
8132	26.4414	37.0881	33.1533	32.6954	29.4410	15.7840	30.7166	23.8026
8133	15.2435	8.3980	36.9845	32.5324	4.2944	9.5402	4.4686	13.5780
8134	5.4468	39.2807	16.4380	34.9006	35.0137	31.0459	19.2226	17.5102
8135	3.8651	16.6546	22.4964	16.4490	34.7042	2.7035	21.7779	2.8376
8136	12.2065	10.7465	26.4692	4.6510	2.4532	19.8285	19.0394	7.1865
8137	32.9553	29.2296	6.3284	1.0385	31.0115	3.4999	5.0260	38.8527
8138	25.0901	21.3644	34.5019	5.6106	4.0350	24.8623	3.0490	22.1320
8139	8.2373	15.0045	16.2865	27.0772	30.8478	12.6666	33.2830	18.5331
8140	39.2074	15.2962	32.0776	33.6545	31.2369	3.6871	29.2292	15.4415
8141	17.5110	8.6864	7.3277	24.1737	1.3300	36.7581	19.2128	24.3246
8142	33.4078	33.8059	39.5089	6.8101	16.1365	5.5410	2.6764	18.9387
8143	21.5672	2.8696	34.3979	22.7140	7.8354	39.9143	22.1101	27.8467
8144	1.0051	14.5008	32.9534	33.7926	23.8671	33.8039	19.6831	27.1128
8145	0.2730	4.3835	16.2242	15.5055	26.2470	27.4982	39.8665	26.7010
8146	31.4434	36.5432	2.7452	24.5734	20.0242	37.2701	15.7484	29.4932
8147	17.8957	1.6203	24.4287	20.4576	10.2171	22.8776	28.0151	7.3888
8148	11.5495	9.1172	39.6392	16.8828	9.2267	11.5305	14.2819	11.9165
8149	34.5272	14.0495	17.4656	4.0120	9.9628	24.2801	39.8341	12.6954
8150	26.2491	35.6393	6.3434	31.0323	17.0565	18.4444	9.3819	34.1243
8151	11.3877	19.3585	35.4734	10.4443	11.9526	9.4715	3.7708	34.5487
8152	25.7826	8.3765	6.5480	2.2328	3.1554	33.2147	30.5698	17.2282
8153	10.5432	12.7261	0.0694	14.4954	7.9483	11.1994	34.2254	39.0207
8154	6.9911	35.9714	34.5744	8.1894	7.0599	34.9650	15.7407	30.6617
8155	14.7448	0.2795	38.9614	27.3550	3.2717	4.2569	8.0328	14.0031
8156	21.5017	6.9554	28.8630	4.8554	25.6194	6.0917	9.4923	13.7132
8157	37.0563	30.7096	28.8262	19.8604	9.0908	28.8313	0.0726	20.8690
8158	29.9648	7.0228	21.9977	26.1553	37.9562	2.3263	17.9192	28.2100
8159	5.9094	20.2936	24.0808	21.2548	5.9337	0.7789	3.3592	30.8856
8160	0.1155	11.7521	4.2424	38.1949	16.0260	12.8618	36.6820	38.8080
8161	5.0806	34.9300	27.5454	28.4580	16.3825	27.2466	17.5677	0.0692
8162	36.6769	32.9394	33.0221	33.9549	23.7852	33.0911	15.1103	33.1562
8163	26.8849	39.9529	31.5202	18.5829	16.9301	7.6828	27.4879	29.1036
8164	4.5516	14.9817	4.0107	33.2767	36.1337	36.7199	26.1405	36.4110
8165	38.4091	20.3238	36.6201	22.0815	0.3581	20.5337	10.5965	19.0657
8166	4.7753	29.9399	14.4791	14.5436	27.0304	25.9701	22.7120	25.8091
8167	19.6966	15.9966	12.9868	27.0809	37.5466	12.4114	26.6156	14.8844
8168	14.4525	37.2662	38.2660	30.6266	25.4282	21.5492	29.5686	16.9210
8169	18.8000	17.1074	3.6328	18.4921	22.9587	37.6105	1.0375	32.5503
8170	2.3512	1.6888	30.1530	24.6755	20.3865	29.7765	18.5923	5.6250
8171	13.2005	6.3021	15.0282	7.8113	15.3359	4.5300	5.3356	9.7362
8172	10.5808	14.4541	3.5484	27.0594	34.7432	31.2845	29.9648	38.6084
8173	9.8367	2.3542	39.1792	7.8619	24.5725	34.6641	30.3284	11.6087
8174	37.9071	33.7344	27.0870	22.6481	28.0427	19.7732	9.8909	7.1586
8175	1.2977	35.5994	33.1537	38.0390	38.2002	33.7543	38.5198	0.7107
8176	36.2460	2.3558	38.5053	38.4943	9.1340	7.9265	9.9113	30.2710

8177	14.7399	8.0997	25.2488	31.2349	29.1055	16.4537	35.3558	10.4331
8178	26.5016	3.4948	1.1396	30.8566	7.9618	21.8944	18.9215	19.4999
8179	38.1450	22.9229	19.8298	19.9561	17.6600	19.3661	14.3241	38.3106
8180	3.7651	17.4796	7.3491	4.4695	13.5596	34.2194	2.8127	21.4184
8181	27.4126	3.5525	24.5748	36.6170	26.7598	22.5233	14.1710	4.3533
8182	16.4976	31.0289	15.1364	34.2111	39.2954	34.0592	23.7588	27.7620
8183	4.4715	29.3571	25.3204	7.7946	26.7896	17.5492	14.1554	12.7747
8184	19.6734	5.8091	33.8856	15.1639	5.6112	13.9348	26.0273	15.0426
8185	18.4621	24.6331	30.0544	7.9823	18.4315	15.9881	21.8140	8.8560
8186	35.0823	31.9283	15.0192	25.3283	0.9095	12.8713	8.9365	15.7936
8187	26.4096	32.8092	27.5212	4.4392	14.5943	7.9752	9.5354	35.9429
8188	11.1186	14.7450	2.1068	38.8357	13.3228	17.1928	12.7846	36.9465
8189	26.2867	32.3975	32.2024	37.7327	37.0857	35.3336	24.1727	23.0547
8190	25.8166	2.4272	29.0068	25.7546	12.7282	23.9555	23.4826	2.5417
8191	35.8889	3.6300	1.4256	17.1415	25.3827	31.6455	18.2468	9.1359
8192	35.2935	32.2234	20.0690	33.4456	29.1989	1.8416	10.6657	23.8369
8193	13.4069	21.8792	37.1239	5.7843	8.4043	35.0773	10.7961	28.8391
8194	17.0801	35.8050	9.9198	6.6486	15.3656	8.1513	39.7180	36.5781
8195	32.1506	7.4744	36.8451	36.5791	4.6915	0.8667	25.8349	18.4601
8196	36.9238	33.2984	6.0776	19.2568	5.1616	35.1007	19.2073	6.6323
8197	34.6690	2.5056	9.1940	15.9667	27.3544	30.9679	25.1443	13.7953
8198	19.9860	24.0012	33.4358	20.8000	28.4505	33.8713	33.9564	32.4602
8199	23.3793	2.4029	31.2973	36.6808	16.1827	37.6568	9.7302	5.7103
8200	36.9903	9.3413	27.9963	26.6383	0.2582	21.1627	5.5526	30.0195
8201	6.6447	8.7883	33.9155	4.0025	13.8900	3.1895	11.4605	30.9596
8202	14.9744	10.3700	5.1856	37.1583	22.9578	0.9519	25.1110	17.5565
8203	10.3763	23.9714	6.4588	9.0387	21.2981	5.0828	8.1256	3.7313
8204	33.2192	27.7966	8.0674	33.1220	9.1283	24.9330	26.2993	15.5321
8205	2.8071	36.8851	21.2859	36.9809	39.2081	2.2251	16.2841	1.4226
8206	11.2412	26.2230	35.0994	32.4209	9.5948	9.3818	33.8697	37.3759
8207	22.2018	25.3457	23.5142	28.9139	18.6615	22.0351	30.5992	30.0160
8208	24.7587	16.0719	13.0342	5.7051	25.3438	19.7980	39.5281	23.4001
8209	16.3915	14.3137	27.4287	29.0147	39.7856	26.1782	33.8057	25.6549
8210	10.3371	18.1595	3.0195	30.8642	33.5511	39.4897	36.4135	7.3997
8211	16.6698	33.5573	11.4874	27.4309	19.9845	1.2937	19.3092	7.6735
8212	15.2822	15.6005	23.7097	37.8669	7.9110	2.3160	19.0519	20.6317
8213	3.8534	35.5870	6.9276	28.6018	1.5862	38.6815	14.5008	23.0130
8214	5.2514	32.4579	8.4394	9.8175	23.0478	13.2799	36.5750	27.2417
8215	24.3959	32.8901	35.6598	7.4058	9.3352	21.2969	2.4003	13.6061
8216	26.2477	34.5958	35.1186	10.4350	17.2818	36.1355	39.7126	12.7017
8217	33.6398	13.5498	7.4223	3.1410	3.1690	33.0080	15.2920	36.9592
8218	19.9927	2.5306	12.0701	1.2414	35.3440	11.9383	15.2749	14.4020
8219	12.2879	10.6745	13.9720	19.3590	18.7895	24.7639	24.6207	10.2272
8220	6.7906	3.7028	7.0801	16.2852	21.6289	36.0069	11.9866	28.3911
8221	1.9506	26.8524	7.1961	18.7468	4.6373	35.4265	25.4860	24.8512
8222	9.0327	37.7966	31.2579	25.1178	32.5063	3.1372	33.7056	11.4386

8223	14.6788	26.2524	27.0488	11.7081	18.9076	25.1468	36.7371	29.5368
8224	22.7139	16.6544	21.5821	4.2182	25.0607	37.0618	15.9640	28.7225
8225	22.7886	9.6643	24.8524	21.3080	22.9334	28.4088	15.6365	1.9838
8226	31.1511	14.0397	10.4904	36.5844	21.5349	18.1467	4.3033	17.3992
8227	23.6097	27.9896	12.4873	24.4531	24.4634	32.9596	18.6276	34.2881
8228	4.4265	19.5487	34.1484	35.3354	6.3296	5.8802	24.2483	28.5325
8229	25.6271	32.8149	2.3636	12.9121	31.6205	11.2317	19.0774	7.7935
8230	2.8808	13.1985	13.4545	8.9524	5.8360	29.2247	35.6717	39.5994
8231	14.6338	3.7627	37.6996	29.1914	24.7712	16.4047	16.0988	25.1286
8232	11.4921	33.2705	5.2096	14.0686	12.9298	16.2550	5.2395	34.7199
8233	29.6575	17.0671	30.2574	15.0173	27.8078	36.5068	9.4469	5.9517
8234	2.6795	10.2780	2.6639	11.6889	24.4650	30.2279	25.5663	27.7996
8235	30.2866	24.2357	37.9205	13.4524	14.3447	19.4776	0.0257	23.5640
8236	9.6285	23.0850	26.1468	33.3890	6.2488	9.4973	0.2924	6.1678
8237	9.2364	38.4579	4.1299	37.2385	10.3069	14.9288	39.9459	21.4311
8238	0.4795	2.5351	16.0107	26.4158	18.7456	35.1552	29.6015	7.7487
8239	5.2990	28.7774	22.8084	32.0177	9.2925	34.0749	34.0455	29.6312
8240	8.3755	36.0804	25.2796	11.5924	2.2314	3.8170	12.0519	11.6413
8241	8.5673	7.8074	10.7907	39.2047	20.9568	8.9305	6.3565	38.3284
8242	36.2318	35.9067	29.6038	37.3864	36.7920	29.7933	21.9175	32.5041
8243	11.1109	32.8642	14.9500	10.1156	18.2258	17.5316	17.1555	8.9366
8244	3.8841	26.6378	30.9190	28.0806	18.7732	34.6138	5.9819	8.1529
8245	31.3129	26.2670	29.7094	9.0882	15.5898	3.7539	12.7463	37.5740
8246	26.7008	21.4217	37.7591	28.1040	39.3779	31.6081	31.3271	2.0355
8247	24.5069	5.9216	3.9121	14.8955	2.3919	27.4608	30.1463	32.5481
8248	13.7517	24.6149	12.7741	30.5743	31.4820	27.0998	17.9391	0.4713
8249	27.6136	14.9566	0.0244	23.1527	15.2831	12.9969	23.0848	3.4291
8250	9.3424	17.5493	19.9172	13.8649	24.2847	16.9217	37.6243	28.6536
8251	12.3212	23.9666	38.7348	25.9158	21.9911	0.3270	10.7755	39.3112
8252	14.2920	35.1188	0.9616	31.6042	21.4423	0.3276	16.8402	23.2207
8253	8.6679	4.6365	19.9972	8.5727	12.4135	23.7211	21.9085	2.0372
8254	27.0912	39.4296	16.6497	38.5687	35.6223	1.7131	4.6384	26.0558
8255	29.4406	34.2902	33.0872	7.3774	7.7298	3.8923	39.3979	13.2482
8256	0.4007	17.6642	28.7486	27.5382	8.1170	5.2147	15.6294	7.8498
8257	14.2366	33.2648	1.6789	35.1138	29.3641	25.7736	20.1770	4.3041
8258	9.5982	11.7058	30.5827	38.4193	6.2789	30.5059	32.1442	21.3834
8259	37.1043	20.4374	7.3211	17.9219	31.1591	34.8230	38.5680	0.2229
8260	30.5195	30.0495	23.8423	26.4064	1.2744	20.6628	22.7660	2.7321
8261	15.1755	15.2131	23.9813	7.1164	16.8847	39.0203	4.4589	11.0881
8262	27.4383	10.6507	33.8927	22.8081	6.1678	25.9933	29.2390	39.3744
8263	27.9549	10.5042	34.3751	35.0146	20.3070	14.3624	33.2266	15.5493
8264	23.6159	23.3050	6.1258	7.5585	7.0106	35.6297	36.9470	23.6151
8265	38.4822	17.7248	29.4195	20.8253	16.0635	28.7700	18.0070	23.4648
8266	31.2538	17.8591	34.5447	16.5162	37.4483	21.5841	33.5243	13.3730
8267	21.5371	39.1479	37.9759	15.6304	11.6548	2.6236	25.8867	25.6488
8268	3.7655	31.6544	15.3295	18.4345	5.7575	38.9627	13.9882	6.2206

