TIGHT PROJECTIVE 5-DESIGNS AND EXCEPTIONAL STRUCTURES

5-DESIGNS PROJECTIFS SERRÉS ET STRUCTURES EXCEPTIONNELLES

A Thesis Submitted to the Division of Graduate Studies
of the Royal Military College of Canada
by

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In Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy

July, 2023

Acknowledgements

The exploration and growth that led to this thesis has been a personal pleasure and rich adventure. I sincerely thank my two thesis supervisors, Charles Paquette and Claude Tardif, for their patience, curiosity, and encouragement along the way. I also thank Gregg Wade for his support at the beginning of this program, without which I may not have even begun. Finally, I thank my wife Amber and children (Miles, Lucy, and Jane) for their patience and understanding over the course of several years.

Abstract

A t-design on a sphere or projective space is a finite subset such that the integral of any degree t polynomial over the sphere or projective space is equal to the average value of that polynomial evaluated at the points of the t-design. Tight t-designs are optimal in that they use the minimum possible number of points to achieve a particular value of t. Although t-designs are abundant, tight t-designs are rare structures in combinatorics that continue to resist a full classification. However, there are precisely four tight projective 5-designs: the vertices of a regular hexagon, the vertices of a regular icosahedron, the lines spanning the short vectors of the Leech lattice, and a set of points in the octonion projective plane forming a generalized hexagon finite geometry. This thesis explores the four tight projective 5-designs and their connections to various exceptional structures. The regular hexagon provides a starting point from which to recover Lie and Jordan theory. We explore an exceptional sequence of Lie algebras that terminates in the Lie algebra of the standard model of particle physics and provides a three generation representation of standard model fermions. The regular icosahedron is unique among tight t-designs, apart from most polygons on the unit circle, in that it has an irrational angle set. We provide proof of this fact by using Jordan algebras to generalize and correct certain previous attempts to prove that tight t-designs have rational angles. Finally, we explore the relationship between the two remaining tight 5-designs by examining octonion constructions of the Leech lattice and their octonion reflection symmetries. We introduce a common construction technique that yields the two remaining tight 5-designs and explore the role that octonion integers and exceptional Jordan algebra integers can play in this common construction.

Résumé

Un t-design sur une sphère ou un espace projectif est un sous-ensemble fini tel que l'intégrale de tout polynôme de degré t sur la sphère ou l'espace projectif est égale à la valeur moyenne de ce polynôme évaluée en des points du t-design. Les t-designs serrés sont optimaux en ce sens qu'ils utilisent le plus petit nombre possible de points pour atteindre une valeur particulière de t. Bien que les t-designs soient abondants, les t-designs serrés sont des structures rares en combinatoire qui continuent à résister à une classification complète. Cependant, il existe précisément quatre 5-designs projectifs serrés: les sommets d'un hexagone régulier, les sommets d'un icosaèdre régulier, les lignes couvrant les vecteurs courts du treillis de Leech, et un ensemble de points dans le plan de Cayley formant une géométrie hexagonale finie généralisée. Cette thèse explore les quatre 5-designs projectives serrées et leurs liens avec diverses structures exceptionnelles. L'hexagone régulier constitue un point de départ pour retrouver les théories de Lie et de Jordan. Nous explorons une séquence exceptionnelle d'algèbres de Lie qui se termine par l'algèbre de Lie du modèle standard de la physique des particules et qui fournit une représentation à trois générations des fermions du modèle standard. L'icosaèdre régulier est unique parmi les t-designs serrés, hormis de la plupart des polygones sur le cercle unitaire, du fait qu'il possède un ensemble d'angles irrationnels. Nous apportons la preuve de ce fait en utilisant les algèbres de Jordan pour généraliser et corriger certaines tentatives antérieures de prouver que les t-designs serrés ont des angles rationnels. Enfin, nous explorons la relation entre les deux 5-designs serrés restantes en examinant les constructions par octonions du treillis de Leech et leurs symétries de réflexion d'octonions. Nous introduisons une technique de construction commune qui permet d'obtenir les deux 5-designs serrées restantes et nous explorons le rôle que les octonions entiers et les entiers exceptionnels de l'algèbre de Jordan peuvent jouer dans cette construction commune.

Contents

Acknowledgements iii

Abstract v

Résumé vii

List of Tables xi

List of Figures xiii

List of Symbols xv

Chapter 1. Introduction 1
- 1.1. Understanding Exceptionality 1
- 1.2. Tight t-Designs 5
- 1.3. Root Systems 7
- 1.4. Lie, Jordan, and Composition Algebras 15

Chapter 2. Literature Survey 23
- 2.1. The Standard Model and Exceptional Explanation 23
- 2.2. Tight t-Designs and Rational Angle Sets 33
- 2.3. Octonions and Tight Projective 5-Designs 34

Chapter 3. An Exceptional Combinatorial Sequence and Standard Model Particles 41
- 3.1. Introduction 41
- 3.2. Star-Closed Line Systems 41
- 3.3. Nested Sequences of Binary Decompositions 44
- 3.4. The Exceptional Sequence 47
- 3.5. Lie Algebras of Star-Closed Line Systems 48
- 3.6. Connection to the Standard Model 50
- 3.7. Discussion 55

Chapter 4. Rational Angles and Tight T-Designs 57
- 4.1. Introduction 57
- 4.2. Jordan Algebras and T-Designs 57
- 4.3. Bose-Mesner Algebras and the Idempotent Basis 60

4.4. Idempotent Ranks and Complex Automorphisms 62
4.5. Completing the Proof 65
4.6. Conclusion 68

Chapter 5. Octonions and the Two Strictly Projective Tight 5-Designs 71
5.1. Introduction 71
5.2. Tight t-Designs 72
5.3. Octonions and Isometries 75
5.4. The Common Construction 78
5.5. Leech Lattice Symmetries and the Octonion Projective Plane 80
5.6. Conclusion 82

Chapter 6. Octonion Integers and Tight 5-Designs 85
6.1. Introduction 85
6.2. Octonion Integers 86
6.3. Octonion Integer Leech Lattices 92
6.4. Leech Lattice Symmetries 99
6.5. The Common Construction and Octonion Integer Triples 102
6.6. Albert Isotope Integer Rings and Tight 5-Designs 106
6.7. Isotope Rings, Orbits, and Squares 109
6.8. Conclusion 110

Chapter 7. Conclusion 111
7.1. The Regular Hexagon 111
7.2. The Regular Icosahedron 113
7.3. The Leech Lattice and the Octonion Projective Plane 114

Bibliography 117

Appendix A. Tight T-Designs 125

Appendix B. Jacobi Polynomials 131
B.1. Application to the Spherical Cases 131
B.2. An Annihilator Polynomial Identity 132

List of Tables

1.1 Known tight t-designs with cardinality $|X|$, angle set A, and isometry group G. 8

1.2 Properties of irreducible root systems. 9

1.3 Glue vectors of irreducible root lattices 11

1.4 Classification of simple Lie algebras over $\mathbb{C}$. 16

1.5 Classification of simple Jordan pairs over $\mathbb{C}$. 18

3.1 The 3-gradings on finite irreducible root systems. 45

3.2 The Lie algebras of simply-laced root systems. 49

3.3 Fermion particle nomenclature. 53

5.1 The common construction applied to orbits of combinations of S, V_∞, and V_t for $t \neq t' \in \mathbb{F}_7$. 81

6.1 Commutative vectors r in $\Lambda(\overline{\lambda}, \lambda)$ defining reflection symmetries $\mathtt{W}_r$ of $\Lambda(\overline{\lambda}, \lambda)$. 99

6.2 The common construction of [**Nas22**] applied to orbits of combinations of S_λ under $\mathrm{PSL}_2(7) \subset \mathrm{Aut}(\mathsf{O})$. 105

List of Figures

1.1 One line extension construction of irreducible root lattices 12
1.2 Three-gradings of irreducible root systems 13

3.1 Nested 3-gradings of simply-laced root irreducible root systems in $\mathbb{R}^7$. 46

List of Symbols

$\operatorname{ann}(x)$	annihilator polynomial
d	Jordan algebra degree
$\mathfrak{f}_4$	Lie algebra of type F_4
$\mathfrak{g}$	Lie algebra
$\mathfrak{g}_2$	Lie algebra of type G_2
h	Coxeter number
i_t	octonion standard basis vector
s	degree of design X
$\operatorname{srg}(v, k, \lambda, \mu)$	strongly regular graph
t	strength of design X
$\overline{x}$	conjugate of composition algebra element x
$x^\dagger$	conjugate transpose of vector x
A	Albert algebra ring $\operatorname{Herm}(3, \mathsf{O})$
A_5	alternating group
$\mathtt{A}_n$	root system or lattice of type A
$A(X)$	angle set of X
$\mathtt{B}_n$	root system of type B
$\mathbb{C}$	complex numbers
$\mathtt{C}_n$	root system of type C
$\mathrm{Co}_1, \mathrm{Co}_2, \mathrm{Co}_3$	Conway's sporadic simple groups
$\mathbb{CP}^{\rho-1}$	complex projective space
$\mathtt{D}_n$	root system or lattice of type D
$\mathtt{E}_n$	root system or lattice of type E
$\operatorname{End}(V)$	endomorphism group of V
$\mathbb{F}$	division composition algebra $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$
$\mathbb{F}_7$	finite field of order 7
$\mathtt{F}_4$	root system of type F_4
F_4	Lie group of type F_4
$\mathtt{G}_2$	root system of type G_2
G_2	Lie group of type G_2
$G_2(q)$	finite group of Lie type G_2
$\mathrm{Gh}(2, 8)$	generalized hexagon
$\mathrm{Gq}(2, 4)$	generalized quadrangle

$\mathbb{H}$	quaternion algebra
$\mathrm{Herm}(\rho,\mathbb{F})$	$\rho\times\rho$ Hermitian matrices over $\mathbb{F}$
HJ	Hall-Janko sporadic simple group
$\mathbb{HP}^{\rho-1}$	quaternion projective space
$\mathrm{Im}(x)$	imaginary component of x
I_ρ	identity matrix of rank ρ
$\mathcal{J}(V)$	primitive idempotent manifold of V
L_i	Bose-Mesner algebra idempotent basis matrix
$\mathtt{L}_u, \mathtt{R}_u, \mathtt{B}_u$	octonion translation maps
McL	McLaughlin sporadic simple group
$N(x)$	norm of x
$\mathbb{O}$	octonion algebra
O	octonion integers
$\mathrm{O}_8^+(2)$	orthogonal group
$\mathbb{OP}^2$	octonion projective plane
$P(x)$	Jordan quadratic operator of x
$P_n^{(\alpha,\beta)}(x)$	Jacobi polynomial
PL(7)	Projective line on $\mathbb{F}_7$
$\mathrm{PSL}_n(q)$	projective special linear group
$\mathrm{PSU}_n(q)$	projective special unitary group
$\mathbb{Q}$	rational numbers
$Q_k^\varepsilon(x)$	renormalized Jacobi polynomial
$\mathbb{R}$	real numbers
$\mathrm{Re}(x)$	real component of x
$\mathbb{RP}^{\rho-1}$	real projective space
S_n	symmetric group on n points
$\mathrm{Sp}_{2m}(q)$	symplectic group
$\mathrm{SU}_n(q)$	special unitary group
Suz	Suzuki sporadic simple group
V	simple Euclidean Jordan algebra
V_4	Klein four group
$V^{(a)}$	a-homotope or isotope of Jordan algebra V
$W(\mathtt{A}_n)$	Weyl reflection group of type $\mathtt{A}_n$
X	design in $\mathcal{J}(V)$
$\mathbb{Z}$	integers
ρ	Jordan algebra rank
$\delta_{i,j}$	Kronecker delta
λ	zero of x^2+x+2
Λ_{24}	Leech lattice
ω	cubic root of unity
Ω_{d+1}	unit sphere in $\mathbb{R}^{d+1}$
$\overline{\Phi}$	line system

Φ	root system
$(a)_i$	Pochhammer symbol
${}^3D_4(q)$	Steinberg-Tits-Hertzig twisted group of type D_4
$[i]$	lattice glue vector
$[x]$	idempotent matrix constructed from vector x
$[x, y]$	Lie product of x and y
$x \circ y$	Jordan product of x and y
$x \circ_a y$	Jordan a-homotope product of x and y

CHAPTER 1

Introduction

This thesis is motivated by exceptional structures in mathematics—both explanations for their existence and their potential explanatory power. Most of the structures that we will examine are well known and well studied. The focus will be addressing connections that are not yet fully understood and forming new explanations surrounding certain rare structures. We focus primarily on the four tight projective 5-designs. A t-design is a finite subset of a unit sphere or projective space with special properties. Tight t-designs are optimal in a certain sense, and the four tight projective 5-designs are particularly interesting. These four structures consist of the vertices of a regular hexagon, of a regular icosahedron, the lines spanned by the short vectors of the Leech lattice, and an exceptional structure in the octonion projective plane. These four objects are closely related to numerous other exceptional structures in mathematics and deserve closer investigation.

1.1. Understanding Exceptionality

1.1.1. Defining Exceptionality. To understand what counts as an exceptional object, it helps to begin with the concept of a classification. A classification of a family of mathematical structures often begins with a set of axioms. We then raise the question of what are all of the objects that satisfy these axioms. A classification is the answer to such a question. Having completed a classification, we can organize the objects according to their specific properties or methods of construction. An exceptional object stands apart from the others in some sense.

For instance, in the classification of finite simple groups the objects form infinite families of groups as well as twenty six exceptions, known as the sporadic finite simple groups. These sporadic groups are exceptional because they lack an infinite family. In contrast, the classification of division composition algebras over the rational numbers has only four examples. The largest example, the octonion algebra, is exceptional because it alone is non-associative. Another example is the classification of symmetric permutations groups. The exceptional symmetric group is the group of all permutations on six points, since it alone has a non-trivial outer automorphism. In this case the exception has a special property that sets it apart from an infinite number of remaining examples. These examples illustrate how an object within a classification can qualify as exceptional for different reasons.