8269	35.4403	8.4589	3.7389	33.6771	36.5323	0.2444	34.1983	26.4422
8270	10.2998	37.9428	7.9586	4.2201	36.9799	39.7381	27.4942	28.7523
8271	18.5403	2.3420	14.4350	23.3526	7.0677	15.5892	35.9324	10.5169
8272	13.2319	0.9579	36.9657	16.7055	1.5173	33.4365	4.3831	18.1763
8273	30.2709	8.3394	17.5629	11.4351	15.3675	32.1952	27.8743	33.3418
8274	32.1140	11.7714	15.0846	34.1153	30.4411	34.5499	8.2387	20.8071
8275	32.1247	14.6411	18.9921	12.9183	20.3633	34.2518	31.2885	33.4975
8276	35.0655	34.0026	3.3856	1.8936	19.9442	36.8929	21.9820	27.7806
8277	3.4990	37.6704	19.1158	29.4700	26.3846	34.0365	7.6130	7.8821
8278	26.7165	2.5520	30.8515	1.0423	6.2634	35.3549	4.3867	1.8568
8279	20.0969	16.3830	5.9922	0.2391	28.3077	0.5445	22.1442	33.3549
8280	13.4225	37.8980	12.1422	17.0241	22.6483	8.7688	23.1350	3.7872
8281	23.5425	8.9144	39.3957	4.5725	33.3778	7.9890	10.8225	19.0285
8282	14.0773	1.9026	15.8463	4.4294	31.7557	38.3169	32.3133	6.5448
8283	8.2687	0.2889	26.5481	24.0150	1.0056	15.3881	25.1670	12.9615
8284	10.4862	8.7987	30.8754	14.8469	17.8975	36.7570	27.4498	3.9264
8285	3.9460	26.9376	24.5524	7.9105	11.8523	29.9931	37.1808	39.6000
8286	6.0025	5.2742	7.1181	21.7751	1.6956	15.4585	8.3709	35.7572
8287	12.8408	28.1739	0.7804	16.2240	32.2785	38.2531	16.8416	14.9591
8288	23.7756	15.3954	7.9102	35.4637	27.1425	20.6858	17.4028	1.1750
8289	6.1495	14.4865	21.9744	13.0107	18.8787	5.5298	7.4198	13.1924
8290	2.5050	38.5040	12.4193	14.8054	8.8790	15.8788	29.4437	15.9334
8291	0.3113	3.5028	8.0724	17.6290	11.9842	34.8603	10.4127	34.8380
8292	30.9704	10.2264	19.9779	4.4336	21.4788	12.3080	36.2766	20.2656
8293	27.2818	16.5047	24.3665	21.2217	10.1643	28.0856	16.9542	12.7744
8294	4.8201	28.1571	32.3113	5.5689	10.1197	24.1783	7.9897	28.2763
8295	28.7701	29.9191	24.6170	25.9348	30.0842	26.8793	15.1539	17.7614
8296	6.4742	1.4975	16.2249	38.3040	10.3627	14.3654	21.7433	39.8660
8297	4.8064	31.2538	18.8449	26.5701	0.4699	18.9098	31.7179	8.7009
8298	27.2461	12.6787	0.2923	31.8321	18.3372	22.2723	27.7927	14.3714
8299	5.7931	16.5769	17.4643	16.9534	8.9617	14.1100	21.6521	37.7515
8300	2.3131	3.5161	6.4145	5.8637	34.2806	20.3022	18.5905	24.5760
8301	5.3977	13.9750	18.7679	19.3225	25.7148	37.8811	21.2473	26.5499
8302	0.5912	5.1769	32.6576	33.1943	5.5176	9.5402	10.5226	18.9194
8303	7.5647	12.1731	21.8838	12.2811	31.5999	27.9416	17.1156	17.6176
8304	2.8345	37.8487	1.6543	36.0098	22.7783	23.5065	23.1887	17.1802
8305	1.5144	30.8051	11.1918	21.4532	4.8884	20.9358	30.8838	4.2500
8306	12.3148	32.8523	2.1096	2.6700	23.2560	11.3565	12.2298	21.5793
8307	20.6294	4.5913	14.0634	6.9521	11.3586	27.2003	5.3718	34.8813
8308	15.7404	8.1324	5.6144	2.6950	37.1851	25.5942	34.7878	32.6514
8309	39.8078	23.7039	14.3934	32.8726	14.8454	11.7504	35.7100	18.5101
8310	17.5840	0.4277	8.0860	11.2648	1.2440	27.1973	36.7961	20.6039
8311	6.0576	10.8725	11.6504	32.1193	36.3908	35.4935	14.6200	18.2583
8312	31.1220	26.4361	29.9879	5.0170	6.6533	36.4273	13.6282	7.7775
8313	29.6469	8.0300	31.5995	15.2343	16.8123	35.9093	36.8351	34.4259
8314	3.5057	27.4289	39.7660	8.2895	14.0532	0.7670	39.8844	0.6976

8315	18.8170	4.2914	35.0760	7.3575	20.3758	29.2838	38.3685	13.6823
8316	11.7132	17.8798	8.2683	29.3822	1.1443	32.4492	4.0423	10.5679
8317	16.2740	5.0816	8.4116	30.3595	8.8489	8.8204	14.9248	5.8876
8318	17.2095	20.7910	37.7050	39.0667	27.4958	6.0723	24.5303	38.0191
8319	34.8505	10.2710	26.6762	27.3126	27.4317	38.2565	30.3798	12.7490
8320	28.2900	6.3972	3.5925	32.9049	20.3393	32.0423	25.1583	33.6733
8321	0.5719	16.8686	19.5610	6.4460	0.9235	8.6446	36.4966	26.3404
8322	0.9833	22.3477	6.1783	2.8961	28.4004	29.9711	1.3460	13.3854
8323	9.4798	12.0264	35.0211	19.1104	3.9044	3.3197	34.0930	1.4650
8324	23.9306	29.4232	35.3929	14.5975	8.8624	3.3505	0.3310	20.2352
8325	37.0479	13.6070	26.3707	26.1041	21.3567	14.7132	1.7519	24.3704
8326	13.3765	1.7877	1.0800	31.2551	24.7750	37.7766	8.0013	26.6146
8327	17.7213	16.3274	38.3218	24.9387	17.3336	6.9215	33.1177	3.4888
8328	3.9655	12.8873	25.8383	1.7104	21.6858	32.6988	14.7387	6.8545
8329	28.6060	14.3326	3.6002	22.1527	36.3987	10.4126	18.5285	39.7882
8330	28.8655	2.1688	11.2800	32.9803	2.9601	0.2510	2.0168	22.8818
8331	31.2988	29.9403	30.6781	20.9806	7.1477	26.2174	13.0996	39.3116
8332	37.6818	3.1895	7.8361	38.9814	31.0615	4.1563	3.7551	9.4262
8333	1.9027	11.7017	0.9972	21.2695	25.4302	25.1948	17.4039	23.9259
8334	32.0140	16.6627	5.2696	37.6698	5.6843	32.7822	37.0947	36.9032
8335	18.6715	27.9913	13.9603	30.8460	25.0427	27.3459	29.8394	4.6807
8336	11.1656	9.1638	27.5813	39.5376	3.7429	12.0738	18.8563	15.7735
8337	1.7888	31.9593	3.8243	21.6870	31.0096	28.3171	9.3518	31.6654
8338	31.5020	30.4317	19.9688	27.7800	31.6549	26.4198	17.2281	36.2368
8339	35.9650	13.6301	5.2790	18.8794	11.1448	36.7680	24.2527	7.0847
8340	31.0274	9.2413	25.0739	10.1901	22.9753	36.0101	17.0357	4.5950
8341	33.4980	28.3513	21.7544	7.0172	37.0640	12.9292	10.1963	1.2339
8342	13.7768	16.2579	24.1518	35.5506	13.6478	13.3200	20.4159	12.7496
8343	7.6477	37.7994	32.1816	28.1863	22.7501	30.4155	34.1222	34.3803
8344	15.1376	26.1475	20.8958	3.5199	29.7892	35.9307	35.2703	13.3844
8345	10.2534	4.1362	4.0419	26.4473	32.5698	27.0480	27.0559	19.0294
8346	0.8585	31.2319	27.1363	36.3755	36.2143	12.1676	26.5807	37.7622
8347	11.7312	27.8987	32.7418	29.3576	35.3559	1.3871	38.2004	1.3461
8348	9.6169	4.3112	15.0407	10.2009	31.7601	6.7103	18.3199	25.2709
8349	8.1036	8.2537	0.0029	6.5224	38.5847	38.7392	32.7949	32.5179
8350	13.9288	37.3391	0.6514	10.1142	38.5077	17.2544	20.1509	20.3568
8351	21.4860	8.1882	10.3811	26.6012	11.5945	31.6380	38.7934	38.8335
8352	4.9726	1.6211	37.7107	19.1168	35.3090	17.7452	19.5146	34.7287
8353	7.7095	21.6960	16.0341	19.5011	5.5984	28.7403	28.6457	9.5518
8354	19.5448	0.1753	5.9978	11.4896	31.4361	6.4209	3.1753	13.7690
8355	16.2804	23.0359	12.1985	26.6263	2.5058	25.3668	17.2679	1.8593
8356	29.3968	17.8456	6.6276	20.4876	0.0081	2.7677	29.0915	22.7099
8357	11.9962	33.8605	14.9658	25.0161	12.2422	20.2207	13.7575	23.3912
8358	32.2854	23.7367	26.5692	19.1019	9.4140	33.6558	28.4192	1.3221
8359	36.5466	2.1317	29.2664	13.3371	18.5726	0.0755	19.2363	33.4736
8360	30.6043	27.5037	5.3988	35.7410	10.8023	25.0627	0.7401	26.3503