Sometimes a classification is incomplete or otherwise difficult to achieve. As we will see, the classification of tight t-designs over spheres and projective spaces is incomplete. Even tight 5-designs are not yet classified. In the case of tight projective 5-designs, however, we do have a classification that includes exactly four examples. These four examples are perhaps exceptional because any tight projective t-design (other than those contained in the real projective line) is known to have $t \leq 5$, so these four examples achieve a type of bound. In another sense, a tight t-design is exceptional among t-designs because it is tight, namely it is optimal in certain senses defined below. We see then that a structure may be considered exceptional on account of being a finite example lacking an infinite family, on account of a unique property, or on account of being optimal in some way.

1.1.2. Exceptionality, Explanation, and Physics. In a certain sense, exceptional structures are simply given, conjured by the axioms that define the family of structures to which they belong. Yet the existence of exceptional objects within a family may seem surprising. An explanation has *explanatory power* when it lessens the surprise about the existence of an exceptional object (see for example [**SS11**]). Certain exceptional structures exist because others do. A construction yields the one from the other, or the one exists as a substructure of the other. In either case, the construction or reduction allows the existence of the one to explain the existence of the other.

When it comes to applying mathematics to physics, certain symmetries are useful in the construction of physical models. For instance, the Lie algebra of the standard model of particle physics selects one of the infinite available Lie algebras as physically manifest or actual. Why is this particular Lie algebra manifest in physics, in contrast to all the others? The standard model Lie algebra is exceptional in that it is physically manifest, but it is difficult to explain why this particular case is the exceptional one.

The existence of an exceptional object could hold explanatory power over something else, such as the manifestation in physics of one specific symmetry among other possibilities. As described in Chapter 2, many researchers have attempted to describe the symmetries of the standard model of particle physics via embedding in some larger symmetry group. The choice of embedding requires some justification and many have sought to find that justification using exceptional mathematical objects and their special properties. In this way, exceptionality may serve an explanatory role. It turns out that standard model symmetries can be embedded in all simple Lie algebras of rank at least 5, so the mere fact that an embedding is possible explains little about the choice of embedding. An exceptional embedding explains more that the mere fact of an embedding.

1.1.3. Where to Begin? It is not simple to determine where to start when working with Lie, Jordan, and composition algebras. McCrimmon relates a dictum from Kantor: "There are no Jordan algebras, there are only Lie algebras," but replies with his own: "nine times out of ten, when you open up a Lie algebra you find a Jordan algebra inside which makes it tick" [**McC04**, p. 10]. Perhaps the same could be said of composition algebras, which determine the properties of a unital Jordan algebra about three-quarters of the time (i.e., for three of the four infinite families). Indeed, Jordan triple systems (used to construct Jordan algebras) can be constructed from 3-graded Lie algebras, and the reverse is also true. Yet not all simple Lie algebras admit a 3-grading, and not all Jordan triple systems admit a unital Jordan algebra construction. Only the unital Jordan algebras of rank at least 3 always admit a composition algebra construction. These nuances prevent us from treating Lie, Jordan, and composition algebras as interchangeable structures, yet they should not be treated as independent structures given their many connections (the category theory relating Lie and Jordan structures is described well in [**CS14**]).

For example, the octonions are the largest and uniquely non-associative composition algebra. The only simple Jordan algebra of rank greater than 2, constructed using octonions, is the exceptional Jordan algebra $\mathrm{Herm}(3, \mathbb{O})$, which admits a Jordan triple system corresponding to a three-grading on the exceptional Lie algebra $\mathfrak{e}_7$. This means that the octonion algebra $\mathbb{O}$, the exceptional Jordan algebra $\mathrm{Herm}(3, \mathbb{O})$, and the simple Lie algebra $\mathfrak{e}_7$ form a natural family of related exceptional objects. We will pay careful attention to these three structures in what follows. Chapter 3 will draw attention to exceptional properties of $\mathfrak{e}_7$ and Chapters 5 and 6 will explore the relation of $\mathbb{O}$ and $\mathrm{Herm}(3, \mathbb{O})$ to the third and fourth tight projective 5-designs.

Instead of beginning with Lie, Jordan, and composition algebras it is possible to focus on the combinatorial structures that govern their properties and classification. Lie algebras admit a Cartan grading and Jordan triple systems admit a Jordan grid, both of which are governed by the structure of root systems. Root systems, in turn, define root lattices. Root lattices can be manipulated to construct unimodular lattices. In certain cases, unimodular lattices correspond to important integer subrings of composition or Jordan algebras. We will examine integer subrings of $\mathbb{O}$ and $\mathrm{Herm}(3, \mathbb{O})$ in Chapter 6. Wherever possible, we will attend to the underlying combinatorics of root systems when dealing with Lie or Jordan structures.

In what follows we elect to focus on the combinatorics of the four tight projective 5-designs. We will characterize a t-design as a finite subset (with certain special properties) of the primitive idempotents of a simple Euclidean Jordan algebra. Simple Euclidean Jordan algebras, and t-designs subsets, are directly related to the geometry of symmetric cones. Specifically, simple

Euclidean Jordan algebras are in one-to-one correspondence with connected symmetric cones in Euclidean vector spaces [**FK94**, chaps. 2-3]. The primitive idempotents of the algebra form the boundary of the corresponding symmetric cone. Equipped with the standard Jordan inner product, the manifold of primitive idempotents forms a *compact connected symmetric space of rank* 1, and all such spaces can be constructed in this way [**FK94**, p. 79], [**Hog92**, p. 259]. Indeed, these Jordan primitive idempotent manifolds also constitute the classification of *compact and connected two-point homogeneous spaces*, in which the isometry group is transitive on pairs of points at each fixed distance [**Wan52**]. Although subsets of Jordan primitive idempotents may appear to require a great deal of algebraic preamble to understand, t-designs on the symmetric spaces that they model are conceptually quite simple.

1.1.4. Outline. This thesis is organized around the four tight projective 5-designs, which are exceptional in their own way, and aims to explore connections between them and other exceptional structures. One objective of this thesis is to show that the standard model Lie algebra is also exceptional for combinatorial reasons, lessening the surprise at it being physically manifest. Another objective is to strengthen the explanation of certain exceptional objects in terms of properties of others, using the four tight projective 5-designs as a focal point.

The remainder of Chapter 1 reviews the definitions of tight t-designs. The known examples are given in Appendix A. We then provide an exposition of the basic structure of Lie, Jordan, and composition algebras as well as some well known connections between the three. In particular, we will see how the smallest tight projective 5-design provides the minimal structure needed to construct all irreducible root systems, and thereby to recover all of Lie, Jordan, and composition algebra theory. The structure of root lattices also permits one to construct the Leech lattice, which defines our third example of a tight projective 5-design.

Chapter 2 provides a literature review of recent work related to the research described in Chapters 3-6. We describe certain notable attempts to use the octonion algebra to explain the seemingly accidental structure of the standard model of particle physics and provide some commentary on the difficulties remaining with these models. We also identify a gap in the literature relating to theorems about the angle sets of tight t-designs, prior to addressing that gap in chapter 4. Finally, we describe the most important examples in the literature of attempts to use octonions to construct the Leech lattice or to connect the two strictly projective tight 5-designs.

Chapter 3 explores the connections between the regular hexagon—our first tight projective 5-design—and the structure of simply-laced irreducible root systems, with their corresponding simple Lie algebras. We explore sequences of three-gradings and identify one particular sequence as exceptional. This sequence begins with the exceptional $\mathtt{E}_7$ root system and terminates in the $\mathtt{A}_1\mathtt{A}_2$ root system of the standard model Lie algebra.

Chapter 4 explores the properties of tight t-designs and their angle sets. The main aim is to repair and extend respectively incorrect and incomplete proofs in the literature. This allows us to verify that, apart from certain exceptions, the angle set of a tight t-design is rational. The main exception is the vertices of an icosahedron on the complex projective line, the second of our for tight projective 5-designs.

Chapter 5 provides a common construction of the two remaining tight projective 5-designs in terms of the exceptional octonion algebra. This chapter has been published elsewhere as [**Nas22**]. Chapter 6 continues work with this common construction with a focus instead on octonion integer rings. Both chapters provide methods to generate certain sporadic simple groups using octonion reflection matrices.

Chapter 7 concludes with some open questions and areas for potential exploration.

1.2. Tight t-Designs

In this section we define tight t-designs and provide a catalogue of the known examples. We then describe the classification of tight projective 5-designs. These definitions are also given where needed in chapters 4 and 5.

1.2.1. Concepts. Let V be a simple Euclidean Jordan algebra of rank ρ and degree d (defined in [**FK94**, chap. 2] and describe in terms of Lie algebras in Section 1.4.2). Algebra V is equipped with a positive definite inner product $\langle x, y\rangle$. The manifold of primitive idempotents $\mathcal{J}(V)$ with inner product $\langle x, y\rangle$ is one of the following spheres or projective spaces [**FK94**, chap. 2]:

$$\Omega_{d+1}, \quad \mathbb{RP}^{\rho-1}, \quad \mathbb{CP}^{\rho-1}, \quad \mathbb{HP}^{\rho-1}, \quad \mathbb{OP}^2, \quad d \geq 1,\ \rho \geq 3.$$

In the non-spherical cases, $d = [\mathbb{F} : \mathbb{R}] = 1, 2, 4, 8$, where $\mathbb{F}$ is one of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. A *design* $X \subset \mathcal{J}(V)$ is a finite subset of the manifold of primitive idempotents. We call X a *spherical design* when $X \subset \Omega_{d+1}$, which is precisely whenever $\rho = 2$. We call X a *projective design* when $X \subset \mathbb{FP}^{\rho-1}$, which is precisely whenever $d = 1, 2, 4, 8$. A design with $\rho = 2$ and $d = 1, 2, 4, 8$ is both spherical and projective. We call X a *strictly projective design* when it is projective and not spherical, which is precisely when $\rho > 2$.

Each design has some degree s and strength t. To determine the degree, we use the *angle set* $A(X)$ of design X:

$$A(X) = \left\{\langle x, y\rangle \mid x, y \in X \subset \mathcal{J}(V), x \neq y\right\}.$$

The *degree* s of design X is the cardinality of $A(X)$, written $s = |A(X)|$. We also write $\varepsilon = |\{0\} \cap A(X)|$. A design with $0 \in A(X)$, so that $\varepsilon = 1$, is known as an *antipodal design*.

The strength t of a design has to do with its ability to approximate functions on a sphere or projective space. Since we are using Jordan algebra idempotents, we may adapt the common definition of a t-design as follows. A t-design is a finite subset X of Jordan algebra primitive idempotents $\mathcal{J}(V)$ such that the integral of any degree t polynomial over $\mathcal{J}(V)$ is equal to the average value of that polynomial evaluated on the points of X (cf. [**Sei90**], [**Sei01**]). To determine the strength t we use the following renormalized Jacobi polynomials:

$$Q_k^\varepsilon(x) = \left(\frac{1}{2}\rho d + 2k + \varepsilon - 1\right) \frac{(\frac{1}{2}\rho d)_{k+\varepsilon-1}}{(\frac{1}{2}d)_{k+\varepsilon}} P_k^{(\frac{1}{2}d(\rho-1)-1,\frac{1}{2}d-1+\varepsilon)}(2x-1).$$

In this definition, we use k a non-negative integer, $\varepsilon = 0$ or 1, Pochhammer symbol $(a)_i = a(a+1)(a+2)\cdots(a+i-1)$, and Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ as defined in [**AS72**, 22.2.1]. The *strength* of design X is the maximum non-negative integer t that satisfies,

$$\sum_{x\in X}\sum_{y\in X} Q_k^0(\langle x, y\rangle) = 0, \quad k = 1, 2, \ldots, t.$$

A *t-design* is just a design with strength t.

Given any design X, we can determine A, s, and t using the definitions above. It is more difficult to construct or obtain a set X with a predetermined angle set, degree, or strength. Indeed, there are strict limits on the cardinality of $|X|$ for a given A or t. If we specify s then $|X|$ has an absolute upper limit. If we specify t then $|X|$ has an absolute lower limit. To reach either limit, X must have a specific angle set $A(X)$. A design is *tight* when it meets both limits simultaneously, with $t = 2s - \varepsilon$ [**Hog82, BH85**].

A simple way to describe the properties of a tight t-design $X \subset \mathcal{J}(V)$ is via the annihilator polynomial. The *annihilator polynomial* $\mathrm{ann}(x)$ of design X is the unique degree s polynomial that satisfies $|X| = \mathrm{ann}(1)$ and $\mathrm{ann}(\alpha) = 0$ for each α in $A(X)$. When a design is tight, we have $t = 2s - \varepsilon$ and annihilator polynomial $\mathrm{ann}(x) = x^\varepsilon R_{s-\varepsilon}^\varepsilon(x)$, where we have

$$R_{s-\varepsilon}^\varepsilon(x) = Q_0^\varepsilon(x) + Q_1^\varepsilon(x) + \cdots + Q_{s-\varepsilon}^\varepsilon(x) = \frac{(\frac{1}{2}\rho d)_s}{(\frac{1}{2}d)_s} P_{s-\varepsilon}^{(\frac{1}{2}d(\rho-1),\frac{1}{2}d-1+\varepsilon)}(2x-1).$$

The equivalence between these two expressions for $R_{s-\varepsilon}^\varepsilon(x)$ is verified in Appendix B. These definitions ensure that a tight $(2s-\varepsilon)$-design has cardinality $|X| = R_{s-\varepsilon}^\varepsilon(1)$ and an angle set $A(X)$ given by the roots of the annihilator polynomial $\mathrm{ann}(x) = x^\varepsilon R_{s-\varepsilon}^\varepsilon(x)$. Furthermore, a tight t-design defines an association scheme and corresponding Bose-Mesner algebra—concepts that generalize strongly regular graphs and to which we return in Chapter 4. The important parameters of an association scheme (the subdegrees and intersection numbers) are fixed by the value of t, rank ρ, and degree d [**Hog92**].

1.2.2. Examples of Tight t-Designs. It is very difficult to obtain new tight t-designs or complete the full classification. As described in Chapter 5, the classification of tight t-designs is incomplete. A complete classification would need to determine whether any additional tight 2-designs or 3-designs exist in $\mathbb{RP}^{\rho-1}$, $\mathbb{CP}^{\rho-1}$, or $\mathbb{HP}^{\rho-1}$. However, the classification of tight projective 5-designs is complete and corresponds to the $t = 5$ case of Example A.2, as well as Examples A.5, A.10, and A.19. We will discuss this partial classification more specifically in Chapter 5. The known tight t-designs and their basic data are given in table Table 1.1 and also described in Appendix A.

1.3. Root Systems

In this section we describe the fundamental role that our smallest tight projective 5-design, the regular hexagon, plays in the theory of root systems, and therefore in the theory of Lie, Jordan, and composition algebras. We will describe integral lattices, root lattices, and special properties of their dual lattices. This will allow us to describe the processes of one-line extension and three-grading, which allow us to construct larger root lattices from smaller ones, as well as obtain certain sublattices from larger ones. We will see that the smallest tight projective 5-design has the structure required to recover all root lattices. We will also see how the Leech lattice, our third tight projective 5-design, can be constructed from certain root lattices.

1.3.1. Integral Lattices. Integral lattices, root lattices, and root systems are described at length in many places. This treatment follows the description given in [**CS13**] and [**Ebe13**].

We begin with a real vector space $\mathbb{R}^n$ equipped with the standard Euclidean inner product, denoted (x, y) and with vectors $x = (x_1, x_2, \ldots, x_n)$ written in orthonormal coordinates relative to (x, y). A *lattice* Γ is a subset of $\mathbb{R}^n$ equipped with a (non-unique) basis $\{e_1, e_2, \ldots, e_n\}$ such that any x in Γ is a $\mathbb{Z}$-linear combination of the basis vectors. The corresponding *dual lattice* Γ^* is the subset of $\mathbb{R}^n$ with integral inner product to every lattice point in Γ:

$$\Gamma^* = \{x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z}, \forall y \in \Gamma\}.$$

Given a lattice basis $\{e_1, e_2, \ldots, e_n\}$ we define the *generator matrix* M as the matrix with the e_i vectors as rows. In the orthonormal coordinates chosen, the *Gram matrix* $G = MM^T$ is the matrix of all inner products of basis vectors. The *lattice determinant* $\det \Gamma$ is defined as the determinant of the Gram matrix. The matrix $G^{-1}M$ has for rows the *dual basis* $\{e_1^*, e_2^*, \ldots, e_n^*\}$, which satisfy $(e_i^*, e_j) = \delta_{i,j}$. That is, $G^{-1}M$ is the generator matrix of dual lattice Γ^*. The determinant of the dual lattice is the inverse of the determinant of the lattice:

$$\det \Gamma^* = \frac{1}{\det \Gamma}.$$

Example	ρ	d	t	A	$\lvert X\rvert$	G
A.1	ρ	d	1	$\{0\}$	ρ	
A.2	2	1	t	$\left\{\cos^2\left(\frac{n\pi}{t+1}\right) \mid n \in \mathbb{Z}\right\}$	$t+1$	$W(\mathtt{I}_2(t+1))$
A.3	2	d	2	$\left\{\frac{d}{2(d+1)}\right\}$	$d+2$	$W(\mathtt{A}_{d+1})$
A.4	2	d	3	$\{0, \frac{1}{2}\}$	$2d+2$	$W(\mathtt{D}_{d+1})$
A.5	3	1	2	$\{\frac{1}{5}\}$	6	A_5
	2	2	5	$\left\{0, \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{5}}\right)\right\}$	12	$W(\mathtt{H}_3)$
A.6	7	1	2	$\{\frac{1}{9}\}$	28	$\mathrm{Sp}_6(2)$
	2	5	4	$\{\frac{1}{4}, \frac{5}{8}\}$	27	$W(\mathtt{E}_6)$
	2	6	5	$\{0, \frac{1}{3}, \frac{2}{3}\}$	56	$W(\mathtt{E}_7)$
A.7	23	1	2	$\{\frac{1}{25}\}$	276	Co_3
	2	21	4	$\{\frac{3}{8}, \frac{7}{12}\}$	275	$\mathrm{McL}:2$
	2	22	5	$\{0, \frac{2}{5}, \frac{3}{5}\}$	552	$2 \times \mathrm{Co}_3$
A.8	8	1	3	$\{0, \frac{1}{4}\}$	120	$\mathrm{O}_8^+(2):2$
	2	7	7	$\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$	240	$W(\mathtt{E}_8)$
A.9	23	1	3	$\{0, \frac{1}{9}\}$	2300	Co_2
	2	22	7	$\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$	4600	Co_2
A.10	24	1	5	$\{0, \frac{1}{16}, \frac{1}{4}\}$	98280	Co_1
	2	23	11	$\{0, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}\}$	196560	$2 \cdot \mathrm{Co}_1$
A.11	ρ	2	2	$\left\{\frac{1}{\rho+1}\right\}$	ρ^2	Weyl-Heisenburg
A.12	3	2	2	$\{\frac{1}{4}\}$	9	$\mathrm{SU}_3(2)$
A.13	8	2	2	$\{\frac{1}{9}\}$	64	$2^6:(\mathrm{PSU}_3(3):2)$
A.14	4	2	3	$\{0, \frac{1}{3}\}$	40	$\mathrm{PSU}_4(2):2$
A.15	6	2	3	$\{0, \frac{1}{4}\}$	126	$\mathrm{PSU}_4(3):2$
A.16	5	4	3	$\{0, \frac{1}{4}\}$	165	$\mathrm{PSU}_5(2)$
A.17	3	4	2	$\{\frac{2}{7}\}$	15	
A.18	3	8	2	$\{\frac{4}{13}\}$	27	
A.19	3	8	5	$\{0, \frac{1}{4}, \frac{1}{2}\}$	819	${}^3D_4(2)$

Table 1.1. Known tight t-designs with cardinality $|X|$, angle set A, and isometry group G.

Lattice Γ is an *integral lattice* when all inner products are integers, i.e. when $\Gamma \subset \Gamma^*$. The *norm* of a lattice point is the square of its length, computed as (x, x). An integral lattice must only have lattice points of integer valued norm.

1.3.2. Root Systems and Root Lattices. Certain integral lattices have reflection symmetries defined by a subset of their vectors. Specifically, vector r defines the following reflection in $\mathbb{R}^n$.

$$s_r : x \mapsto x - 2\frac{(x,r)}{(r,r)}r.$$

A root system is a subset of integral lattice vectors that define reflections. As described in [**CS13**, p. 97], a *root vector*, or root, is a vector r in integral lattice Γ for which the reflection map s_r is a reflection symmetry of Γ. A *root system* Φ is a set of roots that span Γ such that for all α, β in Φ,

(1) If α, β are linearly dependent then $\alpha = \pm\beta$.
(2) $s_\alpha(\beta)$ and $s_\beta(\alpha)$ are also in Φ.

Similar definitions are available in numerous treatments of the topic. The traditional requirement that $2(x, r)/(r, r)$ be an integer is captured above in the requirement that the roots of Φ define reflection symmetries of Γ. A root system is *irreducible* if we cannot partition it into mutually orthogonal components. The irreducible root systems are classified, with proofs available in numerous places. We summarize the classification and various properties in Table 1.2, based on [**Wil09a**, p. 33], [**Ebe13**, p. 25], and [**Car72**, chap. 3].

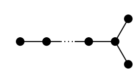
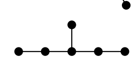
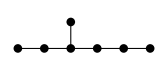
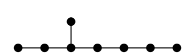
Φ	$\lvert\Phi\rvert$	h	$\Gamma(\Phi)$	$W(\Phi)$	Diagram
$\mathtt{A}_n$ $(n \geq 1)$	$n(n+1)$	$n+1$	$\Gamma(\mathtt{A}_n)$	S_{n+1}	
$\mathtt{B}_n$ $(n \geq 2)$	$2n^2$	$2n$	$\mathbb{Z}^n$	$C_2 \wr S_n$	
$\mathtt{C}_n$ $(n \geq 3)$	$2n^2$	$2n$	$\Gamma(\mathtt{D}_n)$	$C_2 \wr S_n$	
$\mathtt{D}_n$ $(n \geq 4)$	$2n(n-1)$	$2n-2$	$\Gamma(\mathtt{D}_n)$	$C_2^{n-1} : S_n$	
$\mathtt{E}_6$	72	12	$\Gamma(\mathtt{E}_6)$	$\mathrm{O}_6^-(2)$	
$\mathtt{E}_7$	126	18	$\Gamma(\mathtt{E}_7)$	$\mathrm{O}_7(2) \times 2$	
$\mathtt{E}_8$	240	30	$\Gamma(\mathtt{E}_8)$	$2 \cdot \mathrm{O}_8^+(2)$	
$\mathtt{F}_4$	48	12	$\Gamma(\mathtt{D}_4)$	$2^{1+4} : (S_3 \times S_3)$	
$\mathtt{G}_2$	12	6	$\Gamma(\mathtt{A}_2)$	D_{10}	

TABLE 1.2. Properties of irreducible root systems.

Let $\Gamma(\Phi)$ be the integral lattice spanned by root system Φ. A *simple system of roots* $\Pi \subset \Phi$ is a basis for $\Gamma(\Phi)$ with the property that every root in Φ has either all non-negative or all non-positive coefficients in the Π basis. Each root system has a corresponding simple system of roots, which we depict using a *Coxeter-Dynkin diagram*. As described in [**Wil09a**, p. 34], in such a diagram each simple root is a vertex. Orthogonal simple roots are not connected by an edge. Simple roots of equal length at an angle of 120 degrees are joined by a single undirected edge. A pair of roots of relative length $\sqrt{2}$ and angle of 135 degrees are joined by a double directed edge, pointing toward the shorter root. Finally, a pair of roots of relative length $\sqrt{3}$ and angle 110 degrees are joined by a triple directed edge, pointing toward the shorter root.

We can obtain the full root system Φ by taking the closure of its simple system under reflection. The finite group generated by all reflections defined by the roots in Φ is known as the *Weyl group* $W(\Phi)$. The *Coxeter number* h of a root system is $|\Phi|/n$, where Φ spans $\mathbb{R}^n$. Finally, the integral lattices spanned by two root systems are not necessarily distinct. In certain cases, two root systems can span the same integral lattice.

A *root lattice* is an integral lattice spanned by a root system. A root lattice is always spanned by its norm 1 and 2 vectors, although in certain systems there are roots longer than norm 2. Likewise, an integral lattice spanned by its norm 1 and 2 vectors is also a root lattice [**CS13**, chap. 4].

1.3.3. Dual Lattices and Glue Vectors. As described above, an integral lattice is always a sublattice of its dual, $\Gamma \subseteq \Gamma^*$. The quotient Γ^*/Γ is a group of order $\det \Gamma$. In the context of root lattices, Γ^*/Γ is called the *glue group* and certain special representatives $[i]$ of coset $[i] + \Gamma$ are called glue vectors [**CS13**, chap. 4]. Given a simply-laced root system for lattice $\Gamma(\Phi)$, we can define a glue vector $[i]$ in terms of a simple root system, using a modified Coxeter-Dynkin diagram. That is, *glue vector* $[i]$ is the unique vector in Γ^* orthogonal all black vertices in the diagram and having inner product 1 with the white vertex. Put another way, if we define the dual basis relative to a simple root system, the dual to the indicated root in the Coxeter-Dynkin diagram is the corresponding glue vector. Except in the case of $\mathtt{D}_n$ with n even, we can compute $([i]+\Gamma)+([j]+\Gamma) = [i+j]+\Gamma$, where $i+j$ is evaluated modulo $\det \Gamma(\Phi)$. In the case of $\mathtt{D}_n$ with n even the glue group is the Klein group V_4, with $[i]+[i] = [0]$ and $[1]+[2] = [3]$ for all permutations of $1, 2, 3$. The glue group properties for irreducible root lattices are collected in Table 1.3.

1.3.4. One Line Extensions. There are numerous ways to classify root lattices with minimal norm 2 in the literature. One interesting method proceeds as follows: since the inner products between linearly independent roots must have values $(x, y) = -1, 0, 1$, the problem is equivalent to classifying systems of lines in $\mathbb{R}^n$ where any two lines are either orthogonal or form an angle of 60 degrees. This method was first employed in [**CGSS76**], in connection

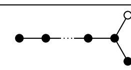
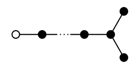
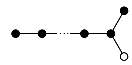
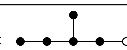
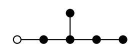
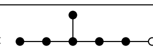
Φ	$\det\Gamma(\Phi)$	$\Gamma^*(\Phi)/\Gamma(\Phi)$	$[i]$	$([i],[i])$
$\mathtt{A}_n$	$n+1$	C_{n+1}	$[i] =$	$\frac{i(n+1-i)}{n+1}$
$\mathtt{D}_n$	4	V_4 (n even)	$[1] =$ 	$\frac{n}{4}$
		C_4 (n odd)	$[2] =$ 	1
			$[3] =$ 	$\frac{n}{4}$
$\mathtt{E}_6$	3	C_3	$[1] =$ 	$\frac{4}{3}$
			$[2] =$ 	$\frac{4}{3}$
$\mathtt{E}_7$	2	C_2	$[1] =$ 	$\frac{3}{2}$
$\mathtt{E}_8$	1	1		

TABLE 1.3. Glue vectors of irreducible root lattices

with the problem of classifying graphs with least eigenvalue -2. It is also described at length in [**CVL91**, chap. 3], [**GR01**, chap. 12], and [**CRS04**, chap. 3].

The treatment in [**CRS04**, chap. 3] makes use of a process called *one line extension*, in which a single line is added at 60 or 90 degrees to all previous lines and then a closure operation is conducted. The corresponding process in a irreducible root lattice involves finding a glue vector $[i]$ of norm at most 2. For u a norm 1 vector orthogonal to all of the roots, we construct a new norm 2 vector of the form $u\sqrt{2-([i],[i])}-[i]$ and append it to the simple roots. Together they span a new irreducible root system. Indeed, this new root defines the line of the one-line extension in [**CRS04**, chap. 3]. All possible one line extensions are given in Figure 1.1 with a representative glue vector labeling the arrow from the old to the new root system.