8361	13.3685	5.1456	9.9371	14.4413	2.5592	2.5727	7.5420	36.9037
8362	28.7735	37.6607	39.0552	21.7578	33.2273	37.1958	17.6522	37.4890
8363	1.3437	1.5592	28.1538	7.3813	35.7423	33.3246	36.2502	0.8740
8364	24.6250	28.2181	17.1037	38.5863	3.4850	32.8541	37.5840	3.5539
8365	4.4874	30.6333	5.7524	34.3131	30.9753	30.5914	22.5323	7.4660
8366	2.0241	28.0459	18.3416	8.4574	32.1732	27.8986	24.4519	36.1353
8367	2.2084	15.7906	24.0102	20.6779	37.8224	31.9821	35.6516	34.3982
8368	1.0488	35.7410	30.1655	2.0454	6.7839	35.9702	20.2393	22.2471
8369	38.2174	18.7899	7.4496	16.5639	36.3838	20.7910	8.6031	9.7190
8370	21.0086	24.6236	32.9190	13.4216	35.6768	9.3232	28.0933	35.2370
8371	30.2502	5.2295	9.1538	13.8923	28.2944	0.1875	32.2649	25.8159
8372	23.5927	33.4100	36.2370	1.9681	18.0522	15.2113	26.8450	12.1592
8373	29.5562	17.8709	38.1410	9.2447	0.0263	22.0005	21.1942	18.7525
8374	12.7767	36.8658	20.0607	8.4583	1.9070	17.5989	36.5234	13.3777
8375	33.9252	30.8161	13.2848	8.1763	8.5352	24.2460	12.1187	14.5108
8376	19.7459	8.6038	7.7808	28.0785	19.0376	3.7352	12.2311	30.3176
8377	4.5934	5.6518	3.1646	7.2683	1.9195	17.4381	20.7842	1.4101
8378	32.4909	4.5192	6.5455	13.1532	27.8696	38.8100	15.1941	35.9721
8379	26.6732	8.9885	25.3145	10.1740	32.1333	9.5922	16.7351	7.9563
8380	15.2713	8.8762	30.3352	20.7516	18.7661	26.4695	28.2582	39.0655
8381	28.7524	8.8781	19.4429	22.7860	15.9181	6.3418	18.7883	3.0755
8382	16.7451	17.8821	10.6965	9.6780	23.8490	12.6107	37.5514	8.7219
8383	2.2373	12.3883	24.8607	6.2188	24.3907	2.5778	30.3454	5.6220
8384	23.4000	32.7283	33.5992	34.2328	5.6513	9.9244	19.6719	15.5777
8385	11.9531	23.4912	7.9289	10.9240	30.7173	17.4775	2.3515	28.4828
8386	16.3825	34.8479	31.4793	21.4845	31.6777	28.0417	38.5538	1.3059
8387	23.4436	29.4190	21.5356	25.6254	1.5232	36.3522	19.6826	25.9905
8388	12.6249	28.0840	20.3614	34.6810	30.0942	2.6497	19.9893	19.6438
8389	5.8943	34.1733	22.2786	38.8908	15.4049	14.2950	23.1975	38.7404
8390	39.6612	5.3569	12.7612	22.8479	16.7219	10.7810	12.7825	37.3471
8391	4.2997	5.1401	16.4519	30.6116	14.6133	36.0022	12.6553	11.8718
8392	13.3703	18.2972	36.1780	28.3598	12.7348	5.0102	9.0499	7.9411
8393	1.5184	11.5801	34.6535	38.6403	17.9863	29.7104	5.8674	8.7687
8394	36.8963	26.8349	36.9014	25.1400	0.3402	38.7658	21.4023	39.2182
8395	14.9257	9.2665	22.5495	21.8763	15.8322	2.4717	29.1218	0.9659
8396	39.5664	8.3609	7.4530	22.4722	20.5677	24.9475	22.3327	15.7658
8397	21.7382	24.3948	27.0637	8.0378	33.0335	22.8304	4.3641	9.8747
8398	22.5327	24.4927	14.2920	26.9823	9.9346	2.0546	35.5411	2.3892
8399	7.5159	39.8008	7.2072	7.3471	0.3198	33.7103	7.6315	11.1489
8400	8.6198	39.5910	16.5385	25.6928	11.1597	29.4458	32.0574	3.5412
8401	19.4123	28.9137	32.0385	17.5065	1.9652	32.2384	22.6578	1.2816
8402	8.7800	33.1357	10.0513	26.8641	2.8530	22.4638	9.7687	37.1176
8403	37.4749	14.6221	26.5995	33.5992	3.6071	36.1572	13.3866	5.5179
8404	11.1050	5.8425	7.0786	31.6325	24.7644	22.2494	27.9115	29.6978
8405	38.1241	31.2996	17.2834	30.6379	27.1111	25.4472	32.9584	26.8723
8406	32.5973	25.8578	3.8328	21.9396	12.2241	18.3541	11.0288	37.0360

8407	13.9840	39.6594	12.0381	0.4392	36.8993	8.4964	18.6773	8.3729
8408	0.8642	31.0014	28.7881	32.3729	0.4425	39.8919	2.6604	21.1493
8409	13.6405	38.8177	15.3416	32.5697	7.0415	10.3377	13.3527	8.1151
8410	15.3759	5.8344	17.5358	26.0410	32.0395	8.9897	30.7256	2.7286
8411	3.7316	28.7223	0.3637	2.1538	34.0891	37.3368	15.3798	32.7286
8412	38.7701	11.4990	6.9012	17.7397	32.5045	13.1453	25.8513	18.0450
8413	7.6018	24.1227	3.3280	11.9386	33.1318	35.1448	11.8110	32.5022
8414	6.5848	18.7254	26.2051	17.2987	39.2933	13.0834	17.0574	17.6333
8415	20.2813	22.7454	17.3963	39.8209	12.4347	18.0148	24.2059	13.3043
8416	20.3295	36.6139	25.6994	38.2972	9.1859	20.1082	37.0212	38.1625
8417	2.0591	16.6650	20.7166	24.9749	19.6576	21.6113	39.4757	38.8483
8418	14.6805	3.5513	5.2649	14.1967	28.2213	20.4405	30.8216	16.8384
8419	23.3919	30.6986	17.1716	37.4862	16.9905	2.6453	22.0077	39.7469
8420	39.7900	14.8844	39.7380	5.5370	5.6929	24.0155	15.4635	15.2580
8421	17.4053	19.5401	15.2263	10.9270	21.2875	24.6200	2.8291	34.6421
8422	27.8399	11.7146	33.5576	10.7501	36.7060	15.3687	31.8370	6.2713
8423	0.6590	24.4872	1.9476	26.7868	4.3374	21.7945	0.5342	13.9883
8424	6.1016	33.9663	7.1920	2.9090	9.0956	16.5236	19.2713	16.5485
8425	35.9816	25.5438	28.1284	6.6846	36.4759	0.3324	2.4152	28.7233
8426	1.8551	19.9150	11.9142	20.9127	1.0801	25.4420	1.9551	22.5359
8427	1.1731	36.8656	16.9298	8.2950	27.9386	29.3919	15.5168	14.0032
8428	22.9778	13.8873	21.3510	26.6148	6.8460	31.6433	21.2557	11.6436
8429	13.6162	0.8832	30.8654	38.1316	23.3681	18.4769	32.5930	19.3822
8430	8.7720	1.9326	9.5206	16.5353	34.5649	10.8628	39.3194	34.4411
8431	36.1596	31.8461	25.0243	1.8679	33.2878	9.1589	30.6116	35.6089
8432	27.3235	38.8357	28.7201	18.5199	15.0487	35.3582	18.1369	21.3439
8433	14.8659	29.6662	8.7548	32.3477	24.8451	26.0712	9.5177	38.5004
8434	16.0447	11.1514	19.5712	38.5783	11.8382	1.0184	33.5862	36.2589
8435	8.1988	28.5631	35.4365	5.1164	19.2448	15.3412	26.4506	14.9593
8436	9.5080	3.9245	39.9828	16.3173	20.6964	35.2064	7.3789	6.4091
8437	35.3744	36.5593	31.4624	24.4596	6.3772	15.4052	19.6021	38.2887
8438	36.8497	31.3868	6.5132	6.8467	19.4830	34.2205	25.1501	23.8129
8439	31.2660	6.0726	25.0769	27.2231	33.6762	33.8703	1.8034	14.2177
8440	9.6766	16.8435	13.0556	19.9223	26.5470	16.7469	22.6432	4.5798
8441	23.3439	38.1937	23.9814	16.9596	15.8155	19.6383	5.1168	38.2736
8442	8.0742	7.5795	29.9442	39.4840	5.4112	36.8803	1.4512	0.2708
8443	5.0827	10.9367	34.5013	17.6627	15.9844	36.3167	17.4481	25.9348
8444	27.5068	2.1509	2.4611	31.4883	8.4457	16.7843	23.7969	7.6492
8445	36.4296	5.9372	24.2922	21.8319	12.6900	27.2948	34.6382	29.2072
8446	30.0376	3.2211	16.6028	34.3898	33.3238	6.6046	20.8198	16.5565
8447	32.9164	28.4605	16.5628	36.1981	21.6927	0.0041	39.9524	11.1401
8448	9.0048	2.8008	14.9211	31.4310	6.3874	23.5362	2.3126	6.3851
8449	15.4699	20.9659	9.7042	35.2472	5.3258	31.9842	25.9691	14.8226
8450	16.7475	31.1267	20.3049	3.9300	30.2011	33.7698	29.9968	3.1974
8451	37.4314	2.8598	39.1223	9.0302	25.4148	4.0299	39.0525	5.8672
8452	4.2222	26.4538	18.3154	25.9932	27.9655	18.4074	3.8447	32.8855

8453	21.2516	17.3947	39.0522	18.8685	38.9178	28.0210	11.8006	18.6598
8454	24.8477	15.7673	1.7430	28.0662	16.5390	16.3004	32.7183	36.5895
8455	11.2137	24.1755	22.0860	5.9217	9.3932	7.3592	37.0215	7.0116
8456	7.9105	39.8527	35.0151	8.0357	6.4299	27.2275	29.6482	31.2857
8457	16.3355	5.1120	11.9791	15.0735	5.4748	19.0527	22.7813	1.9751
8458	14.5042	7.9198	2.5178	13.3036	5.1055	34.8850	1.5782	10.3272
8459	6.4761	35.5106	23.7596	13.9026	24.1627	11.5050	32.0812	28.8753
8460	35.7410	2.4067	7.6290	24.4908	38.6082	24.0276	18.1139	11.7326
8461	17.6191	3.3308	11.8299	20.2450	7.0843	16.8369	37.3153	16.3584
8462	26.8124	23.4333	9.4158	10.2149	2.2854	31.2961	32.3009	11.4705
8463	8.4171	1.5168	25.2321	39.7185	27.9634	22.1116	37.8560	5.6581
8464	18.8458	4.1413	1.5108	21.3478	4.5856	0.7399	31.5186	37.9856
8465	1.9966	15.9664	17.9635	12.5153	10.5744	13.2700	34.5293	16.8505
8466	19.8373	21.5084	8.5499	16.1695	0.5962	30.1100	39.0097	36.0170
8467	10.7888	8.6353	15.5115	31.5603	3.3662	27.1210	16.7699	31.1584
8468	27.3670	15.4267	1.3464	0.8531	15.3680	18.1389	0.6888	36.8788
8469	29.9198	36.5512	37.7814	4.7949	35.3987	1.9929	8.1129	22.4122
8470	15.6657	14.4853	18.3745	11.5332	37.3300	31.9897	36.2928	31.1519
8471	22.3531	21.6682	31.6206	36.0842	7.4347	31.8912	8.2994	2.2298
8472	29.2898	7.3131	4.2768	20.2801	20.1035	29.5697	24.4992	29.6984
8473	24.4649	13.6601	30.4190	36.4314	5.3554	31.2322	34.3530	11.6626
8474	23.2492	34.0510	38.9097	10.7919	13.5754	3.2580	8.7767	0.9957
8475	2.8242	38.5732	3.6850	33.3623	23.3304	32.9558	2.1097	0.8360
8476	14.9699	21.4671	15.0275	24.8545	8.4985	21.7684	22.6858	11.1902
8477	1.3436	35.5047	20.6319	23.4399	14.5364	14.5899	30.8232	21.2851
8478	16.2457	29.9884	0.1583	24.8707	32.3790	25.2303	30.4656	30.0529
8479	26.3183	5.5896	37.8954	14.3766	1.4902	14.5568	12.6318	6.1473
8480	23.0096	0.1839	18.0610	18.9981	38.9902	3.2353	13.0842	38.2649
8481	39.0382	31.6502	39.3257	12.5583	32.9382	1.9939	27.2830	7.2091
8482	17.6656	20.5597	9.3250	2.6585	12.0594	26.2665	10.1977	15.9497
8483	31.1135	9.2414	14.6957	9.7208	23.0103	37.4519	32.4057	21.6658
8484	22.7783	28.1451	10.5135	24.2337	19.6337	37.1140	26.4848	6.2366
8485	0.8440	26.0299	0.5299	17.2803	16.1143	16.2417	24.5440	37.0639
8486	20.4909	34.7264	2.8151	17.4053	0.0924	31.2666	22.2471	7.7642
8487	16.3021	13.9112	15.1307	3.7997	12.9896	21.1809	16.3948	19.9963
8488	8.7935	13.2259	22.3951	2.3766	14.4388	38.2510	20.3935	7.2741
8489	9.2184	11.7508	9.4420	34.9262	28.3287	29.1700	4.0498	16.0426
8490	17.5420	0.5334	23.6348	6.6525	37.6590	36.8595	37.5151	27.1344
8491	36.7958	11.1873	35.7404	5.6627	9.9958	31.3209	32.2079	2.5146
8492	12.7615	34.3130	26.0389	26.2049	23.3107	22.7739	2.0275	39.8355
8493	11.3734	6.0441	20.1556	3.4294	30.4742	5.0683	1.7577	11.9171
8494	14.5512	22.0998	25.7757	30.9972	38.6241	21.4354	16.5655	27.0524
8495	19.9727	2.4398	30.3704	20.9407	29.6500	20.0678	21.0161	34.2296
8496	32.5410	33.8487	13.1570	9.6889	0.1944	7.8341	25.2947	7.5193
8497	9.3927	38.5117	21.9272	5.1307	11.0501	8.7092	11.0221	34.3549
8498	0.8906	12.0440	30.4663	4.8612	4.7340	24.7362	24.2083	27.9781