It is interesting to examine the one-line extensions that yield the exceptional root systems $\mathtt{E}_6$, $\mathtt{E}_7$, and $\mathtt{E}_8$. For example, the root systems of type $\mathtt{A}_n$ have the symmetric groups for Weyl reflection groups, $W(\mathtt{A}_n) = S_{n+1}$. The exceptional symmetric group, S_6, is the Weyl group of type $\mathtt{A}_5$, which is the first root system with a glue vector of type [3]. This new glue vector defines a third class of one-line extension, yielding $\mathtt{E}_6$. Indeed, an examination of Table 1.3 and Figure 1.1 shows that the $\mathtt{E}_n$ series does not continue beyond $\mathtt{E}_8$ because we lack glue vectors with norm at most 2 in larger root systems to carry out the needed one-line extensions. As we move up the $\mathtt{A}_n$ and $\mathtt{D}_n$ series, the temporary existence of glue vectors of suitable norms explains the exceptional one-line extensions yielding root systems of types $\mathtt{E}_6$, $\mathtt{E}_7$, and $\mathtt{E}_8$.

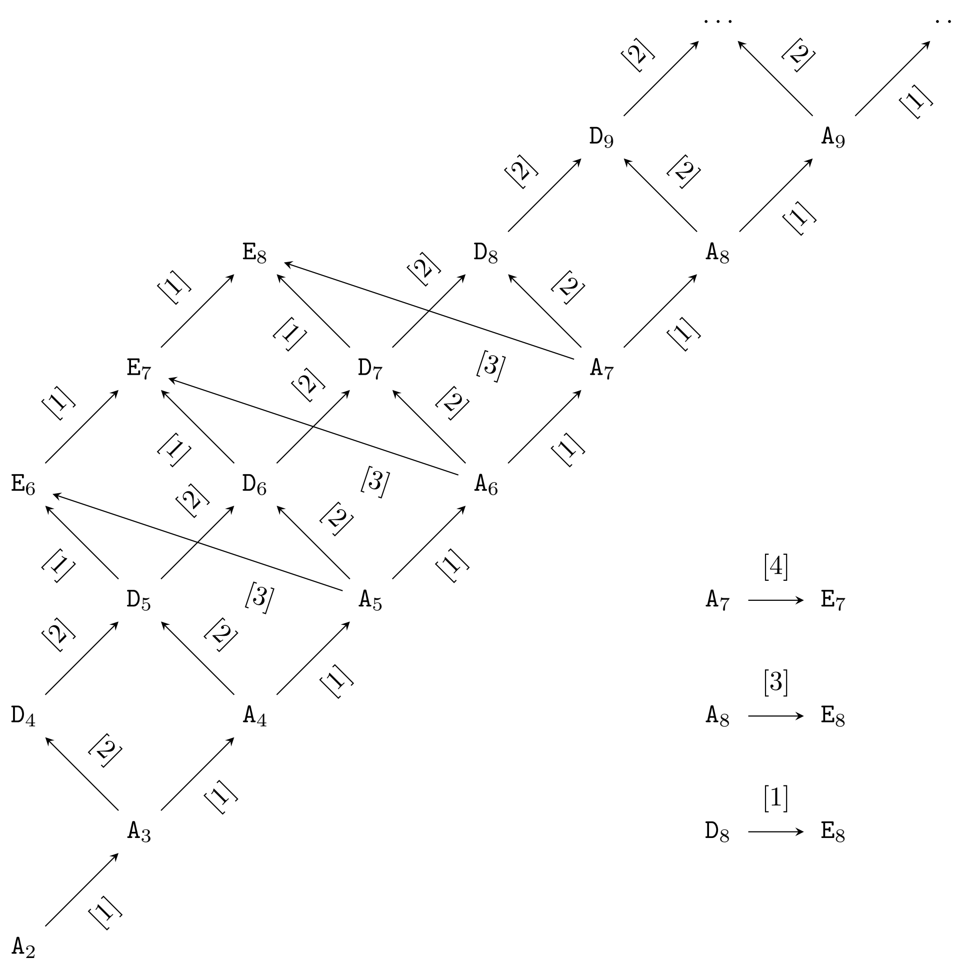


FIGURE 1.1. One line extension construction of irreducible root lattices

1.3.5. Three-Gradings. Three-gradings of root systems are described in detail in [**LN04**]. Each glue vector of an irreducible root system has the property that it is orthogonal to all simple roots except for one. Accordingly, for root system Φ with glue vector $[i]$, there exists a partition,

$$\Phi = \Phi_1 \dot\cup \Phi_0 \dot\cup \Phi_{-1}, \quad \Phi_n = \{\alpha \in \Phi \mid (\alpha, [i]) = n\}\,.$$

This partition is a *three-grading* on root system Φ, since we have $\Phi_i + \Phi_j \subset \Phi_{i+j}$, where $\Phi_k = \varnothing$ for $k \neq -1, 0, 1$. In fact, all three-gradings on a root system are defined in this way—in terms of a glue vector of the corresponding root lattice.

In addition to the glue vectors given in Table 1.3, we need to address three-gradings on root systems of type $\mathtt{B}_n$ and $\mathtt{C}_n$, which both contain $\mathtt{D}_n$ as a subsystem. In the case of $\mathtt{B}_n$ the three-grading due to the glue vector [2] of $\mathtt{D}_n$ also defines a three-grading on $\mathtt{B}_n$. In the case of $\mathtt{C}_n$ the three-grading due to the glue vectors [1] or [3] of $\mathtt{D}_n$ also defines a three-grading on $\mathtt{B}_n$. Respectively, these three-gradings correspond to the following diagrams:

The classification of three-gradings on irreducible root systems is depicted in Figure 1.2. Of note, the only irreducible root systems that lack a three-grading are the systems of types $\mathtt{E}_8$, $\mathtt{F}_4$, and $\mathtt{G}_2$.

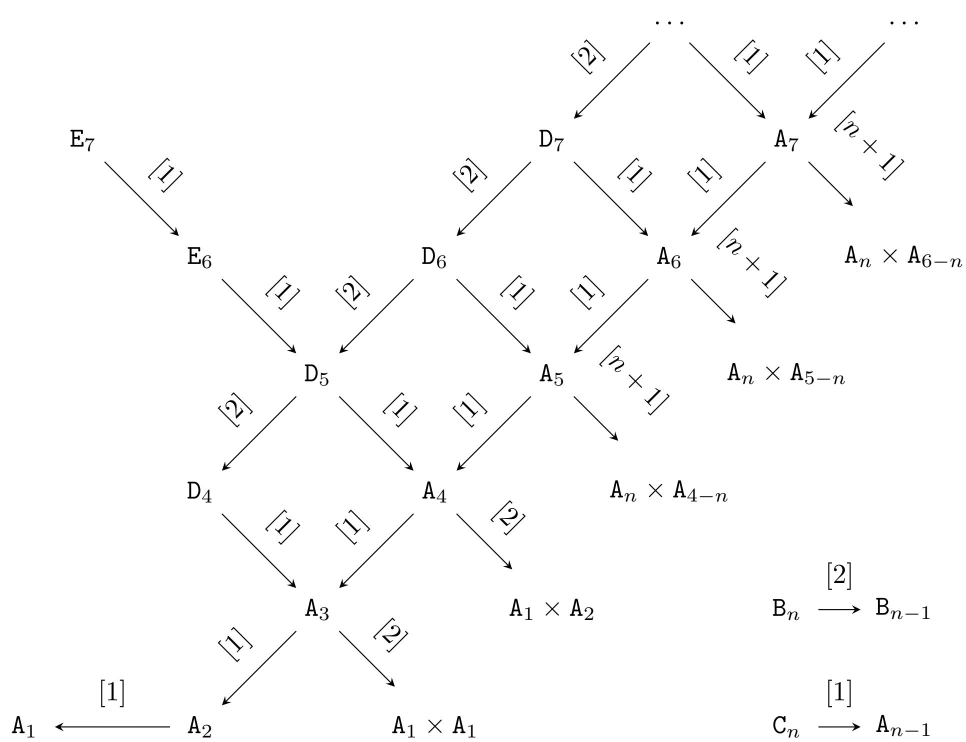


FIGURE 1.2. Three-gradings of irreducible root systems

In Chapter 3 we will examine the structure of Figure 1.2 more closely and define sequences of three-gradings on root systems. Our aim will be to obtain an exceptional sequence that terminates in the root lattice of the Lie algebra of the standard model of particle physics.

1.3.6. Construction of Unimodular Lattices. In special cases an integral lattice can be self-dual, so that not only do we have $\Gamma \subseteq \Gamma^*$ but also $\Gamma^* \subseteq \Gamma$. A self-dual integral lattice is called a *unimodular lattice*. An integral lattice is *even* when the norm of each vector is an even integer, i.e. when (x, x) is an even integer for each lattice point x in Γ. We are interested in constructing *even unimodular lattices*. Even unimodular lattices only exist in $\mathbb{R}^n$ when $n \equiv 0 \pmod 8$ [**CS13**, p. 192].

Begin with $n = 8$. If Γ is an even unimodular lattice then Γ is the $\mathtt{E}_8$ root lattice [**Ebe13**, p. 52]. We can construct an even unimodular lattice by taking the $\mathbb{Z}$-span of the $\mathtt{D}_8$ roots and the norm 2 glue vector [1] or [3], but not both.

$$\mathtt{E}_8 \cong \mathtt{D}_8 \cup (\mathtt{D}_8 + [1]) \cong \mathtt{D}_8 \cup (\mathtt{D}_8 + [3]).$$

Likewise, we can construct an even unimodular lattice by taking the $\mathbb{Z}$-span of the $\mathtt{A}_8$ roots and the glue vector [3] (or equivalently [6]).

$$\mathtt{E}_8 \cong \mathtt{A}_8 \cup (\mathtt{A}_8 + [3]) \cong \mathtt{A}_8 \cup (\mathtt{A}_8 + [6]).$$

In general, for $n = 8, 16, 24$, the sublattice of an even unimodular lattice generated by the roots (which is not necessarily a unimodular sublattice) is such that the irreducible components each have the same Coxeter number h and the number of roots $|\Phi| = hn$ [**Ebe13**, p. 89]. This ensures that for $n = 16$ there are only two possible even unimodular lattices, with root sublattices either $\mathtt{E}_8 \times \mathtt{E}_8$ or $\mathtt{D}_{16}$. The root lattice $\mathtt{E}_8 \times \mathtt{E}_8$ is already unimodular. To obtain an even unimodular lattice from $\mathtt{D}_{16}$ we use either the [1] or [3] glue vector (which are both norm 4 glue vectors):

$$\Gamma = \mathtt{D}_{16} \cup (\mathtt{D}_{16} + [1]) \cong \mathtt{D}_{16} \cup (\mathtt{D}_{16} + [3]).$$

For $n = 24$ there are 23 non-empty root systems that span $\mathbb{R}^{24}$ and have the same Coxeter number for each irreducible component. The even unimodular lattices constructed from each of these are called the *Niemeier lattices* with roots (the Leech lattice lacks any roots and is also a Niemeier lattice). Apart from the Leech lattice, there are 23 Niemeier lattices, and their construction via glue vectors is given in [**CS13**, chap. 16]. Furthermore, each Niemeier lattice with roots admits a construction of the Leech lattice. Indeed, each set of simple roots for the root sublattice of a Niemeier lattice supplies a construction of the Leech lattice. These constructions are described in [**CS13**, chap. 24].

1.3.7. Summary. We have seen that the smallest tight projective 5-design (the regular hexagon of Example A.2) is fundamental to the theory of root systems. Using the one-line extension process, as depicted in Figure 1.1, we can recover all indecomposible simply-laced root systems from the $\mathtt{A}_2$ root system. These root systems define the root lattices, which in turn allow us to construct the 24 Niemeier lattices, including the Leech lattice. As described in Appendix A, it turns out that orbits of various glue vectors of root systems define infinite families of tight 2-designs and 3-designs, as well as a tight 4-design, and a tight 5-design. The Leech lattice and its substructures are responsible for most of the remaining known spherical tight t-designs.

1.4. Lie, Jordan, and Composition Algebras

Root systems determine the structure of an important family of non-associative algebras, the Lie algebras. In this section we review the relation of root systems to Lie algebras. We then describe how to recover Jordan structures (including Jordan algebras) and composition algebras. This will clarify the known connections between various exceptional structures. These connections are clearest when we work with Lie, Jordan, and composition structures over the complex numbers $\mathbb{C}$.

1.4.1. Lie Algebras and Root Systems. A Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is an algebra in which all squares vanish and multiplication is an algebra derivation. Every simple Lie algebra $\mathfrak{g}$ admits a grading known as a *Cartan decomposition*, which has the form,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{r \in \Phi} \mathfrak{g}_r.$$

Here $\mathfrak{h}$ is a *Cartan subalgebra* of $\mathfrak{g}$, Φ is an irreducible root system, and $\mathfrak{g}_r$ are the *root spaces* of the decomposition. The dimension of $\mathfrak{h}$ is equal to the dimension of the space $\mathbb{R}^n$ spanned by the roots Φ, whereas the dimension of each root space $\mathfrak{g}_r$ is 1. The *rank* of the Lie algebra is the dimension of $\mathfrak{h}$.

We can construct the simple Lie algebra corresponding to any root system by defining a basis that exhibits this grading. Such a basis is known as a *Chevalley basis*, and may be constructed as follows. Each subspace $\mathfrak{g}_r$ is spanned by basis vector e_r, where r is a root in Φ. The Cartan subalgebra $\mathfrak{h}$ is spanned by the *co-roots* of Φ, written $h_r = 2r/(r,r)$. In general, we choose a simple system of co-roots for a basis of Cartan subalgebra $\mathfrak{h}$. For any x not in $\Phi \cup \{0\}$ we have $\mathfrak{g}_x = 0$. The products involving co-roots are defined entirely in terms of the geometry of the roots r, s in Φ:

$$[h_r, h_s] = 0, \qquad [h_r, e_s] = \frac{2(r,s)}{(r,r)} e_s, \qquad [e_r, e_{-r}] = h_r.$$

If two roots are linearly independent, then the corresponding basis vectors involve structure constants $N_{r,s}$:

$$[e_r, e_s] = N_{r,s} e_{r+s}.$$

We have $N_{r,s} = 0$ whenever $r + s$ is not a root. Otherwise, $N_{r,s} = \pm(p+1)$ for $p = \max\{q \in \mathbb{Z} \mid s - qr \in \Phi\}$. The axioms of a Lie algebra place certain additional constraints on the structure constants $N_{r,s}$, but this construction of a simple Lie algebra from the roots of an irreducible system is unique up to isomorphism.

THEOREM 1.1. [**Car72**, pp. 42-43] *Let Φ be an irreducible root system. Then there exists, up to Lie algebra isomorphism, a unique simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ with a Chevalley basis.*

This means that we can represent a simple Lie algebra using a Coxeter-Dynkin diagram of an irreducible root system. Table 1.4 lists some data about the simple Lie algebras, taken from [**Car72**, p. 43]. We see that the simple Lie algebras are in bijection with the irreducible root systems.

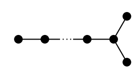
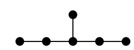
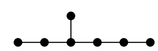
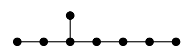
Type	$\mathfrak{g}$	dim $\mathfrak{g}$	rank $\mathfrak{g}$	$\lvert\Phi\rvert$	Dynkin diagram
$\mathtt{A}_n$ $(n \geq 1)$	$\mathfrak{sl}(n+1)$	$n(n+2)$	n	$n(n+1)$	
$\mathtt{B}_n$ $(n \geq 2)$	$\mathfrak{so}(2n+1)$	$n(2n+1)$	n	$2n^2$	
$\mathtt{C}_n$ $(n \geq 3)$	$\mathfrak{sp}(2n)$	$n(2n+1)$	n	$2n^2$	
$\mathtt{D}_n$ $(n \geq 4)$	$\mathfrak{so}(2n)$	$n(2n-1)$	n	$2n(n-1)$	
$\mathtt{E}_6$	$\mathfrak{e}_6$	78	6	72	
$\mathtt{E}_7$	$\mathfrak{e}_7$	133	7	126	
$\mathtt{E}_8$	$\mathfrak{e}_8$	248	8	240	
$\mathtt{F}_4$	$\mathfrak{f}_4$	52	4	48	
$\mathtt{G}_2$	$\mathfrak{g}_2$	14	2	12	

TABLE 1.4. Classification of simple Lie algebras over $\mathbb{C}$.

An important property of the Chevalley basis is that the structure constants of the Lie algebra with respect to this basis are integers [**Car72**, p. 57]. This property makes it possible to construct several families of finite simple groups, known as the *Chevalley groups*, as described in [**Car72**]. Indeed for each simple Lie algebra over $\mathbb{C}$, and for each field K, there exists a corresponding Chevalley group. These groups are finite when the field K is finite. This means that for each prime power q and each irreducible root system Φ we can construct a finite simple group, the corresponding Chevalley group.

1.4.2. Jordan Structures. We now review Jordan structures from the standpoint of the Cartan grading on the corresponding Lie algebra. We will show how to construct a Jordan pair, Hermitian Jordan triple system, and Jordan algebra from a three-graded simple Lie algebra.

As depicted in Figure 1.2, every simple Lie algebra except for $\mathfrak{g}_2$, $\mathfrak{f}_4$, and $\mathfrak{e}_8$ admits a three-grading, corresponding to a three-grading on the corresponding irreducible root system. Let $\Phi = \Phi_1 \dot{\cup} \Phi_0 \dot{\cup} \Phi_{-1}$ be a three-grading on irreducible root system Φ. The corresponding simple Lie algebra three-grading is a coarsening of the familiar Cartan grading:

$$\mathfrak{g} = \left(\bigoplus_{r\in\Phi_1} \mathfrak{g}_r \right) \oplus \left(\mathfrak{h} \oplus \bigoplus_{s\in\Phi_0} \mathfrak{g}_s \right) \oplus \left(\bigoplus_{t\in\Phi_{-1}} \mathfrak{g}_t \right) = \mathfrak{g}(1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(-1).$$

This three-grading satisfies,

$$[\mathfrak{g}(i), \mathfrak{g}(j)] \subseteq \mathfrak{g}(i+j).$$

Jordan structures (including Jordan pairs, Jordan triple systems, and Jordan algebras) can be understood in terms of this coarsening of a Cartan grading. In particular, Jordan structures emerge from the fact that we also have,

$$[[\mathfrak{g}(i), \mathfrak{g}(j)], \mathfrak{g}(k)] \subseteq \mathfrak{g}(i+j+k).$$

This means that the $\mathfrak{g}(\pm 1)$ components satisfy,

$$[[\mathfrak{g}(\pm 1), \mathfrak{g}(\mp 1)], \mathfrak{g}(\pm 1)] \subseteq \mathfrak{g}(\pm 1).$$

This expression shows that $\mathfrak{g}(1)$ and $\mathfrak{g}(-1)$ have a special relationship. The concept of a Jordan pair, introduced axiomatically by Loos, captures this relationship [**Loo75**].

A *Jordan pair* consists of two vector spaces and two trilinear maps satisfying certain axioms. The two vector spaces are the pair $(\mathfrak{g}(1), \mathfrak{g}(-1))$. The needed trilinear maps must have structure $\{\cdot,\cdot,\cdot\}_\sigma : \mathfrak{g}(\sigma) \times \mathfrak{g}(-\sigma) \times \mathfrak{g}(\sigma) \to \mathfrak{g}(\sigma)$, for $\sigma = \pm 1$. We define both maps at once in terms of the Lie product as,

$$\{x, y, z\}_\sigma = \frac{1}{2}[[x, y], z].$$

The factor of $\frac{1}{2}$ makes defining idempotents and tripotents more convenient below. This construction satisfies the axioms of a Jordan pair. Indeed, the simple Jordan pairs are classified in [**Loo75**, pp. 195-201] and listed in Table 1.5. They correspond precisely to the three-gradings of irreducible root lattices.

A *Jordan pair idempotent* (f^+, f^-) is an element (pair of vectors) that satisfies $\{f^\sigma, f^{-\sigma}, f^\sigma\}_\sigma = f^\sigma$ [**Loo75**, p. vii]. Using the Cartan grading of a simple Jordan pair described above, we see that (e_s, e_{-s}) with $s \in \Phi_1$ is an idempotent since for any s in Φ_σ,

$$\{e_s, e_{-s}, e_s\}_\sigma = \frac{1}{2}[[e_s, e_{-s}], e_s] = \frac{1}{2}[h_s, e_s] = \frac{1}{2}\left(\frac{2(s,s)}{(s,s)}\right) e_s = e_s.$$

We define a *Jordan grid* as the idempotents $\{(e_r, e_{-r})_\sigma \mid r \in \Phi_\sigma\}$ corresponding to the Cartan decomposition of $\mathfrak{g}(\sigma)$. Neher introduced Jordan grids and

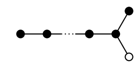
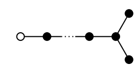
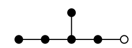
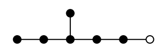
Type	Roots	dim $\mathfrak{g}(\sigma)$	rank $\mathfrak{g}(\sigma)$	Unital	Diagram
$\mathrm{I}_{p,q}$	$\mathtt{A}_{p+q-1} \to \mathtt{A}_{p-1}\mathtt{A}_{q-1}$	pq	p	$p = q$	
II_n	$\mathtt{D}_n \to \mathtt{A}_{n-1}$	$\binom{n}{2}$	$\left\lfloor \frac{n}{2} \right\rfloor$	n even	
III_n	$\mathtt{C}_n \to \mathtt{A}_{n-1}$	$\binom{n+1}{2}$	n	true	
IV_{2m}	$\mathtt{D}_{m+1} \to \mathtt{D}_m$	$2m$	2	true	
IV_{2m-1}	$\mathtt{B}_m \to \mathtt{B}_{m-1}$	$2m-1$	2	true	
V	$\mathtt{E}_6 \to \mathtt{D}_5$	16	2	false	
VI	$\mathtt{E}_7 \to \mathtt{E}_6$	27	3	true	

TABLE 1.5. Classification of simple Jordan pairs over $\mathbb{C}$.

classified Jordan triple systems axiomatically using this concept in [**Neh87**]. Further details are available in [**LN04**].

A *Jordan triple system* is a Jordan pair with the additional structure of an involution that swaps the vector spaces of the pair. This permits us to use one vector space and one triple product, rather than two of each. Different choices of involutions yield different Jordan triple systems, so a Jordan pair is more basic than a Jordan triple system. Given involution $\theta : \mathfrak{g}(\sigma) \to \mathfrak{g}(-\sigma)$, the corresponding Jordan triple system is the single vector space $\mathfrak{g}(\sigma)$ and single triple product $\{\cdot,\cdot,\cdot\} : \mathfrak{g}(\sigma) \times \mathfrak{g}(\sigma) \times \mathfrak{g}(\sigma) \to \mathfrak{g}(\sigma)$ of the form,

$$\{x, y, z\} = \frac{1}{2}[[x, \theta(y)], z].$$

The Cartan grading on $\mathfrak{g}(\sigma)$ and Chevalley basis provides a natural definition of θ. Specifically, we set $\theta(\lambda e_r) = \overline{\lambda} e_{-r}$, where λ is a complex scalar and e_r is the Chevalley basis vector spanning the root space $\mathfrak{g}_r$. When a Jordan triple system is defined in terms of this involution θ, it is called a *Hermitian Jordan triple system*. Simple Hermitian Jordan triple systems correspond precisely to simple Jordan pairs and three-graded simple Lie algebras [**FKK$^+$00**, p. 525], so Figure 1.5 also lists the Hermitian Jordan triples systems.

Although Jordan algebras were invented first [**JvNW34**], they are the last Jordan structure we discuss in our constructive approach. Given a Jordan triple system V containing vector a, we define a *Jordan algebra* on V with product $\circ_a$ as follows:

$$x \circ_a y = \{x, a, y\}.$$

Likewise, given a Jordan algebra we can recover the underlying triple system via the expression,

$$\{x, y, z\} = (x \circ_a y) \circ_a z + x \circ_a (y \circ_a z) - (x \circ_a z) \circ_a y.$$

So defined, this Jordan algebra may or may not include an identity element e. Regardless of whether the algebra $(V, \circ_a)$ has an identity element, it is commutative and satisfies the traditional Jordan identity, $x(x^2y) = x^2(xy)$ [**FKK$^+$00**, chap. V:II]. This identity specifies that multiplication by x commutes with multiplication by x^2.

A *unital Jordan algebra* is a Jordan algebra with identity element e. To construct a unital Jordan algebra we require that the underlying Jordan triple system contain a *unitary tripotent*: an element e in V that satisfies $\{e, e, e\} = e$ and also $\{e, e, x\} = x$ for all x in V. Given unitary tripotent e, the Jordan product $x \circ_e y = \{x, e, y\}$ defines a unital Jordan algebra on V. The Lie algebra 3-gradings that admit the construction of a unital Jordan algebra are specified in Table 1.5.

It turns out that every complex simple unital Jordan algebra is the complexification of an underlying simple Euclidean Jordan algebra [**FK94**, p. 155]. To recover a Euclidean Jordan algebra from a unital Jordan algebra over $\mathbb{C}$ we simply take all $\mathbb{R}$-linear combinations of primitive idempotents. Of note, the manifolds of primitive idempotents correspond to the following Jordan pair classification types, as depicted in Table 1.5:

$$\Omega_{d+1} = \mathrm{IV}_{d+2}, \quad \mathbb{RP}^{\rho-1} = \mathrm{III}_{\rho}, \quad \mathbb{CP}^{\rho-1} = \mathrm{I}_{\rho,\rho}, \quad \mathbb{HP}^{\rho-1} = \mathrm{II}_{2\rho}, \quad \mathbb{OP}^2 = \mathrm{VI}.$$

Euclidean Jordan algebras are particularly interesting because there exists a one-to-one correspondence between them and symmetric cones [**FK94**, chap. III]. For our purposes, the primitive idempotents of a simple Euclidean Jordan algebra form a compact Riemannian symmetric space of rank 1, and all such spaces can be obtained in this way as manifolds of primitive idempotents [**FK94**, p. 99]. We are interested in the properties of tight t-designs on these spaces.

1.4.3. Quadratic Maps and Jordan Isotopes. A Jordan algebra is commutative and power associative, but non-associative in general. In order to avoid complications related to non-associativity, it is convenient to use the quadratic representation of a Jordan algebra. Let V be a simple Euclidean Jordan algebra with identity e and let $L(x)y = x \circ y$, so that $L(x)$ is the left-translation map of x. The *quadratic operator* of x is defined as [**FK94**, p. 32],

$$P(x) = 2L(x)^2 - L(x^2).$$

The quadratic operator satisfies the following identity [**FK94**, p. 33]:

$$P(P(x)y) = P(x)P(y)P(x).$$

Whenever $P(x)$ is an invertible map in $\mathrm{End}(V)$ we can define a *Jordan inverse* $x^{-1} = P(x)^{-1}x$. (There are subtle differences between the concept of a Jordan inverse and the more familiar algebra inverse element [**FK94**, pp. 30-31]. We denote by x^{-1} the Jordan inverse in what follows.) The subset of *invertible elements* in V is precisely the subset of elements for x which the quadratic map $P(x)$ is invertible.

A Euclidean Jordan algebra also has well defined trace and determinant. In terms of quadratic map $P(x)$ and translation map $L(x)$, we can define the *Jordan trace* as $\mathrm{tr}(x) = \frac{\rho}{n}\mathrm{Tr}L(x)$ and the *Jordan determinant* as $\det(x) = (\mathrm{Det}P(x))^{\rho/2n}$, where ρ is the rank of V and $n = \dim(V)$ [**FK94**, p. 52]. Accordingly, the trace of the identity is $\mathrm{tr}(e) = \rho$ and the invertible elements of V are precisely the elements x with $\det(x) \neq 0$. Another important property of a Jordan algebra is the *composition rule* [**FK94**, p. 52], [**McC04**, p. 75]:

$$\det(P(x)y) = \det(x)^2 \det(y).$$

This composition rule ensures that the set of invertible elements in V is closed under the mapping $(x, y) \mapsto P(x)y$ [**FK94**, p. 33]. Likewise, the elements with determinant 1 are also closed under this mapping.