8499	12.0449	9.3285	6.0592	1.8193	5.3026	30.4788	32.9835	0.3810
8500	11.9225	37.4039	36.6210	31.6128	15.2464	4.5152	32.3993	39.7200
8501	33.0371	33.6859	29.4521	3.1283	28.8398	27.6876	25.1476	10.8220
8502	5.4558	31.1916	36.3453	18.4658	22.9808	2.3927	15.9999	32.9042
8503	32.5832	20.3040	24.3552	34.1329	36.5745	29.0005	17.8400	27.9698
8504	34.4435	23.2977	25.0237	26.1372	26.7293	10.0467	11.0174	32.3581
8505	22.1177	23.4585	14.3075	32.0687	17.4377	1.2785	35.2991	11.1466
8506	0.7412	16.8211	35.4221	39.4037	39.5292	24.7132	10.2898	3.3972
8507	20.3129	33.3235	10.5564	28.3098	9.3312	20.7588	13.0214	0.8964
8508	28.5458	6.2097	4.1847	14.8987	9.0673	0.4824	4.9601	8.0572
8509	27.9099	4.3740	12.9922	28.1708	16.7503	33.9146	0.7354	13.0674
8510	35.7382	20.9293	21.0116	32.1928	7.8432	14.8872	6.7408	17.3046
8511	28.0474	22.1128	7.2397	3.5383	5.5258	10.6491	8.3875	23.8238
8512	37.4916	6.5487	21.6399	2.7685	10.2933	31.8235	33.2420	2.6388
8513	0.4357	5.9750	11.7240	30.9267	11.4703	19.6557	21.0322	33.4459
8514	32.7896	2.4102	38.5995	34.6009	16.4806	30.9016	6.6921	15.8645
8515	1.3707	17.3007	14.0428	14.1500	8.1999	18.9106	27.6136	31.1129
8516	20.6464	19.5942	17.3319	27.3858	13.5810	1.9142	8.4306	1.9225
8517	26.1332	0.2103	33.8024	24.1771	20.1370	21.7194	24.5706	2.4140
8518	30.1651	10.4575	13.6133	7.8195	36.8302	2.6344	23.4267	4.8979
8519	11.1999	13.7032	26.0648	25.9459	36.9859	14.3180	1.9177	7.8894
8520	24.1220	0.5382	21.0959	13.1647	17.1646	6.9364	17.1473	12.7767
8521	35.5512	28.0330	2.6988	16.9566	16.3866	5.4240	34.1120	29.0280
8522	25.7294	33.9695	12.1026	32.3772	39.8375	18.8412	16.8817	22.0837
8523	31.4425	16.8230	19.7600	28.3643	38.5088	26.6836	9.4035	17.4130
8524	19.6456	32.4396	26.7367	12.0157	32.7895	9.4579	26.0705	12.5357
8525	16.1384	39.5966	39.8525	3.9722	26.3455	7.7711	2.8261	0.9791
8526	35.0976	17.3276	25.4164	14.0351	0.5962	27.5611	5.9298	36.7639
8527	28.3273	38.4869	30.8438	34.5662	15.4086	23.3300	0.5592	7.8091
8528	33.0597	17.4097	10.3419	35.2141	15.8178	31.0851	17.3511	26.5467
8529	0.4268	8.2977	7.9554	39.2192	3.4300	15.4290	34.0699	5.9269
8530	3.8223	8.0971	9.5452	17.3733	39.8654	0.1879	23.4422	8.9260
8531	21.7314	30.2387	20.7678	7.5918	16.0802	4.3147	23.8312	8.1189
8532	34.2487	37.7725	24.7475	14.8495	17.1356	3.7065	26.8961	31.1013
8533	10.5742	34.4192	3.9953	15.4650	36.2417	30.4485	35.0755	14.2579
8534	31.0219	16.2682	16.2496	8.0204	38.7449	31.7336	2.6124	26.1960
8535	25.5168	38.0555	38.8475	33.3000	29.1449	19.4427	8.3426	1.8747
8536	29.0024	18.7698	26.0219	14.1138	20.5792	13.7430	18.3299	1.7639
8537	35.7859	28.2911	31.3165	17.2916	20.2310	29.6493	7.9192	1.5270
8538	30.7195	19.6263	3.1993	13.5120	34.1349	4.5844	26.4196	23.4909
8539	8.2739	14.6262	19.2224	5.9719	16.4546	14.5706	15.6011	26.6233

Total numbers of image points of set F = 8539

Number The Coordinates of the Minimal Points

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6350	35.3782	9.5626	34.6912	29.4844	7.9640	4.1235	4.7849	4.7678
5787	35.3997	2.1828	33.7802	19.2029	4.1930	1.5399	17.9376	24.2469
1230	35.4565	37.8855	6.5008	10.7519	33.7774	9.0277	5.7009	0.7555
908	35.4680	14.4676	12.1031	0.4737	10.3833	20.6481	36.0228	5.3111
789	35.4706	34.8609	1.1775	3.4861	18.7967	36.4804	38.1606	0.9330
92	35.5153	2.4069	0.7981	14.9999	38.8683	20.4887	37.5448	18.3875
5935	35.5360	29.6375	7.9929	34.2694	0.3718	7.1484	2.1691	6.0175
4824	35.5489	1.8677	10.9015	21.6169	10.1600	3.1675	5.7893	37.3126
5784	35.5579	13.7659	38.9147	3.1845	21.7757	13.7873	32.3406	1.9350
5678	35.6211	7.0215	2.8111	9.1864	7.9772	4.8173	11.5176	12.1029
7295	35.6508	16.0672	15.6926	14.3989	4.3023	7.2268	19.1509	17.8138
6429	35.6563	6.8143	22.6295	19.2509	18.7828	6.8094	28.3337	2.3124
2700	35.7182	18.8643	4.2472	30.0039	0.1443	16.3217	13.1430	23.6589
6546	35.7230	29.7079	15.5919	8.6536	7.8395	16.9393	0.2592	22.9934
1186	35.7805	23.7458	5.8650	28.1278	13.9896	6.9598	34.3254	0.0610
1182	35.7824	15.8635	10.2058	31.1965	7.3196	0.6657	31.4185	25.6781
2979	35.8293	0.2615	5.0222	5.4015	38.3945	39.2743	10.5541	20.3189
4013	35.8379	22.7486	0.6299	25.7866	18.9534	11.3368	1.4112	5.8219
8191	35.8889	3.6300	1.4256	17.1415	25.3827	31.6455	18.2468	9.1359
6844	35.9324	0.2382	9.4954	29.7235	11.6214	28.7472	0.3150	5.0027
6996	35.9330	10.9369	0.3390	21.3233	18.4664	8.9644	34.5798	38.7131
1538	35.9370	38.2219	13.6300	35.6026	15.3105	24.1643	0.1778	18.6765
6187	35.9402	5.4041	27.1694	22.0419	17.5997	35.2282	0.6781	1.3003
7083	35.9487	7.5259	12.1947	12.6031	9.7001	2.7385	22.6193	4.9209
2524	35.9771	17.1687	4.1444	5.5162	0.7321	11.3736	19.4677	26.5818
2451	35.9814	25.0275	15.2254	9.5801	0.7867	7.2395	8.7821	13.3962
8425	35.9816	25.5438	28.1284	6.6846	36.4759	0.3324	2.4152	28.7233
4721	36.0136	20.4196	0.7089	9.0374	27.1345	6.6344	0.5171	32.7150
1937	36.0339	20.1664	19.4348	5.4572	8.6254	1.0765	18.0005	23.2108
7214	36.0568	3.2795	6.9285	23.8662	13.2465	1.0380	29.2908	20.9879
7953	36.0990	6.1641	31.1735	32.0277	13.7453	0.0470	1.9000	20.8156
3821	36.1028	4.7450	37.9333	14.4274	20.5393	27.2368	1.3212	19.3718
3631	36.1149	37.2728	19.0884	13.1974	0.9345	4.9188	1.5123	19.8892
3491	36.1203	0.0533	21.0865	8.6108	36.9714	5.4611	4.1190	18.4293
6240	36.1651	37.1280	0.1845	1.9614	8.2444	4.9482	21.5287	13.7450
4801	36.1805	8.2587	1.8073	8.4093	38.8240	11.5234	10.1390	18.7220
3107	36.1876	7.7958	4.4592	2.2477	34.8506	23.8531	33.1232	4.0461
5216	36.1900	5.9832	38.2718	3.0388	17.4171	27.8227	35.2192	2.0596
2901	36.1969	3.4366	18.4322	2.2390	22.9128	13.4443	8.3445	21.9675
4952	36.3002	3.7037	29.0492	27.4624	4.8345	3.1598	3.1999	7.2476
6748	36.3420	38.9591	33.6623	20.0468	23.4481	1.8367	10.1958	1.4562
1152	36.3758	16.8145	7.1223	30.2959	3.1230	18.2710	4.1750	18.9690