For any a in V we can define a new Jordan algebra $V^{(a)}$ with a modified product $\circ_a$ on V as follows [**McC04**, p. 86],

$$x \circ_a y = \{x, a, y\} = x \circ (a \circ y) + (x \circ a) \circ y - a \circ (x \circ y),$$

The algebra $V^{(a)}$ is called the *a-homotope* of V. The algebra $V^{(a)}$ is a Jordan algebra but it is only unital when a is invertible. When a is invertible the element a^{-1} is the identity of $V^{(a)}$ and we call $V^{(a)}$ the *a-isotope* of V. In either case, the algebra $V^{(a)}$ has a simple expression for the corresponding quadratic operator:

$$P^{(a)}(x) = P(x)P(a).$$

A square in V (which has identity e) has the form $x \circ_e x = P(x)e$ for some x in V. In $V^{(a)}$ (which has identity a^{-1}) a square relative to product $\circ_a$ is defined as $P(x)a = P(x)P(a)a^{-1} = P^{(a)}(x)a^{-1}$. Homotopy and isotopy are reflexive and transitive relations:

$$V^{(e)} = V, \quad (V^{(a)})^{(b)} = V^{(P(a)b)}.$$

When a is invertible $P(a)^{-1}$ exists and we can use $b = P(a)^{-1}e$ such that $(V^{(a)})^{(P(a)^{-1}e)} = V^{(e)}$. That is, for a invertible, $V^{(a)}$ is the a-isotope of V and V is the $P(a)^{-1}e$-isotope of $V^{(a)}$.

The *symmetric cone* Ω in V is the orbit of identity e under the action of $P(x)$, for any invertible x in V [**FK94**, p. 48]. Any element a^{-1} in the symmetric cone is the identity element of the a-isotope algebra. For any a^{-1}

in Ω there exists an invertible x such that $a^{-1} = P(x)e$. This map $P(x)$ gives the isomorphism of Jordan algebras between V and $V^{(a)}$.

1.4.4. Composition Algebras. A *composition algebra* is an algebra over some field equipped with a non-degenerate quadratic form N on the underlying vector space that satisfies the composition law $N(xy) = N(x)N(y)$. A *division composition algebra* has the additional property that $N(x) = 0$ only when $x = 0$. There are precisely four division composition algebras over $\mathbb{R}$, namely $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. These are normally used to construct the Euclidean Jordan algebras $\mathrm{Herm}(\rho, \mathbb{F})$ (with $\rho = 2, 3$ for $\mathbb{F} = \mathbb{O}$).

However, for a unital Jordan algebra of rank $\rho \geq 3$, there are many ways to recover the underlying composition algebra. This process of obtaining a composition algebra from a Jordan algebra V of rank $\rho = 3$ is described carefully in [**SV00**, chap. 5] and also for higher ranks in [**FK94**, chap. 5]. We do not define this composition algebra construction here, but the process is similar to defining explicit matrices for a vector space of linear transformations. This process involves selecting a non-unique basis (resulting in a Pierce decomposition of the Jordan algebra) and specifying an identity element (since the off-diagonal entries had dimension degree $d = 1, 2, 4, 8$). There are many ways to do this, all of which result in the same composition algebra up to isomorphism for a particular Jordan algebra $\mathrm{Herm}(3, \mathbb{F})$. One might say, in the pattern of Kantor and McCrimmon, that there are no composition algebras but only Jordan algebras; yet usually (i.e. for $\rho \geq 3$) when you open up a Jordan algebra you find a composition algebra inside that makes it tick.

1.4.5. Summary. This section has described the path from three-gradings on root systems through Lie algebras, to Jordan structures, and finally to composition algebras. A more familiar path is to begin with composition algebras, construct Jordan algebras, and then recover Lie algebras. In Chapter 3 we will examine the correspondence of exceptional sequences of root systems to sequences of Lie algebras. In Chapter 5 we will use primitive idempotents u of Euclidean Jordan algebras to define reflection elements of the form $e - 2u$ that act by right multiplication on row vectors in $\mathbb{F}^{\rho}$ and by the quadratic map $P(e - 2u)$ on the corresponding projective space $\mathbb{FP}^{\rho-1}$. In Chapter 6 we will explore integer subrings of the octonions and the Jordan algebra $\mathrm{Herm}(3, \mathbb{O})$. We will also use the concept of a-isotope algebras to generate non-isomorphic Jordan integer rings that exhibit structure related to the two strictly projective tight 5-designs.

CHAPTER 2

Literature Survey

This chapter describes the literature pertinent to the research outlined in the following chapters. We begin with a description of recent efforts to use exceptional structures in mathematics to explain certain features of the standard model of particle physics, preparing for the approach described in Chapter 3. We then briefly discuss a gap in the literature regarding the angle sets of tight t-designs, which we address in Chapter 4. Finally, to prepare for the techniques explored in Chapters 5 and 6, we explore octonion and exceptional Jordan algebra approaches to the Leech lattice and the tight 5-design in the octonion projective plane.

2.1. The Standard Model and Exceptional Explanation

In what follows we will describe the basic terminology and structure of the standard model of particle physics. We will then describe a number of attempts to explain seemingly accidental or surprising features of the standard model using exceptional mathematical objects.

2.1.1. The Standard Model Lie Algebra and Nomenclature. The standard model is described fairly clearly in [**Sch18**], and in a form that best suits our purposes by Baez and Huerta in [**BH10**]. Our focus is to describe the standard model particles as a suitable complex representation of the Lie algebra of the standard model and to link that representation to various exceptional structures in mathematics. We will not address the dynamics, particle masses, or Higgs mechanism of the standard model.

The standard model of particle physics describes a particular Lie group G_{SM} with the following 12 dimensional Lie algebra over $\mathbb{C}$:

$$\mathfrak{g}_{SM} = \mathbb{C} \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_3.$$

In the standard model, physical particles correspond to specific representations of this Lie algebra. A *Lie algebra representation* of Lie algebra $\mathfrak{g}$ is a vector space V and homomorphism $\phi : \mathfrak{g} \to \mathrm{End}(V)$ such that for any x, y in $\mathfrak{g}$ we have,

$$\phi([x, y]) = [\phi(x), \phi(y)] = \phi(x)\phi(y) - \phi(y)\phi(x).$$

Here $[x, y]$ denotes the Lie product in $\mathfrak{g}$ and $\phi(x)\phi(y)$ denotes the composition of the two endomorphisms $\phi(x)$ and $\phi(y)$.

Since the rank of $\mathfrak{g}_{SM}$ is 4, any representation (ϕ, V) must contain four commuting endomorphisms in the image of the four-dimensional Cartan subalgebra of $\mathfrak{g}_{SM}$. A conventional basis uses a *hypercharge operator* B that spans the $\mathbb{C}$ component, a *weak isospin operator* W_0 that spans the Cartan subalgebra of the $\mathfrak{sl}_2$ component, and the pair of orthogonal *colour charge operators* λ_3 and $\sqrt{3}\lambda_8$ that span the Cartan subalgebra of the $\mathfrak{sl}_3$ component. For V a representation of $\mathfrak{g}_{SM}$, a simultaneous eigenvector v in V of these four commuting endomorphisms corresponds to a *particle* with some particular hypercharge, isospin, and colour. The corresponding *antiparticle* of v is the vector with the opposite eigenvalues.

In general, particles of the adjoint representation $V = \mathfrak{g}_{SM}$ are called *bosons* and (aside from the Higgs boson), particles of other representations are called *fermions*. A fermion is called *right-handed* if it belongs to a trivial representation of the $\mathfrak{sl}_2$ component, with isospin 0, and called *left-handed* if it belongs to a two dimensional representation of $\mathfrak{sl}_2$, with isospin $\pm\frac{1}{2}$. A fermion is called a *lepton* (either an electron or neutrino) if it belongs to the trivial representation of the $\mathfrak{sl}_3$ component, being colourless. A fermion is called a *quark* (either up or down) if it belongs to a three-dimensional representation of $\mathfrak{sl}_3$, in which case the eigenvalue pairs of λ_3 and $\sqrt{3}\lambda_8$ form one of three *colours* or their anti-colour. The hypercharge eigenvalue of B determines how a fermion represents the $\mathbb{C}$ component. The *electric charge* of a particle is the eigenvalue of $Q = \frac{B}{2} + W_0$. As described in [**BH10**], standard model fermions are the following representations of $\mathfrak{g}_{SM} = \mathbb{C} \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_3$:

- Left-handed leptons: $\mathbb{C}_{-1} \otimes \mathbb{C}^2 \otimes \mathbb{C}$,
- Left-handed quarks: $\mathbb{C}_{\frac{1}{3}} \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$,
- Right-handed neutrino: $\mathbb{C}_0 \otimes \mathbb{C} \otimes \mathbb{C}$,
- Right-handed electron: $\mathbb{C}_{-2} \otimes \mathbb{C} \otimes \mathbb{C}$,
- Right-handed up quarks: $\mathbb{C}_{\frac{4}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3$,
- Right-handed down quarks: $\mathbb{C}_{-\frac{2}{3}} \otimes \mathbb{C} \otimes \mathbb{C}^3$.

The *fermion representation* of $\mathfrak{g}_{SM}$ is the direct sum of these six irreducible representations, a sixteen dimensional reducible representation.

A *generation* of particles corresponds to a representation ρ_{SM} of $\mathfrak{g}_{SM}$ that contains the sixteen dimensional fermion representation and their corresponding antiparticles. Accordingly, ρ_{SM} is a 32-dimensional representation of $\mathfrak{g}_{SM}$. The standard model includes three generations of fermions, so the full fermion content of the standard model is a 96-dimensional representation $\rho_{SM} \oplus \rho_{SM} \oplus \rho_{SM}$. The *second generation* replaces electrons with muons, electron neutrinos with muon neutrinos, up quarks with charm quarks, and down quarks with strange quarks. The *third generation* replaces the electron with the tau, electron neutrino with tau neutrino, up quarks with top quarks, and down quarks with bottom quarks. Although the particle masses differ between the generations, the corresponding particles in each of the three generations have the same hypercharge, weak isospin, and colour.

2.1.2. Open Questions. The standard model of particle physics is empirically motivated and verified, although right-hand neutrinos (due to their null eigenvalues) have not been observed. This experimental foundation leaves some explanatory questions unanswered. As described in [**Sch18**], [**Boy20**], [**Fur18**], and [**Kra21**] (among others), the following questions are still open:

- Why does experiment support this symmetry group G_{SM}, with Lie algebra $\mathfrak{g}_{SM}$, and not some other?
- Why does experiment observe this representation ρ_{SM} for a generation of fermions and not some other?
- Why does experiment observe three generations ρ_{SM} of fermions and not some other number of generations?

In each case, certain facts about the standard model seem arbitrary or surprising.

2.1.3. Grand Unified Theories and Explanation. In order to better explain these seemingly arbitrary facts about the standard model, many researchers have sought to construct grand unified theories by embedding $\mathfrak{g}_{SM}$ in a larger Lie algebra. Suppose that $\mathfrak{g}$ is a finite dimensional simple Lie algebra and that it contains $\mathfrak{g}_{SM}$ as a subalgebra. The project of constructing a grand unified theory involves obtaining a suitable representation of $\mathfrak{g}$ and determining how it splits into representations of the subalgebra $\mathfrak{g}_{SM}$. Ideally, the representation of $\mathfrak{g}$ would split perfectly into a representation $\rho_{SM}\oplus\rho_{SM}\oplus\rho_{SM}$, providing the full fermion content of the standard model as a restriction on the symmetries of $\mathfrak{g}$, although this is rarely the case. Often only one generation ρ_{SM} is represented and sometimes only a partial generation emerges. In many cases, irreducible representations of $\mathfrak{g}_{SM}$ distinct from those listed above appear, which represent new particles required by the grand unified theory.

An extensive recent paper by Yamatsu provides an overview of the landscape of possible grand unified theories [**Yam20**]. Yamatsu provides detailed calculations that analyse finite dimensional Lie algebras for the purpose of obtaining representations of the standard model Lie algebra. For $\mathfrak{g}_{SM}$ a subalgebra of $\mathfrak{g}$, the manner in which irreducible representations of $\mathfrak{g}$ split into irreducible representations of $\mathfrak{g}_{SM}$ is governed by *branching rules*. Yamatsu compiles the branching rules for, what is likely, all potentially useful representations of simple Lie algebras $\mathfrak{g}$ containing $\mathfrak{g}_{SM}$ [**Yam20**]. This paper provides a suitable context for the other attempts to link the standard model to exceptional structures that we will discuss below.

Since the standard model Lie algebra $\mathfrak{g}_{SM}$ has rank 4, the smallest available examples of $\mathfrak{g}$ are simple Lie algebras of rank 4 or 5. As described in [**Yam20**], $\mathfrak{g}_{SM}$ is a subalgebra of the simple Lie algebras of types $\mathtt{A}_4$, $\mathtt{B}_4$, $\mathtt{C}_4$, and $\mathtt{F}_4$ (but not $\mathtt{D}_4$). An important historical $\mathtt{A}_4$ approach is known as the Georgi and Glashow SU(5) grand unified theory, which is described clearly in [**BH10**]. Krasnov develops a $\mathtt{B}_4$ approach using octonions in [**Kra21**]. We are not aware of any $\mathtt{C}_4$ approaches, but the branching rules given in [**Yam20**] suggest that it might be difficult to construct the desired fermion representation of $\mathfrak{g}_{SM}$ using $\mathtt{C}_4$. An $\mathtt{F}_4$ approach is developed in [**TD18**], using octonions and the exceptional Jordan algebra. Of note, the standard model can be embedded in $\mathtt{F}_4$ in two ways [**Yam20**], only one of which is discussed in [**TD18**]. We discuss these $\mathtt{F}_4$ and $\mathtt{B}_4$ approaches in relation to octonions below. The rank 5 Lie algebras that contain $\mathfrak{g}_{SM}$ as subalgebras are those of types $\mathtt{A}_5$, $\mathtt{B}_5$, $\mathtt{C}_5$, and $\mathtt{D}_5$, namely *all* simple Lie algebras of rank 5 [**Yam20**]. Grand unified theories for all but $\mathtt{C}_5$ are given in the references of [**Yam20**], and the $\mathtt{D}_5$ theory is described well in [**BH10**].