198	36.3765	4.5284	7.6715	4.6134	12.7141	0.3704	26.9427	19.3968
324	36.3786	15.0895	11.5510	38.4902	0.6408	38.9435	15.6705	1.0976
1177	36.3852	15.9223	18.5655	5.6032	0.2932	36.6333	23.3787	24.8956
3204	36.4186	25.8031	9.7671	2.1970	1.5145	27.0593	27.1324	4.2398
5953	36.5121	38.2132	13.9234	0.2910	20.6223	37.8796	10.7373	3.1479
2175	36.5408	20.5189	0.2978	8.6391	6.6112	20.1382	20.4524	35.7802
8359	36.5466	2.1317	29.2664	13.3371	18.5726	0.0755	19.2363	33.4736
3120	36.5593	17.9053	0.2471	37.6752	37.0372	22.0002	4.6218	21.7541
5630	36.5622	39.0986	10.0454	15.4567	9.7821	1.2236	9.2761	1.5738
7022	36.5797	26.3997	33.9357	0.1824	26.4714	4.0098	0.4975	12.5035
7998	36.7137	25.8418	31.7520	6.8793	4.2816	5.7576	20.2421	5.6997
74	36.7569	6.7405	4.4185	19.1343	26.4038	7.8698	23.3416	5.9397
2426	36.7712	5.9091	0.2716	33.0164	38.2343	29.8735	16.5860	1.4256
7446	36.8010	7.3233	0.4321	21.8699	3.3400	10.5975	34.9993	7.3720
3372	36.8013	19.7595	21.7097	35.5613	22.3499	17.0803	1.1276	0.4254
1268	36.8017	5.3144	3.1154	9.4556	24.7641	34.4651	13.8005	6.9524
972	36.8535	2.2833	6.4149	37.8266	12.3817	32.9798	2.9195	20.5580
7316	36.9421	15.5245	36.1502	0.0500	23.3414	30.3234	16.7438	20.3257
706	36.9583	38.8897	2.4878	7.2581	22.2092	5.2589	14.9818	0.2692
8200	36.9903	9.3413	27.9963	26.6383	0.2582	21.1627	5.5526	30.0195
261	36.9909	26.7677	16.6689	7.1717	2.0919	13.3271	12.7364	3.9265
8058	36.9970	12.2373	20.0326	13.0109	2.8940	4.4529	17.1418	21.6345
1330	37.0093	34.5509	34.6371	19.3831	17.6045	2.4367	0.9826	4.8701
8157	37.0563	30.7096	28.8262	19.8604	9.0908	28.8313	0.0726	20.8690
4500	37.0734	36.9936	3.7307	2.1487	3.2109	1.2112	13.5697	37.9679
187	37.0845	14.1866	2.3244	35.8027	1.6592	29.7955	26.9538	7.6552
4887	37.0971	0.7087	32.7396	9.3347	15.2152	7.2120	16.4030	11.9361
8259	37.1043	20.4374	7.3211	17.9219	31.1591	34.8230	38.5680	0.2229
1890	37.1552	19.2421	16.3148	32.3973	0.2944	13.1485	30.2418	3.6800
8118	37.1574	31.6229	35.4921	3.6298	1.4487	1.6349	30.9182	35.7147
1139	37.1774	2.1593	27.3104	17.7883	6.1652	4.2134	17.1577	8.4223
5520	37.1803	5.0356	14.0742	30.5992	0.6937	32.9361	3.6091	3.6007
4437	37.2048	0.0019	32.6769	7.2773	19.4483	19.5462	12.2421	30.2556
5447	37.2048	14.1371	3.1538	0.5475	2.4820	21.6177	9.6687	28.0173
4483	37.2121	2.4276	0.9835	23.0810	6.1546	0.7185	35.2069	13.1334
1747	37.2240	12.1669	0.8415	16.5704	27.0485	8.8800	3.5457	2.9552
4585	37.2321	30.8386	9.6244	16.0970	16.8228	30.8329	4.2812	0.5762
7005	37.2491	2.4572	32.2828	22.4288	21.6784	6.6580	1.6087	34.6542
3272	37.2681	20.1361	20.3731	13.0426	17.7781	3.8573	3.1584	7.1065
2250	37.2706	23.8893	39.4804	19.6707	0.3268	2.6679	25.7183	13.2574
4622	37.2775	17.5339	31.3717	7.6763	3.8121	6.8305	16.7434	0.5305
7096	37.2877	22.9899	10.0777	3.1680	0.0678	30.4148	12.7306	14.5245
1767	37.2878	23.9867	27.4857	11.3512	7.5371	35.6034	0.0630	31.4845
7813	37.3201	2.4910	23.0372	18.1839	2.8107	15.5590	0.3276	32.4544
3469	37.3863	10.6636	6.0539	4.9786	0.5736	0.2862	16.8955	9.4526
1945	37.4104	12.3886	0.0917	9.9353	38.2373	23.1228	13.5241	32.6058

2836	37.4233	15.5532	11.9357	13.4793	22.4224	36.5994	0.0940	8.0189
6704	37.4729	19.3032	0.3273	34.0763	35.3639	21.5905	0.4663	0.5998
6851	37.4785	9.0700	3.9601	5.2889	27.3499	30.8581	13.0196	8.7334
8512	37.4916	6.5487	21.6399	2.7685	10.2933	31.8235	33.2420	2.6388
3746	37.5043	18.7870	34.2005	6.0573	0.3257	15.7324	14.7876	21.0942
7227	37.5159	10.3821	0.1542	5.8129	35.0640	29.5630	30.4199	30.1914
4029	37.5209	14.7256	10.7978	0.9020	7.1484	3.1708	3.2482	27.0702
6022	37.5946	6.3575	3.9783	4.9494	1.4049	28.8638	2.6128	27.8204
7244	37.5960	12.8619	2.2617	11.0717	5.2758	9.1843	0.8244	10.6156
3290	37.6202	19.2938	10.4748	32.8251	5.0565	31.6576	22.6620	0.6239
5544	37.6576	9.5534	3.2186	15.7903	8.3965	13.5801	7.2911	5.1277
8332	37.6818	3.1895	7.8361	38.9814	31.0615	4.1563	3.7551	9.4262
5125	37.6944	5.2320	20.0154	4.8952	3.8939	1.8182	22.3613	6.8651
5412	37.7011	20.9355	20.6303	0.1910	1.2666	12.3225	20.1280	39.9335
7781	37.7294	9.4512	11.2362	22.5721	9.5935	23.8274	30.8304	0.1163
1099	37.7320	11.0407	0.6735	25.1701	0.6228	10.3789	2.4258	2.4771
3351	37.7411	18.3460	39.0163	0.1326	18.6053	10.4621	19.5000	10.2349
1886	37.7438	36.6486	15.4519	0.7872	27.5472	10.3497	10.8047	6.9220
1136	37.7444	5.8711	12.7064	13.5322	3.0182	21.1333	14.5457	3.5675
5994	37.7540	17.3606	34.8723	2.8177	7.2813	28.6471	17.4286	1.1379
409	37.7733	9.9342	0.2285	11.6890	0.9795	37.8030	0.3373	33.9461
6306	37.7813	33.4603	12.3016	3.9939	37.2289	32.9539	25.2408	0.5112
5298	37.8316	4.9550	4.1349	16.6058	5.7496	18.4013	4.3573	22.4638
1229	37.8379	10.3653	0.2764	23.0579	6.0355	25.4359	23.1930	36.2869
3958	37.8499	3.1690	2.8608	1.3412	1.0188	27.7854	19.0069	38.0837
3897	37.9186	0.9516	19.1966	3.8039	3.7184	23.6574	27.5555	11.1128
6848	37.9572	17.2953	32.5715	6.4348	30.3892	8.8985	0.1887	31.1260
4440	37.9696	0.2801	4.4364	15.7556	2.4323	36.9915	4.9691	20.9699
580	37.9756	2.4701	25.3373	12.9406	14.5338	0.4204	9.2175	13.7257
1519	38.0218	12.0479	10.4445	1.7486	2.1453	2.2490	19.8072	15.2231
11	38.0303	6.6610	0.2721	27.3453	29.3867	5.3785	9.0765	2.8648
3268	38.0439	30.0892	37.3701	33.0781	32.4937	6.8818	4.0144	0.2496
905	38.0566	29.2082	5.7662	0.7562	18.0430	24.5296	6.1001	20.0874
5303	38.0739	7.3573	14.8327	33.6692	4.7347	29.6714	0.7484	3.3140
3886	38.1306	27.4130	19.8952	3.2327	35.1346	2.5762	0.7960	33.5916
2191	38.1871	0.3043	9.2895	2.6739	39.6842	20.5927	4.7620	37.2517
6100	38.2038	6.9936	2.9603	24.6120	0.4727	34.9930	19.9728	15.9201
7077	38.2219	27.6870	1.6441	15.4822	0.0085	39.3740	21.5702	38.3706
7669	38.2364	32.6411	12.6669	8.3871	1.4730	35.9716	0.8768	3.4101
4925	38.2681	24.0571	0.9276	19.2930	2.5625	8.2493	4.7274	19.9886
6399	38.2716	22.1931	38.5338	0.6308	6.3889	9.2201	19.0934	21.6865
5922	38.2784	1.1706	29.0963	32.1789	4.3048	0.4472	37.9916	34.5055
2596	38.2844	3.0222	16.0722	3.1531	19.0078	32.6165	1.5235	19.5921
2797	38.3079	30.2704	3.3341	0.5103	2.6427	33.8034	17.2660	15.2192
4159	38.3244	17.3441	32.3416	12.9286	0.5209	8.4301	27.5611	6.1728
7854	38.3820	13.8629	4.1743	23.8205	14.1100	1.5056	31.7462	4.6216

7000	38.3912	17.3904	9.4743	5.2943	17.2287	1.4094	10.6371	38.3495
6647	38.4330	2.0764	32.6370	23.3810	7.1538	26.2324	14.0698	1.6129
7856	38.4798	22.1117	23.9029	3.7317	16.9722	0.2610	33.5323	3.4640
7825	38.5136	1.8763	26.0159	4.9286	20.0093	6.6895	10.9285	35.0444
2352	38.5254	33.2649	7.0168	0.3934	29.3956	21.3458	13.3272	8.3272
8086	38.5648	34.7429	23.8684	3.0387	11.9839	0.6756	30.0198	22.0274
48	38.5703	27.1200	2.0822	8.5907	5.9009	27.5932	0.7158	23.4934
1672	38.6642	6.8807	6.4739	9.3513	30.7428	15.8369	2.4426	5.5768
94	38.6721	22.0187	31.9639	25.8818	4.2213	0.0419	35.1742	33.7173
3708	38.6773	21.9896	9.4794	8.4601	34.0796	3.3045	1.4343	6.7898
7839	38.6976	26.4391	20.9597	28.7034	11.0275	2.9440	4.0764	6.7028
5942	38.6979	38.3289	22.6394	3.0528	22.5651	0.5067	37.7434	14.4273
1545	38.7067	38.9537	8.9374	36.0129	13.2126	2.0754	2.3447	15.5728
1842	38.7906	24.9601	0.4182	0.9285	3.1077	39.8053	1.6414	18.9963
1588	38.8116	26.8786	38.2202	0.4947	7.6253	24.2121	10.9900	0.7750
5821	38.8132	25.5074	5.8944	23.5745	12.1028	38.1383	37.7699	0.3367
7630	38.8250	4.5823	20.0145	30.4739	2.4378	4.6405	27.6017	13.1748
7286	38.8357	10.6126	20.6318	35.8843	7.9379	19.4529	0.2799	5.0970
371	38.8444	16.0870	0.9745	36.3836	8.9673	9.2833	13.8608	2.4362
4968	38.9004	20.3298	28.4606	1.8457	29.7112	37.5596	16.8125	0.5547
1534	38.9042	30.2156	16.9998	14.7752	0.6197	0.5246	3.5773	13.6068
7156	38.9074	19.4036	1.2578	7.5395	26.1858	39.3306	3.1025	17.6897
7739	38.9262	0.6225	31.9627	8.8084	2.4680	26.5365	14.4878	38.9894
7185	38.9476	8.0013	4.5072	31.1572	0.5175	2.9347	30.1162	23.2291
1058	38.9753	13.2757	0.7398	0.0072	39.9549	29.5388	35.9693	9.2841
6117	39.0183	5.2851	17.0267	20.1019	18.5021	3.0975	4.1657	3.3035
6685	39.0268	31.7799	8.5971	38.8626	5.3930	1.5535	8.3820	8.0393
379	39.0283	0.1850	9.6838	38.6569	11.9914	23.5652	25.8075	3.0925
7926	39.0541	23.7812	22.7997	22.0656	17.8233	5.0124	14.6219	0.6771
5816	39.1245	8.9305	10.6347	19.6530	4.5775	15.2563	28.9104	19.2821
4065	39.1424	0.0923	29.0336	17.8018	28.9452	31.8591	4.5604	12.6561
5328	39.1780	1.9603	7.5434	9.4188	39.7912	5.6322	30.4968	6.8598
7188	39.2361	36.8189	2.8706	13.2529	26.9141	29.9020	4.7822	0.4828
5655	39.2384	4.2938	26.4437	29.1391	11.0420	8.0157	28.7284	3.0674
881	39.2997	2.8247	11.2971	23.8041	1.1674	15.8322	13.3846	33.7625
4776	39.3119	2.2554	17.6739	18.7393	26.0930	10.0455	21.7163	1.9460
822	39.3214	10.8719	7.0829	6.8682	22.1320	0.1239	37.9305	28.0382
6146	39.3503	2.7211	9.5450	14.0911	3.7542	23.9680	38.7112	12.2224
722	39.3567	17.7362	27.2331	15.0171	15.3683	0.0500	10.3979	39.3319
2147	39.3569	36.9819	4.1668	14.4089	1.6879	5.9956	37.8198	14.9618
5471	39.3984	8.7936	26.7198	27.7883	18.2588	7.0010	0.9406	25.3196
3528	39.4446	26.3862	2.9744	2.4019	2.5608	10.0214	9.8340	29.9519
1681	39.5336	22.9014	34.0264	7.1040	26.4431	3.2804	26.0091	1.0957
3425	39.5448	13.1763	5.5293	4.3399	15.1871	3.2104	7.8653	33.4548
6970	39.5588	29.3590	4.3154	21.9172	2.7733	34.3839	1.0447	6.0581
582	39.5946	26.2251	0.1307	16.3200	26.9572	2.6130	35.2748	38.6377

6917	39.6156	2.5593	32.6375	22.4091	2.0167	0.8136	19.1497	28.2782
7197	39.6723	0.0735	21.7199	25.0553	2.3469	11.3828	17.3844	31.9336
7039	39.6732	3.2643	2.5030	37.3641	19.2030	8.9994	3.9272	3.2937
2117	39.7215	7.6271	15.2427	1.6869	2.0077	36.8264	37.2703	36.5094
4759	39.7494	17.4960	15.0680	19.9484	0.0689	33.1813	35.0483	3.3177
7277	39.7625	2.9101	4.7580	4.1354	32.7477	17.0495	26.5393	1.1849
3699	39.7904	37.8114	17.9657	0.6224	22.4650	15.2704	12.6089	1.8631
4145	39.8412	7.5359	4.6891	16.7538	5.6222	3.7577	1.2516	38.1607
6836	39.8507	10.9461	22.4700	19.2049	2.8458	9.5473	15.8051	13.4701
3494	39.8554	6.4545	0.7077	14.8029	9.6074	4.6223	35.1452	14.0816
4249	39.8585	24.1362	19.9191	15.5685	0.3569	4.0033	9.2310	11.5531
5791	39.8887	25.0767	7.0693	0.1647	13.2324	13.4992	23.2667	19.7272
3463	39.9257	3.6134	12.0383	3.8296	30.6823	29.4104	5.2728	6.5934
2867	39.9438	3.5220	6.6297	28.1923	6.0107	14.4474	31.9473	5.3268
5868	39.9575	1.1390	28.6979	8.9591	9.2408	13.9703	28.4369	24.8818

Total number of the Minimal Points = 2418

Number The Coordinates of the Maximal Points

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252	38.9572	21.9048	12.9666	24.1701	27.2765	9.6325	31.4184	9.9509
5594	38.9582	3.2427	36.3516	13.5102	37.6887	21.0943	31.3408	5.3918
1058	38.9753	13.2757	0.7398	0.0072	39.9549	29.5388	35.9693	9.2841
3556	38.9790	0.4908	20.8822	37.8741	35.8113	39.8578	11.2042	20.9560
7417	38.9828	35.6264	31.5219	20.3428	10.6086	18.4152	36.0046	19.7458
5747	38.9913	14.3783	31.7627	23.2280	37.6989	28.8530	2.8957	16.6106
3006	38.9932	38.3143	3.7738	31.6761	29.6675	39.3703	37.3176	9.4172
7720	39.0117	20.8264	28.9513	16.8502	5.7213	36.3969	31.4391	3.7107
1175	39.0319	7.4547	39.6969	34.9010	3.1156	1.2523	15.9875	32.4229
8481	39.0382	31.6502	39.3257	12.5583	32.9382	1.9939	27.2830	7.2091
2834	39.0464	32.1048	15.4489	1.4696	19.9962	39.0482	27.1442	33.1257
7114	39.0624	16.0007	32.1129	35.9377	9.8185	38.1453	19.5128	22.7073
42	39.0653	36.9077	39.2534	37.9629	10.6524	9.0158	31.0165	11.0159
3639	39.0668	31.6266	22.0574	13.4978	28.1146	31.4220	13.9367	24.6823
5695	39.0879	33.4718	26.9058	16.0056	0.6672	35.9922	23.2216	4.8927
5527	39.0903	29.0982	17.3936	4.3999	17.7790	27.9408	27.7729	27.2991
6161	39.1042	28.9985	2.3402	10.7340	19.6508	6.7196	37.8608	12.3527
2587	39.1290	33.6584	35.6743	33.3828	25.1535	15.7456	28.1103	35.1876
4979	39.1415	24.8703	29.5987	4.6343	16.6242	7.5956	29.9046	20.9595
4065	39.1424	0.0923	29.0336	17.8018	28.9452	31.8591	4.5604	12.6561
4300	39.1444	0.9955	16.8024	26.0324	13.7571	17.7366	33.6289	14.4770
1052	39.1464	37.7000	32.3936	6.5064	11.1386	9.1328	15.2245	38.2081
7660	39.1538	31.6650	14.3719	28.5990	8.4582	10.5131	19.7022	31.5662
5328	39.1780	1.9603	7.5434	9.4188	39.7912	5.6322	30.4968	6.8598

723	39.1916	26.6108	17.3088	13.1228	30.0500	29.7402	31.6105	14.0897
5259	39.1919	2.8117	8.5944	29.5321	7.5430	31.4333	37.5049	23.1080
6919	39.2023	38.5910	8.1313	33.8294	7.0066	26.1925	4.7881	31.4761
6741	39.2055	11.0468	23.0746	15.3056	10.3657	30.2308	30.4764	18.0041
3670	39.2064	16.0661	20.0457	39.4830	11.7598	3.1208	9.6841	26.8793
8140	39.2074	15.2962	32.0776	33.6545	31.2369	3.6871	29.2292	15.4415
4592	39.2158	21.2441	17.5373	29.4551	21.9366	37.5511	1.3496	31.5720
7188	39.2361	36.8189	2.8706	13.2529	26.9141	29.9020	4.7822	0.4828
36	39.2399	33.5579	20.3646	32.6894	12.3306	30.8322	21.5698	22.1413
1623	39.2507	21.8470	11.8730	21.6675	14.5181	35.4210	27.6171	36.1039
6889	39.2529	33.3839	39.4893	29.1080	21.9881	14.0401	12.2713	7.6448
6550	39.2627	9.7214	36.2396	32.3829	9.8940	26.5579	33.5252	3.9177
5488	39.2666	37.1100	0.5040	31.8691	11.9804	9.4906	7.6854	19.0602
8087	39.2767	29.9405	5.4380	1.1068	27.2956	16.1426	36.5509	27.3720
5079	39.2840	36.4996	38.7654	36.3642	35.7953	17.2100	30.3901	13.0709
3168	39.2886	34.3745	23.9857	20.3089	20.4690	21.4967	1.5324	32.9662
4349	39.3075	39.5221	39.4625	12.1062	28.6627	34.8162	33.6895	17.6568
5082	39.3150	19.1918	7.0261	34.6874	19.1393	7.7402	36.1364	15.0463
194	39.3176	5.6716	24.0929	27.1832	15.7925	25.2073	23.9699	4.5070
822	39.3214	10.8719	7.0829	6.8682	22.1320	0.1239	37.9305	28.0382
3946	39.3301	21.7122	7.5746	31.0319	28.6213	19.0703	18.4546	38.6067
974	39.3368	8.5832	24.1915	21.7607	34.2360	21.7192	33.8187	39.0848
1442	39.3399	29.0600	13.3773	20.8033	31.6687	1.2077	4.9064	35.9383
6146	39.3503	2.7211	9.5450	14.0911	3.7542	23.9680	38.7112	12.2224
2764	39.3557	30.5028	38.9781	1.3793	12.6649	8.6374	3.3816	33.5463
722	39.3567	17.7362	27.2331	15.0171	15.3683	0.0500	10.3979	39.3319
2147	39.3569	36.9819	4.1668	14.4089	1.6879	5.9956	37.8198	14.9618
973	39.3606	25.2135	38.6463	33.9324	32.4885	19.8575	16.2154	35.1785
6559	39.3826	13.4962	6.2474	6.1890	20.3100	29.5104	36.7440	21.9988
4835	39.3857	13.5301	34.5145	37.7484	9.6150	10.0185	27.4681	27.4788
3445	39.3945	32.0840	6.9464	29.1529	38.5743	30.2162	22.8729	39.2772
4160	39.4022	5.4150	21.9868	38.5579	36.2386	17.7855	18.0151	2.8130
2199	39.4072	9.1026	26.7334	8.9211	16.3093	18.3574	24.8291	32.0810
7974	39.4183	8.6134	14.0658	27.2591	25.7538	18.7692	4.2953	39.2014
275	39.4183	5.6748	28.6196	27.1580	32.0018	29.6278	39.6769	9.5670
7327	39.4254	12.5714	31.0399	29.2912	23.0725	36.6810	22.2043	0.3682
7719	39.4423	11.3026	35.6710	34.8267	22.8824	11.7114	3.3498	26.8318
105	39.4439	36.8066	5.2309	28.3673	33.4063	36.5815	10.2095	9.0910
3280	39.4484	0.5077	37.2936	33.4742	37.0578	19.6204	25.6157	18.4162
2718	39.4739	35.3041	20.1501	20.1505	0.5275	32.1589	34.2459	3.2120
7982	39.4841	28.7656	31.7124	14.5137	6.3087	17.1411	31.4897	37.1033
771	39.4982	3.8234	39.3805	30.5894	20.0294	27.7235	6.2955	21.1043
4636	39.5057	26.3362	23.0117	17.2780	30.7587	19.1689	38.3925	15.6710
3069	39.5070	37.9383	30.7685	34.4900	14.1222	27.4796	12.5225	10.3511
7013	39.5112	6.1984	30.3079	5.6240	28.9729	28.7190	24.7596	19.4721
4031	39.5181	30.9912	34.1490	37.4106	31.8242	33.8481	35.6125	2.7311