Whether a grand unified theory provides any explanatory power over the features of the standard model is another question. For instance, the SU(5) grand unified theory sets $\mathfrak{g} = \mathfrak{sl}_4$ and begins with a 5 dimensional representation. As described in [**BH10**], by taking the exterior algebra we obtain a reducible representation of $\mathfrak{sl}_4$ with irreducible components of dimensions $1 + 5 + 10 + 10 + 5 + 1$. This $\mathfrak{sl}_4$ representation becomes a ρ_{SM} representation of $\mathfrak{g}_{SM}$ according to the branching rules for $\mathfrak{sl}_4$. That is, branching rules compiled in [**Yam20**] ensure that the 5 and 10 dimensional irreducible representations of $\mathfrak{sl}_4$ split into physically useful quark and lepton representations of $\mathfrak{g}_{SM}$. Even so, the original choice of $\mathfrak{sl}_4$ and the exterior algebra representation are not self-explanatory. Why choose $\mathfrak{sl}_4$ rather than some larger Lie algebra containing $\mathfrak{sl}_4$? Indeed, the references provided in [**Yam20**] reflect numerous attempts by researchers to construct grand unified theories from most simple Lie algebras of rank at least 4 (examples of type $\mathtt{C}_n$ are notably absent).

In Chapter 3, we will explore combinatorial reasons to favour a construction from the Lie algebra of type $\mathtt{E}_7$. Constructions from $\mathtt{E}_7$ exist in the literature, notably in [**KY84**], and have the advantage of including all three generations of fermions in the adjoint representation of $\mathfrak{e}_7$. A survey article by Slansky identifies some challenges facing $\mathtt{E}_7$ grand unification, particularly due to the fact that the irreducible representations of the corresponding compact Lie group are all self-conjugate [**Sla81**]. Even so, mechanisms to overcome these obstacles exist and $\mathtt{E}_7$ remains a viable candidate [**Yam20**]. Our focus will be on the potential explanatory power of exceptional structures to justify the choice of $\mathfrak{g}_{SM}$ as an exceptional subalgebra of $\mathfrak{g} = \mathfrak{e}_7$.

2.1.4. Exceptional Structures and the Standard Model. To provide further context for our approach to $\mathfrak{e}_7$ in Chapter 3, we now discuss some examples of how other researchers have attempted to link the seemingly arbitrary standard model to exceptional structures in mathematics, rather than seemingly arbitrary choices of Lie algebras. The octonion algebra, the exceptional Jordan algebra, and the exceptional Lie algebras all provide potential structures that could render the standard model less arbitrary if it could be linked to any of them. A brief history of the attempts to incorporate octonions and the exceptional Jordan algebra into physics is provided by Gürsey and Tze in [**GT96**, pp. 340-346].

Günyadin and Gürsey wrote an important paper linking the standard model to the octonion algebra over $\mathbb{C}$ [**GG73**]. This paper serves as a dictionary between the terminology related to Lie algebras $\mathfrak{g}_2$, $\mathfrak{so}(7)$, and $\mathfrak{so}(8)$ and the terminology of particle physics, particularly the non-trivial quark representations of $\mathfrak{sl}_3 \subset \mathfrak{g}_2$. The authors provide a convenient basis for $\mathfrak{g}_2$ and $\mathfrak{so}(7)$ that allows them to construct representations of subalgebras that correspond to portions of the standard model representation. Günyadin and Gürsey thereby link particle physics to the octonion algebra over $\mathbb{C}$, which is unique and has the complex Lie algebra $\mathfrak{g}_2$ for its derivation algebra. By constructing components of the standard model representation using octonion derivations, this paper explains aspects of the standard model in terms of an exceptional structure.

The simple Lie algebra $\mathfrak{g}_2$ has both $\mathfrak{sl}_3$ and $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ as maximal subalgebras [**Yam20**]. The main focus of [**GG73**] is the fact that octonion derivations $\mathfrak{g}_2$ have $\mathfrak{sl}_3$ as a subalgebra. The 7 dimensional representation of $\mathfrak{g}_2$ is given by the octonions orthogonal to the identity element 1 and the adjoint representation of $\mathfrak{g}_2$ is 14 dimensional. The branching rules for $\mathfrak{sl}_3 \subset \mathfrak{g}_2$ split these into $7 = 1+3+3$ and $14 = 3+3+8$. The authors of [**GG73**] interpret these three dimensional irreducible representations of $\mathfrak{sl}_3$ as quarks of the standard model. They also interpret the 8 dimensional irreducible representation as mesons (composite, not fundamental, particles). In summary, [**GG73**] focuses mostly on the fact that the $\mathfrak{sl}_3$ component of standard model Lie algebra $\mathfrak{g}_{SM}$ is a subalgebra of the derivations of the complex octonion algebra. This provides some partial clues to the exceptionality of $\mathfrak{g}_{SM}$ while leaving many aspects unanswered.

Another attempt to explain the standard model in terms of exceptional structures is Dixon's analysis of the algebra $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ in [**Dix94**]. Here the four division composition algebras form a tensor product over $\mathbb{R}$, resulting in a 32-dimensional $\mathbb{C}$-algebra. Again, the 8-dimensional $\mathbb{C} \otimes \mathbb{O}$ algebra has derivation Lie algebra $\mathfrak{g}_2$ with subalgebra $\mathfrak{sl}_3$. Algebra $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ has derivation Lie algebra $\mathfrak{sl}_2 \oplus \mathfrak{g}_2$, with the $\mathfrak{sl}_2$ component corresponding to the derivations of $\mathbb{H}$. As in [**GG73**], the $\mathbb{C} \otimes \mathbb{O}$ representation splits into a $1 + 3 + 3 + 1$ dimensional representation of $\mathfrak{sl}_3$, with 3 dimensional quark and 1 dimensional lepton components.

Furey provides a subsequent treatment of the algebra $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ in [**Fur15**] and [**Fur18**]. The left multiplication algebra of $\mathbb{C} \otimes \mathbb{O}$ is equivalent to $\mathrm{Mat}(8, \mathbb{C})$. The orbit of a primitive idempotent in $\mathrm{Mat}(8, \mathbb{C})$ under left multiplication is an eight-dimensional subspace of $\mathrm{Mat}(8, \mathbb{C})$. Furey selects a pair of maximally totally isotropic subspaces of $\mathbb{C} \otimes \mathbb{O}$ and uses them to construct the needed primitive idempotent in $\mathrm{Mat}(8, \mathbb{C})$, using the concept of ladder operators. The stabilizer of the original totally isotropic spaces corresponds to Lie algebra $\mathfrak{sl}_3$. The left ideal forms a $1+3+3+1$ representation of this stabilizer, serving as a representation of two quarks and two leptons. Furey also uses right multiplication and the $\mathbb{C} \otimes \mathbb{H}$ structure to exhibit the weak force properties of these quark and lepton representations [**Fur18**]. Furey also attempts to construct a representation of three generations of quarks and leptons from the vector space $\mathrm{Mat}(8, \mathbb{C})$, the left multiplication algebra of $\mathbb{C} \otimes \mathbb{O}$.

In summary, these three attempts to link standard model symmetries to properties of the division composition algebras—those of the octonions in particular—tend to focus on the fact that $\mathfrak{sl}_3$ is a subalgebra of octonion derivations $\mathfrak{g}_2$ and that $\mathfrak{sl}_2$ is the derivation Lie algebra of the quaternions. In particular, the $\mathfrak{sl}_3$ subalgebra corresponds to fixing or selecting an octonion imaginary unit that is stabilized by the automorphisms corresponding to these derivations. In these approaches, the representations selected do not seem to easily explain the three generations of standard model particles. Whether these approaches provide a simple or natural connection between the standard model and division composition algebras is a largely subjective question.

As described in [**Yam20**], $\mathfrak{g}_{SM}$ is also a subalgebra of $\mathfrak{f}_4$, the simple Lie algebra of type $\mathtt{F}_4$. This Lie algebra is itself exceptional and is also the derivation Lie algebra of the exceptional Jordan algebra (known as the Albert algebra). Exploring $\mathtt{F}_4$ approaches is a natural next step beyond $\mathtt{G}_2$ approaches since the Albert algebra is an algebra of Hermitian octonion matrices under the commutative Jordan product.

An important recent example of an $\mathtt{F}_4$ approach is given by Todorov and Drenska, who obtain the standard model Lie group as Albert algebra automorphisms [**TD18**]. Although they work with Lie groups, their approach corresponds to constructing an $\mathfrak{f}_4$ grand unified theory using the fact that $\mathfrak{g}_{SM}$ is a subalgebra of Albert algebra derivations. Todorov and Drenska begin by noting that the octonion automorphism group G_2 is a subgroup of the Jordan algebra automorphism group F_4. Furthermore, a $\mathrm{SU}(3)$ subgroup of G_2 stabilizes a complex structure $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ (this is also the stabilizer of a single imaginary unit in $\mathbb{O}$). There are two maximal subgroups of F_4 that contain this $\mathrm{SU}(3)$, namely $\mathrm{Spin}(9)$ and $(\mathrm{SU}(3) \times \mathrm{SU}(3))/\mathbb{Z}_3$. The intersection of these two Albert algebra automorphism subgroups is the standard model group. As noted in [**TD18**], the 26-dimension representation of F_4 (i.e. the trace-free elements of $\mathrm{Herm}(3, \mathbb{O})$) does not provide a full generation of quarks and leptons (some right-handed fermions are missing). This is also clear from the branching rules catalogued in [**Yam20**].

The primitive idempotents of $\mathrm{Herm}(3,\mathbb{O})$ form a 16-dimensional manifold known as the octonion projective plane, $\mathbb{OP}^2$. The approach developed in [**TD18**] only describes one of the two possible embeddings of $\mathfrak{g}_{SM}$ in $\mathfrak{f}_4$ [**Yam20**]. The embedding described in [**TD18**] corresponds to restricting to Albert algebra automorphisms (or their derivations) that (1) fix a primitive idempotent in $\mathbb{OP}^2$ and (2) also fix an octonion imaginary unit in $\mathrm{Herm}(3,\mathbb{O})$. Put another way, this embedding simultaneously stabilizes in the Albert algebra (1) a $\mathrm{Herm}(2,\mathbb{O})$ subalgebra and (2) a $\mathrm{Herm}(3,\mathbb{C})$ subalgebra. The other embedding of $\mathfrak{g}_{SM}$ in $\mathfrak{f}_4$ given in [**Yam20**] corresponds instead to stabilizing a $\mathrm{Herm}(3,\mathbb{C})$ subalgebra within a $\mathrm{Herm}(3,\mathbb{H})$ subalgebra of the Albert algebra. This other embedding is not explored by [**TD18**].

In a follow up paper [**DVT19**], Dubois-Violette and Todorov attempt to construct three generations of fermions in the following manner. They take a Jordan frame of three orthogonal primitive idempotents with the three corresponding $\mathrm{Herm}(2,\mathbb{O})$ Jordan subalgebras and use each one to construct a generation. Each generation is obtained by combining and complexifying the two 16-dimensional irreducible representations of the Clifford algebra related to $\mathrm{Herm}(2,\mathbb{O})$. These are obtained by constructing the two natural universal unital associative envelops of the $\mathrm{Herm}(2,\mathbb{O})$ subalgebra orthogonal to the primitive idempotent. This addresses the missing lepton content of the minimal representation of $\mathfrak{f}_4$ described above, by focusing instead on representations of $\mathfrak{so}(9) \subset \mathfrak{f}_4$ corresponding to stabilizing a primitive idempotent. Again, whether this approach provides a simple or natural connection between the standard model and division composition algebras is a subjective question.

The Lie algebra $\mathfrak{so}(9)$ is of type $\mathtt{B}_4$. The $\mathtt{F}_4$ approaches described above, which fix a primitive idempotent, effectively reduce the problem of a $\mathtt{F}_4$ approach to that of a $\mathtt{B}_4$ approach. Krasnov addresses the $\mathtt{B}_4$ approach directly in [**Kra21**], aiming to avoid prior discussion of the Albert algebra. Krasnov observes that the standard model Lie group is the subgroup of $\mathrm{Spin}(9)$, and that $\mathrm{Spin}(9)$ has a construction using octonion multiplication acting on $\mathbb{O}\oplus\mathbb{O}$, providing a 16-dimensional representation. The standard model group is the subgroup of $\mathrm{Spin}(9)$ that stabilizes an imaginary unit i in $\mathbb{O}$, or equivalently preserves a description of the octonions as $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$. Krasnov also observes that only the left-handed fermions of a single generation of particles are given in the $\mathbb{O} \oplus \mathbb{O}$ representation of the standard model Lie group, which is consistent with [**TD18**] and the branching rules for $\mathfrak{so}(9)$ given in [**Yam20**].

Of note, prior to these recent papers on $\mathtt{B}_4$ and $\mathtt{F}_4$ approaches involving octonions, Gürsey and Tze observed that the F_4 automorphism group of the Albert algebra contains a $\mathrm{SU}(3)\times\mathrm{SU}(3)$ subgroup—the first component arising as a subgroup of octonion automorphisms G_2 and the second component due to the 3×3 structure of the Albert algebra matrices [**GT96**, p. 305]. They further observe that $\mathrm{SU}(2) \times \mathrm{U}(1)$ is a maximal subgroup of $\mathrm{SU}(3)$, which ensures that the standard model symmetry group is a subgroup of F_4. Gürsey and Tze also observed that the stabilizer of both a primitive idempotent and an octonion imaginary unit in F_4 is the standard model Lie group [**GT96**, p. 216].