673	39.5184	21.1087	31.9231	33.1078	6.5036	24.7167	10.5744	28.7970
240	39.5225	29.6643	23.8324	34.2588	32.0387	16.6187	5.6416	13.9537
3875	39.5315	14.7406	37.5534	38.6876	10.5668	29.4832	7.7588	15.7310
6220	39.5316	37.6506	0.4514	29.9266	24.7278	30.1423	6.7692	15.0672
6585	39.5393	39.5250	14.4680	39.7568	25.9982	28.6094	35.4349	14.4160
6113	39.5415	5.2480	35.5817	18.1784	11.2397	16.7562	21.5327	8.3624
120	39.5427	38.6629	37.3160	17.5127	13.3998	19.2351	24.3449	13.7988
2898	39.5577	14.3425	12.7355	13.2554	14.9653	21.1012	34.1565	34.3989
1501	39.5581	15.8304	6.3850	14.5347	25.2971	17.6022	36.1649	21.0267
8396	39.5664	8.3609	7.4530	22.4722	20.5677	24.9475	22.3327	15.7658
3900	39.5761	22.1693	21.4265	15.4635	35.7755	18.6520	17.2304	21.9387
4741	39.5908	8.6262	25.2218	33.9548	24.3127	7.9115	35.0093	39.4361
582	39.5946	26.2251	0.1307	16.3200	26.9572	2.6130	35.2748	38.6377
4196	39.6125	13.4291	6.4892	31.9783	34.3059	1.7431	18.0375	19.2589
6917	39.6156	2.5593	32.6375	22.4091	2.0167	0.8136	19.1497	28.2782
5288	39.6212	25.3376	22.7832	10.7726	5.5684	34.9734	6.7342	3.9287
2917	39.6333	38.8815	6.3260	22.8606	10.5958	21.6121	13.2730	1.1308
4764	39.6431	24.4193	31.3823	20.4561	32.3650	12.3112	15.8822	33.7378
565	39.6477	9.1515	14.3693	2.7762	28.0768	37.9466	19.7489	7.3438
473	39.6523	30.1733	0.3506	28.1830	18.1347	21.1408	12.2309	22.3106
8390	39.6612	5.3569	12.7612	22.8479	16.7219	10.7810	12.7825	37.3471
6664	39.6680	13.5747	36.2582	7.4645	8.7522	34.8462	16.0909	15.0266
2712	39.6694	38.3006	0.6333	32.8421	20.5834	18.8246	3.1800	13.0467
7197	39.6723	0.0735	21.7199	25.0553	2.3469	11.3828	17.3844	31.9336
5083	39.6804	26.4374	12.9400	33.3716	16.0056	25.0845	38.2618	29.3717
2287	39.6839	28.9757	13.4094	19.9261	24.3090	36.9738	20.0308	3.8967
1375	39.7048	37.7871	18.9365	21.0449	33.0039	16.2580	2.3600	16.4157
5781	39.7095	10.4574	23.4905	6.2333	19.4620	7.7711	32.4071	23.1200
7110	39.7099	3.5509	3.3948	10.5211	15.0474	22.3391	18.6569	39.4233
3658	39.7100	33.0827	35.1362	37.2965	29.1171	6.8891	30.2499	22.2430
2471	39.7154	27.9146	33.6194	8.7533	32.1562	32.7901	21.1529	33.0528
2117	39.7215	7.6271	15.2427	1.6869	2.0077	36.8264	37.2703	36.5094
7072	39.7274	32.4140	29.0709	39.6951	26.4415	31.3457	20.7836	13.4195
4637	39.7304	32.3340	9.4950	38.4624	24.5738	27.7300	10.8646	14.4738
3	39.7337	28.7325	8.5593	21.8453	31.1924	1.9629	16.5574	24.7111
1844	39.7371	22.7670	34.1228	31.1420	2.3108	31.1136	30.1625	15.5574
2599	39.7486	2.4867	34.8488	21.2228	12.1112	4.8243	32.5643	22.7400
4759	39.7494	17.4960	15.0680	19.9484	0.0689	33.1813	35.0483	3.3177
4053	39.7505	37.9197	34.6244	33.4585	31.0405	6.7741	33.9533	8.7631
6233	39.7558	25.1164	39.5827	5.0579	17.7714	3.6092	17.2076	13.9186
437	39.7561	1.9568	38.4527	15.3348	28.4407	39.2872	20.4163	35.1526
5011	39.7606	37.6727	24.1737	9.9903	13.6662	5.5019	15.7047	20.2138
2763	39.7607	39.7411	17.6762	8.6674	36.8070	24.2800	17.7358	9.2353
3582	39.7630	17.9295	35.1374	36.4722	29.8811	15.2569	38.9025	22.2855
3963	39.7633	7.6488	14.0574	13.5853	9.0263	24.1412	16.4229	29.4013
6205	39.7736	4.0175	18.5566	16.3485	39.3947	37.3555	36.9879	21.5346

8420	39.7900	14.8844	39.7380	5.5370	5.6929	24.0155	15.4635	15.2580
3699	39.7904	37.8114	17.9657	0.6224	22.4650	15.2704	12.6089	1.8631
1547	39.7962	24.9166	31.9312	9.2603	13.7014	15.8341	30.2703	33.0738
45	39.7996	34.6196	14.7584	17.0913	18.9428	27.9241	33.8595	19.0940
8309	39.8078	23.7039	14.3934	32.8726	14.8454	11.7504	35.7100	18.5101
3723	39.8129	24.2068	18.2507	22.9053	21.3399	39.7458	1.5463	19.2162
6612	39.8135	26.0857	18.4236	1.3972	27.7199	20.5682	9.0440	35.6289
6709	39.8252	31.0714	35.7329	31.2972	16.6998	34.7126	18.5866	6.1003
5914	39.8328	12.5801	6.1094	15.7668	38.9350	22.6174	9.7558	7.3837
3898	39.8468	37.5817	11.3592	30.8483	24.0709	35.4618	34.9862	7.0264
3508	39.8469	25.2031	27.2151	23.8414	10.2799	5.9061	11.8089	29.5431
6836	39.8507	10.9461	22.4700	19.2049	2.8458	9.5473	15.8051	13.4701
3494	39.8554	6.4545	0.7077	14.8029	9.6074	4.6223	35.1452	14.0816
8115	39.8569	32.6312	39.1474	21.1557	37.5934	39.1486	1.3452	38.2634
6688	39.8578	37.2416	21.0105	4.8643	25.8638	18.7688	23.3191	17.7523
1338	39.8752	15.1660	6.7886	27.5730	27.7429	35.1997	20.6647	22.1195
2213	39.8752	31.0130	1.5515	29.7298	14.9841	17.7427	29.4978	31.6882
387	39.8817	32.2456	36.8872	20.8012	12.2704	37.7200	6.5130	29.9410
486	39.8855	30.5968	8.1850	7.8308	6.4570	3.1658	29.7089	34.3783
6547	39.8872	21.3420	1.7692	28.6615	23.3534	17.8757	9.9974	4.3628
6653	39.9094	0.1808	32.2014	7.9027	24.9250	35.3618	33.5831	19.5876
3060	39.9159	33.2947	35.3450	30.8866	20.0817	20.9824	39.0506	10.3372
2189	39.9265	34.3744	13.7683	31.9744	13.4064	17.8444	9.8787	5.9615
7952	39.9267	19.2334	16.4343	5.2324	12.7435	6.2259	29.8864	21.4655
7226	39.9309	35.7091	11.2639	20.3998	30.3568	35.1341	29.5955	37.8554
7960	39.9342	3.7138	37.7160	12.9250	20.2541	12.6013	33.5892	7.5462
2528	39.9370	28.7969	11.0430	26.8510	17.4739	35.5394	39.6703	9.0907
770	39.9371	31.4954	29.5163	4.3009	1.9465	7.7883	19.8199	36.2502
3816	39.9437	25.0180	9.3551	14.8552	30.6051	32.1185	29.5722	18.7765
2867	39.9438	3.5220	6.6297	28.1923	6.0107	14.4474	31.9473	5.3268
5868	39.9575	1.1390	28.6979	8.9591	9.2408	13.9703	28.4369	24.8818
7236	39.9591	31.1993	26.1258	21.1649	6.0412	4.9806	36.3691	32.3860
7128	39.9780	22.0442	38.4720	19.5188	23.3545	33.7646	7.0394	4.9825
4038	39.9868	7.3893	25.3762	35.0753	39.3103	8.6520	3.5097	6.6627
232	39.9909	17.6364	14.9514	16.9226	35.8127	33.2064	18.3196	27.4294

Total number of the Maximal Points = 2495

Appendix III

(d) Partial Results for the Special Example

Total numbers of the Minimal Points = 70000

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7642	99.8482	0.1517	0.1517	0.1517	0.1517	0.1517	0.1517	0.1517	0.1517	0.1517
41564	99.8488	0.1511	0.1511	0.1511	0.1511	0.1511	0.1511	0.1511	0.1511	0.1511
25131	99.8498	0.1501	0.1501	0.1501	0.1501	0.1501	0.1501	0.1501	0.1501	0.1501
49839	99.8506	0.1493	0.1493	0.1493	0.1493	0.1493	0.1493	0.1493	0.1493	0.1493
13033	99.8541	0.1458	0.1458	0.1458	0.1458	0.1458	0.1458	0.1458	0.1458	0.1458
2129	99.8544	0.1455	0.1455	0.1455	0.1455	0.1455	0.1455	0.1455	0.1455	0.1455
15212	99.8576	0.1423	0.1423	0.1423	0.1423	0.1423	0.1423	0.1423	0.1423	0.1423
34477	99.8597	0.1402	0.1402	0.1402	0.1402	0.1402	0.1402	0.1402	0.1402	0.1402
10759	99.8634	0.1365	0.1365	0.1365	0.1365	0.1365	0.1365	0.1365	0.1365	0.1365
11304	99.8647	0.1352	0.1352	0.1352	0.1352	0.1352	0.1352	0.1352	0.1352	0.1352
67119	99.8681	0.1318	0.1318	0.1318	0.1318	0.1318	0.1318	0.1318	0.1318	0.1318
58393	99.8689	0.1310	0.1310	0.1310	0.1310	0.1310	0.1310	0.1310	0.1310	0.1310
21185	99.8743	0.1256	0.1256	0.1256	0.1256	0.1256	0.1256	0.1256	0.1256	0.1256
54653	99.8765	0.1234	0.1234	0.1234	0.1234	0.1234	0.1234	0.1234	0.1234	0.1234
63379	99.8787	0.1212	0.1212	0.1212	0.1212	0.1212	0.1212	0.1212	0.1212	0.1212
54070	99.8790	0.1209	0.1209	0.1209	0.1209	0.1209	0.1209	0.1209	0.1209	0.1209
14280	99.8796	0.1203	0.1203	0.1203	0.1203	0.1203	0.1203	0.1203	0.1203	0.1203
51314	99.8802	0.1197	0.1197	0.1197	0.1197	0.1197	0.1197	0.1197	0.1197	0.1197
54152	99.8816	0.1183	0.1183	0.1183	0.1183	0.1183	0.1183	0.1183	0.1183	0.1183
42364	99.8829	0.1170	0.1170	0.1170	0.1170	0.1170	0.1170	0.1170	0.1170	0.1170
38019	99.8832	0.1167	0.1167	0.1167	0.1167	0.1167	0.1167	0.1167	0.1167	0.1167
32983	99.8842	0.1157	0.1157	0.1157	0.1157	0.1157	0.1157	0.1157	0.1157	0.1157
15451	99.8868	0.1131	0.1131	0.1131	0.1131	0.1131	0.1131	0.1131	0.1131	0.1131
30727	99.8875	0.1124	0.1124	0.1124	0.1124	0.1124	0.1124	0.1124	0.1124	0.1124
59271	99.8877	0.1122	0.1122	0.1122	0.1122	0.1122	0.1122	0.1122	0.1122	0.1122
31025	99.8897	0.1102	0.1102	0.1102	0.1102	0.1102	0.1102	0.1102	0.1102	0.1102
36380	99.8931	0.1068	0.1068	0.1068	0.1068	0.1068	0.1068	0.1068	0.1068	0.1068
51300	99.8935	0.1064	0.1064	0.1064	0.1064	0.1064	0.1064	0.1064	0.1064	0.1064
55946	99.8940	0.1059	0.1059	0.1059	0.1059	0.1059	0.1059	0.1059	0.1059	0.1059
56087	99.8967	0.1032	0.1032	0.1032	0.1032	0.1032	0.1032	0.1032	0.1032	0.1032
34638	99.8977	0.1022	0.1022	0.1022	0.1022	0.1022	0.1022	0.1022	0.1022	0.1022
56372	99.8999	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
58582	99.9017	0.0982	0.0982	0.0982	0.0982	0.0982	0.0982	0.0982	0.0982	0.0982
49260	99.9029	0.0970	0.0970	0.0970	0.0970	0.0970	0.0970	0.0970	0.0970	0.0970
30653	99.9034	0.0965	0.0965	0.0965	0.0965	0.0965	0.0965	0.0965	0.0965	0.0965
14718	99.9069	0.0930	0.0930	0.0930	0.0930	0.0930	0.0930	0.0930	0.0930	0.0930
16925	99.9074	0.0925	0.0925	0.0925	0.0925	0.0925	0.0925	0.0925	0.0925	0.0925
56554	99.9074	0.0925	0.0925	0.0925	0.0925	0.0925	0.0925	0.0925	0.0925	0.0925

25586	99.9081	0.0918	0.0918	0.0918	0.0918	0.0918	0.0918	0.0918	0.0918	0.0918
50613	99.9090	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909	0.0909
55453	99.9128	0.0871	0.0871	0.0871	0.0871	0.0871	0.0871	0.0871	0.0871	0.0871
24012	99.9134	0.0865	0.0865	0.0865	0.0865	0.0865	0.0865	0.0865	0.0865	0.0865
46671	99.9148	0.0851	0.0851	0.0851	0.0851	0.0851	0.0851	0.0851	0.0851	0.0851
20855	99.9169	0.0830	0.0830	0.0830	0.0830	0.0830	0.0830	0.0830	0.0830	0.0830
32522	99.9173	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826
9165	99.9214	0.0785	0.0785	0.0785	0.0785	0.0785	0.0785	0.0785	0.0785	0.0785
57252	99.9224	0.0775	0.0775	0.0775	0.0775	0.0775	0.0775	0.0775	0.0775	0.0775
903	99.9230	0.0769	0.0769	0.0769	0.0769	0.0769	0.0769	0.0769	0.0769	0.0769
42088	99.9244	0.0755	0.0755	0.0755	0.0755	0.0755	0.0755	0.0755	0.0755	0.0755
11363	99.9281	0.0718	0.0718	0.0718	0.0718	0.0718	0.0718	0.0718	0.0718	0.0718
39423	99.9291	0.0708	0.0708	0.0708	0.0708	0.0708	0.0708	0.0708	0.0708	0.0708
62983	99.9310	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689
26890	99.9359	0.0640	0.0640	0.0640	0.0640	0.0640	0.0640	0.0640	0.0640	0.0640
9117	99.9391	0.0608	0.0608	0.0608	0.0608	0.0608	0.0608	0.0608	0.0608	0.0608
298	99.9447	0.0552	0.0552	0.0552	0.0552	0.0552	0.0552	0.0552	0.0552	0.0552
43928	99.9452	0.0547	0.0547	0.0547	0.0547	0.0547	0.0547	0.0547	0.0547	0.0547
25344	99.9459	0.0540	0.0540	0.0540	0.0540	0.0540	0.0540	0.0540	0.0540	0.0540
65654	99.9470	0.0529	0.0529	0.0529	0.0529	0.0529	0.0529	0.0529	0.0529	0.0529
21558	99.9473	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526
56461	99.9524	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475
491	99.9531	0.0468	0.0468	0.0468	0.0468	0.0468	0.0468	0.0468	0.0468	0.0468
15277	99.9538	0.0461	0.0461	0.0461	0.0461	0.0461	0.0461	0.0461	0.0461	0.0461
66438	99.9578	0.0421	0.0421	0.0421	0.0421	0.0421	0.0421	0.0421	0.0421	0.0421
60551	99.9581	0.0418	0.0418	0.0418	0.0418	0.0418	0.0418	0.0418	0.0418	0.0418
58910	99.9601	0.0398	0.0398	0.0398	0.0398	0.0398	0.0398	0.0398	0.0398	0.0398
52717	99.9606	0.0393	0.0393	0.0393	0.0393	0.0393	0.0393	0.0393	0.0393	0.0393
60920	99.9616	0.0383	0.0383	0.0383	0.0383	0.0383	0.0383	0.0383	0.0383	0.0383
34452	99.9620	0.0379	0.0379	0.0379	0.0379	0.0379	0.0379	0.0379	0.0379	0.0379
31003	99.9650	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349
50002	99.9681	0.0318	0.0318	0.0318	0.0318	0.0318	0.0318	0.0318	0.0318	0.0318
54527	99.9684	0.0315	0.0315	0.0315	0.0315	0.0315	0.0315	0.0315	0.0315	0.0315
1848	99.9697	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302
34462	99.9737	0.0262	0.0262	0.0262	0.0262	0.0262	0.0262	0.0262	0.0262	0.0262
45213	99.9738	0.0261	0.0261	0.0261	0.0261	0.0261	0.0261	0.0261	0.0261	0.0261
19562	99.9744	0.0255	0.0255	0.0255	0.0255	0.0255	0.0255	0.0255	0.0255	0.0255
4411	99.9803	0.0196	0.0196	0.0196	0.0196	0.0196	0.0196	0.0196	0.0196	0.0196
35013	99.9811	0.0188	0.0188	0.0188	0.0188	0.0188	0.0188	0.0188	0.0188	0.0188
52464	99.9832	0.0167	0.0167	0.0167	0.0167	0.0167	0.0167	0.0167	0.0167	0.0167
50979	99.9837	0.0162	0.0162	0.0162	0.0162	0.0162	0.0162	0.0162	0.0162	0.0162
24286	99.9854	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145
38005	99.9870	0.0129	0.0129	0.0129	0.0129	0.0129	0.0129	0.0129	0.0129	0.0129
23603	99.9876	0.0123	0.0123	0.0123	0.0123	0.0123	0.0123	0.0123	0.0123	0.0123
23379	99.9903	0.0096	0.0096	0.0096	0.0096	0.0096	0.0096	0.0096	0.0096	0.0096
43526	99.9912	0.0087	0.0087	0.0087	0.0087	0.0087	0.0087	0.0087	0.0087	0.0087

17095	99.9931	0.0068	0.0068	0.0068	0.0068	0.0068	0.0068	0.0068	0.0068	0.0068
40864	99.9938	0.0061	0.0061	0.0061	0.0061	0.0061	0.0061	0.0061	0.0061	0.0061
65116	99.9954	0.0045	0.0045	0.0045	0.0045	0.0045	0.0045	0.0045	0.0045	0.0045
63665	99.9988	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
27862	99.9988	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011

Total numbers of the Maximal Points = 70000

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21558	99.9473	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526	0.0526
56461	99.9524	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475
491	99.9531	0.0468	0.0468	0.0468	0.0468	0.0468	0.0468	0.0468	0.0468	0.0468
15277	99.9538	0.0461	0.0461	0.0461	0.0461	0.0461	0.0461	0.0461	0.0461	0.0461
66438	99.9578	0.0421	0.0421	0.0421	0.0421	0.0421	0.0421	0.0421	0.0421	0.0421
60551	99.9581	0.0418	0.0418	0.0418	0.0418	0.0418	0.0418	0.0418	0.0418	0.0418
58910	99.9601	0.0398	0.0398	0.0398	0.0398	0.0398	0.0398	0.0398	0.0398	0.0398
52717	99.9606	0.0393	0.0393	0.0393	0.0393	0.0393	0.0393	0.0393	0.0393	0.0393
60920	99.9616	0.0383	0.0383	0.0383	0.0383	0.0383	0.0383	0.0383	0.0383	0.0383
34452	99.9620	0.0379	0.0379	0.0379	0.0379	0.0379	0.0379	0.0379	0.0379	0.0379
31003	99.9650	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349	0.0349
50002	99.9681	0.0318	0.0318	0.0318	0.0318	0.0318	0.0318	0.0318	0.0318	0.0318
54527	99.9684	0.0315	0.0315	0.0315	0.0315	0.0315	0.0315	0.0315	0.0315	0.0315
1848	99.9697	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302
34462	99.9737	0.0262	0.0262	0.0262	0.0262	0.0262	0.0262	0.0262	0.0262	0.0262
45213	99.9738	0.0261	0.0261	0.0261	0.0261	0.0261	0.0261	0.0261	0.0261	0.0261
19562	99.9744	0.0255	0.0255	0.0255	0.0255	0.0255	0.0255	0.0255	0.0255	0.0255
4411	99.9803	0.0196	0.0196	0.0196	0.0196	0.0196	0.0196	0.0196	0.0196	0.0196
35013	99.9811	0.0188	0.0188	0.0188	0.0188	0.0188	0.0188	0.0188	0.0188	0.0188
52464	99.9832	0.0167	0.0167	0.0167	0.0167	0.0167	0.0167	0.0167	0.0167	0.0167
50979	99.9837	0.0162	0.0162	0.0162	0.0162	0.0162	0.0162	0.0162	0.0162	0.0162
24286	99.9854	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145	0.0145
38005	99.9870	0.0129	0.0129	0.0129	0.0129	0.0129	0.0129	0.0129	0.0129	0.0129
23603	99.9876	0.0123	0.0123	0.0123	0.0123	0.0123	0.0123	0.0123	0.0123	0.0123
23379	99.9903	0.0096	0.0096	0.0096	0.0096	0.0096	0.0096	0.0096	0.0096	0.0096
43526	99.9912	0.0087	0.0087	0.0087	0.0087	0.0087	0.0087	0.0087	0.0087	0.0087
17095	99.9931	0.0068	0.0068	0.0068	0.0068	0.0068	0.0068	0.0068	0.0068	0.0068
40864	99.9938	0.0061	0.0061	0.0061	0.0061	0.0061	0.0061	0.0061	0.0061	0.0061
65116	99.9954	0.0045	0.0045	0.0045	0.0045	0.0045	0.0045	0.0045	0.0045	0.0045
63665	99.9988	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
27862	99.9988	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011

Total numbers of points of set F = 70000

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69970	98.8397	1.1602	1.1602	1.1602	1.1602	1.1602	1.1602	1.1602	1.1602	1.1602
69971	45.5436	54.4563	54.4563	54.4563	54.4563	54.4563	54.4563	54.4563	54.4563	54.4563
69972	93.2181	6.7818	6.7818	6.7818	6.7818	6.7818	6.7818	6.7818	6.7818	6.7818
69973	8.4276	91.5723	91.5723	91.5723	91.5723	91.5723	91.5723	91.5723	91.5723	91.5723
69974	85.0271	14.9728	14.9728	14.9728	14.9728	14.9728	14.9728	14.9728	14.9728	14.9728
69975	11.2682	88.7317	88.7317	88.7317	88.7317	88.7317	88.7317	88.7317	88.7317	88.7317
69976	15.7221	84.2778	84.2778	84.2778	84.2778	84.2778	84.2778	84.2778	84.2778	84.2778
69977	53.9415	46.0584	46.0584	46.0584	46.0584	46.0584	46.0584	46.0584	46.0584	46.0584
69978	74.8747	25.1252	25.1252	25.1252	25.1252	25.1252	25.1252	25.1252	25.1252	25.1252
69979	49.0002	50.9997	50.9997	50.9997	50.9997	50.9997	50.9997	50.9997	50.9997	50.9997
69980	12.5086	87.4913	87.4913	87.4913	87.4913	87.4913	87.4913	87.4913	87.4913	87.4913
69981	62.5052	37.4947	37.4947	37.4947	37.4947	37.4947	37.4947	37.4947	37.4947	37.4947
69982	61.0552	38.9447	38.9447	38.9447	38.9447	38.9447	38.9447	38.9447	38.9447	38.9447
69983	89.7225	10.2774	10.2774	10.2774	10.2774	10.2774	10.2774	10.2774	10.2774	10.2774
69984	82.9691	17.0308	17.0308	17.0308	17.0308	17.0308	17.0308	17.0308	17.0308	17.0308
69985	53.9570	46.0429	46.0429	46.0429	46.0429	46.0429	46.0429	46.0429	46.0429	46.0429
69986	23.6921	76.3078	76.3078	76.3078	76.3078	76.3078	76.3078	76.3078	76.3078	76.3078
69987	64.9779	35.0220	35.0220	35.0220	35.0220	35.0220	35.0220	35.0220	35.0220	35.0220
69988	35.6392	64.3607	64.3607	64.3607	64.3607	64.3607	64.3607	64.3607	64.3607	64.3607
69989	79.7802	20.2197	20.2197	20.2197	20.2197	20.2197	20.2197	20.2197	20.2197	20.2197
69990	70.1166	29.8833	29.8833	29.8833	29.8833	29.8833	29.8833	29.8833	29.8833	29.8833
69991	76.8723	23.1276	23.1276	23.1276	23.1276	23.1276	23.1276	23.1276	23.1276	23.1276
69992	90.2287	9.7712	9.7712	9.7712	9.7712	9.7712	9.7712	9.7712	9.7712	9.7712
69993	91.5653	8.4346	8.4346	8.4346	8.4346	8.4346	8.4346	8.4346	8.4346	8.4346
69994	16.5268	83.4731	83.4731	83.4731	83.4731	83.4731	83.4731	83.4731	83.4731	83.4731
69995	61.7786	38.2213	38.2213	38.2213	38.2213	38.2213	38.2213	38.2213	38.2213	38.2213
69996	29.1335	70.8664	70.8664	70.8664	70.8664	70.8664	70.8664	70.8664	70.8664	70.8664
69997	0.9693	99.0306	99.0306	99.0306	99.0306	99.0306	99.0306	99.0306	99.0306	99.0306
69998	69.8834	30.1165	30.1165	30.1165	30.1165	30.1165	30.1165	30.1165	30.1165	30.1165
69999	48.4570	51.5429	51.5429	51.5429	51.5429	51.5429	51.5429	51.5429	51.5429	51.5429
70000	23.6606	76.3393	76.3393	76.3393	76.3393	76.3393	76.3393	76.3393	76.3393	76.3393