Regarding explanatory power, authors of $\mathtt{F}_4$ approaches involving octonions tend to overstate the simplicity of their models. In particular, because the Albert algebra can be constructed as Hermitian octonion 3×3 matrices with the Jordan product, it is often taken for granted that fixing an imaginary octonion unit is a single simple assumption. However, the octonions are not simply given by the Albert algebra. In order to obtain an octonion product from the Albert algebra product one must follow a process described in [**SV00**, pp. 129-136]. This involves first selecting a primitive idempotent, extending it to a full Jordan frame, and then conducting the corresponding Peirce decomposition (which identifies the three independent off-diagonal components). The stabilizer of this selection is a Lie group of type $\mathtt{D}_4$, the only rank 4 example that *does not* admit the standard model symmetries as a subgroup [**Yam20**]. To complete the process, one must identify a pair of (non-isotropic) vectors in distinct off-diagonal vector spaces and use them to define the octonion product on the remaining off-diagonal vector space. Accordingly, an octonion approach to the standard model involves using both *more* and *less* symmetry than octonions afford. We need to use both symmetries that do not preserve Pierce decomposition (i.e., symmetries outside of $\mathtt{D}_4$) and also neglect those symmetries of the octonion product that do not fix some imaginary unit (i.e., to isolate a $\mathfrak{sl}_3$ subalgebra of $\mathfrak{g}_2$). These facts do not rule out octonion constructions, but they should inform our estimation of the explanatory power of octonions for the standard model.

We conclude this overview of exceptional approaches to the standard model by discussing two recent $\mathtt{E}_8$ approaches. Lisi has developed an embedding of the standard model within $\mathfrak{e}_8$, the largest exceptional simple Lie algebra [**Lis07**]. Lisi's model also includes additional particles, which he employs in an attempt to incorporate gravitation and the Higgs mechanism. We do not explore either aspect of Lisi's $\mathtt{E}_8$ model in what follows since our focus in Chapter 3 will be on the internal symmetries of the standard model and not the particle masses.

In Table 9 of [**Lis07**], Lisi assigns explicit particle labels to the roots of $\mathtt{E}_8$. The $\mathtt{E}_8$ root system contains a $\mathtt{D}_4\mathtt{D}_4$ subsystem. Close inspection of Lisi's labeling reveals that in his model, the 192 roots of $\mathtt{E}_8 \setminus \mathtt{D}_4\mathtt{D}_4$ correspond to the standard model fermions, two per fermion on this construction. There are two roots per fermion because Lisi subsequently combines these two real root spaces into a one dimensional complex vector space per fermion. The three fermion generations are the length $64+64+64$ orbits of these $\mathtt{E}_8 \setminus \mathtt{D}_4\mathtt{D}_4$ roots under the action the reflection group $W(\mathtt{D}_4\mathtt{D}_4)$. The $\mathfrak{sl}_2 \oplus \mathfrak{sl}_3$ component of $\mathfrak{g}_{SM}$ corresponds to root system $\mathtt{A}_1\mathtt{A}_2 \subset \mathtt{D}_4\mathtt{D}_4$. Here the $\mathtt{A}_1$ component is a subset of the first $\mathtt{D}_4$ component, and the $\mathtt{A}_2$ component is a subset of the second $\mathtt{D}_4$ component. The remaining abelian Lie subalgebra of $\mathfrak{g}_{SM}$, which we denote $\mathbb{C}$, is not restricted to the Cartan subalgebra corresponding to either $\mathtt{D}_4$ component, but contains a component in both.

Lisi's choice of hypercharge operator in [**Lis07**] defines an 11-grading on the first and second generations of fermions, which yields appropriate hypercharge eigenvalues for the fermions in these two generations. However, this same hypercharge operator defines a 9-grading on Lisi's third generation of fermions, which yields incorrect hypercharge eigenvalues. This is a problem with Lisi's model since we would expect the same hypercharge for the corresponding fermions of distinct generations. Lisi acknowledges in [**Lis07**] that this is a difficulty with his model, and identifies a triality matrix (a mapping of order 3) to transform fermions from one generation to another. An improved version of this model would correct this hypercharge discrepancy, perhaps using the triality operator or perhaps by selecting a different hypercharge operator.

Another recent $\mathtt{E}_8$ approach to the standard model is that of Manogue, Dray, and Wilson in [**MDW22**]. These authors select one of the three real Lie algebras of type $\mathtt{E}_8$, namely $\mathfrak{e}_{8(-24)}$. They then select a maximal $\mathfrak{so}(12,4)$ real Lie subalgebra of type $\mathtt{D}_8$ so that the adjoint representation can be written $\mathfrak{e}_{8(-24)} = \mathfrak{so}(12,4) \oplus \mathbf{128}$. This real 128-dimensional component defines a real representation of $\mathfrak{so}(12,4)$. In order to obtain a complex representation, the authors decompose the real Lie algebra $\mathfrak{so}(12,4)$ as follows:

$$\mathfrak{so}(12,4) = \mathfrak{so}(10,4) \oplus \mathfrak{so}(2) \oplus (2 \times \mathbf{14}).$$

The $\mathfrak{so}(2)$ component provides a complex structure, allowing the authors to reduce the $\mathbf{128}$ real spinor of $\mathfrak{so}(12,4)$ to a 64-dimensional complex spinor of $\mathfrak{so}(10,4)$, which is interpreted instead as the complex Lie algebra $\mathfrak{so}(14)$. This $\mathfrak{so}(14)$, in turn, contains a $\mathfrak{so}(10)$ subalgebra which the authors use to link their model to the Georgi-Glashow $\mathfrak{so}(10)$ GUT, an important $\mathtt{D}_5$ approach described clearly in [**BH10**]. In terms of fermion generations, the authors do not model a single representation containing all three generations. Instead, different choices in the process of reduction from $\mathfrak{e}_{8(-24)}$ to $\mathfrak{g}_{SM}$ result in different generations. Since the critical step in this reduction involves selecting a choice of axis in $\mathbb{R}^3$, the authors propose a potential mechanism for three generations corresponding to three orthogonal axes. Accordingly, this model describes "overlapping generations" of particles rather than a representation containing all three generations. In this way it is similar to the $\mathtt{F}_4$ approach to representing three generations by, Dubios-Violette and Todorov in [**DVT19**].

In summary, the approach of [**MDW22**] is to select a $\mathtt{D}_8$ subsystem of $\mathtt{E}_8$ and further restrict $\mathtt{D}_8$ to obtain a complex structure, a Lorentz group structure, and a standard model internal symmetry structure. It isn't clear why these reductions or restrictions are warranted or whether they provide a compelling explanation of why the standard model is what it happens to be.

2.1.5. Planned Contribution. We have seen that most attempts to explain the arbitrary features of the standard model in terms of exceptional structures can be classified according to the Lie algebra containing (a component of) $\mathfrak{g}_{SM}$. The early and most popular examples of grand unified theories involve the Lie algebra of types $\mathtt{A}_4$ and $\mathtt{D}_5$, and are well described in [**BH10**]. These approaches do not strictly involve an exceptional structure but are useful because of their low rank. Approaches of type $\mathtt{G}_2$ emphasize octonions but only address part of $\mathfrak{g}_{SM}$, while approaches of types $\mathtt{F}_4$ and $\mathtt{B}_4$ tend to take for granted octonion structure when counting the assumptions needed to select the $\mathfrak{g}_{SM}$ subalgebra. Most approaches, except perhaps $\mathtt{D}_5$, have some trouble obtaining a representation with the correct fermion content. The $\mathtt{E}_8$ approaches of [**Lis07**] and [**MDW22**] may put to rest the question of why the grand unified theory is not embedded in some larger theory, since $\mathtt{E}_8$ is maximal, but have problems respectively with correct hypercharge eigenvalues and obtaining a simultaneous representation of all three generations. The fact that $\mathfrak{g}_{SM}$ is a subalgebra of $\mathfrak{e}_8$ should not be surprising, since it belongs to *every* simple Lie algebra of rank 5. The choice of $\mathfrak{g}_{SM}$ with $\mathfrak{e}_8$ needs some justification when developing an exceptional explanation for the standard model.

Gürsey and Tze describe how the exceptional groups of type $\mathtt{E}_6$, $\mathtt{E}_7$, $\mathtt{E}_8$, can be thought to include lower rank classical groups by truncating their corresponding Coxeter-Dynkin diagrams [**GT96**, p. 305]. For instance, we could define $\mathtt{E}_5$ to be the group $\mathtt{D}_5 = \mathrm{SO}(10)$. However, there are two distinct ways to truncate $\mathtt{E}_5$ resulting in $\mathtt{E}_4'$ as $\mathtt{D}_4 = \mathrm{SO}(8)$ or $\mathtt{E}_4$ as $\mathtt{A}_4 = \mathrm{SU}(5)$. They do not explore the further possibility of $\mathtt{E}_3$ as $\mathtt{A}_2 \times \mathtt{A}_1 = \mathrm{SU}(3) \times \mathrm{SU}(2)$, which we will examine in Chapter 3.

In Chapter 3 we will explore a method to obtain the standard model Lie algebra $\mathfrak{g}_{SM}$ and a representation including three generations by identifying an exceptional sequence of three-gradings on irreducible root systems. This approach is distinct from those described above because not only does it include the standard model symmetries within a larger set of symmetries, but it provides reasons for regarding the standard model as an exceptional substructure of a larger exceptional structure in which it is embedded. This approach will also provide a simultaneous representation of all three generations of fermions, and not just a single representation that can be repurposed or reconfigured to describe a different generation.

2.2. Tight t-Designs and Rational Angle Sets

In this section we briefly describe the similarities between combinatorial t-designs and spherical t-designs, before identifying a gap in the literature regarding the properties of the angle sets of a tight spherical or projective t-design.

2.2.1. Combinatorial t-Designs. A *combinatorial t-design*, specifically a $t-(v,k,\lambda)$ design, consists of size k subsets (blocks) of a v-set (points) with the property that any size t subset belongs to precisely λ blocks. Combinatorial designs correspond to spherical designs in the following manner [**DGS77**]. The sphere Ω_{d+1} corresponds to all k-subsets of the v points (namely the sphere is replaced by Johnson scheme $J(v,k)$). The finite design $X \subset \Omega_{d+1}$ corresponds to the blocks $\mathcal{B} \subset J(v,k)$. The angle set corresponds to the sizes of all pairwise intersections of blocks, corresponding to the set of all inner products between design points on the sphere. The size of the angle set is the degree s of the design, and the strength t is defined in the usual way for both cases. Just as a spherical t-design satisfies certain inequalities, a combinatorial t-design satisfies a number of inequalities.

THEOREM 2.1. [**RCW75**] *A $2s-(v,k,\lambda)$ design and $v \geq k+s$ satisfies $b \geq \binom{v}{s}$, where $b = |\mathcal{B}|$ is the number of blocks.*

THEOREM 2.2. [**RCW75**] *Let $\mathcal{B} \subset J(v,k)$ be the blocks of a design with degree $s = |\{|x \cap y| \mid x \neq y \in \mathcal{B}\}|$ and let $b = |\mathcal{B}|$. Then $b \leq \binom{v}{s}$.*

A *tight combinatorial $2s$-design* is a $2s-(v,k,\lambda)$ design with $|\mathcal{B}| = \binom{v}{s}$. The only non-trivial tight combinatorial 4-designs are the Steiner design $4-(23,7,1)$ and its complement [**RCW75**].

2.2.2. The Angles of Tight t-Designs. The concept of a spherical t-design was introduced by Delsarte, Goethals, and Seidel in [**DGS77**]. That paper included lower bounds on the cardinality of a spherical t-design, and labeled a t-design meeting the lower bound as tight (Theorems 5.11 and 5.12). Bannai states shortly thereafter, in Theorem 1 of [**Ban79**], that for a tight spherical t-design in Ω_{d+1} with $d \geq 2$ certain polynomials that determine the angle set of the design will have rational roots. However, the tight 5-design in Ω_3 defined by the vertices of the icosahedron serves as a counter-example, with the corresponding polynomial having the irrational roots $\pm 1/\sqrt{5}$.

Building on Neumaier's work [**Neu81**], Hoggar introduced the concept of a projective t-design in [**Hog82**]. In [**Hog84**], Hoggar introduced a theorem that for a tight projective t-design not belonging to $\mathbb{RP}^1$, the angle set is rational. Again, as observed by Lyubich [**Lyu09**], the icosahedron provides a counter example, but this time in the projective space $\mathbb{CP}^1 \cong \Omega_3$. In [**BH89**], Bannai and Hoggar use this incorrect result about rational angle sets to prove that tight projective t-designs (other than in $\mathbb{RP}^1$) must have $t \leq 5$. Hoggar then uses the rationality of angle sets to prove that the $t = 4$ case is not realized in tight projective t-designs, and to rule out some $t = 5$ possibilities (including the counter-example of $\mathbb{CP}^1$). In [**Lyu09**], Lyubich repairs Hoggar's proof in [**Hog84**], but does not apply that proof to the spherical or octonion projective cases.

2.2.3. Planned Contribution. In Chapter 4 we generalize Lyubich's repair of Hoggar's proof to apply to all spherical and projective tight t-designs. Fortunately, most of the results concerning tight t-designs in the literature are not affected by this error, once the icosahedron exception is properly accounted for. Independently of our work on this problem, Boyvalenkov, Nozaki, and Safaei very recently proved that all tight spherical t-designs apart from the icosahedron have rational angle sets [**BNS22**]. Their paper does not mention the polygon exceptions in the unit circle. However, these exceptions are well known and can be considered implicit in their paper. They provide a more general result involving designs that satisfy $t \geq 2s - 2$ and $s \geq 3$, but do not consider the projective cases. Our contribution is independent and focuses instead on treating projective and spherical cases together in a unified way.

2.3. Octonions and Tight Projective 5-Designs

This final section describes attempts to construct the Leech lattice using octonions, including techniques involving octonion integers. We also describe a lattice method of constructing the tight 5-design in the octonion projective plane using octonion integers.

2.3.1. Octonions and the Leech Lattice. The Leech lattice is the unique even unimodular lattice in $\mathbb{R}^{24}$ without any roots. It also defines the most dense sphere packing possible in 24 dimensions [**CKM$^+$17**] and manifests the sporadic symmetry of the Conway group Co_1. The most valuable reference on the Leech lattice is Conway and Sloane's book [**CS13**], which describes numerous constructions and special properties. The lines spanned by the shortest vectors of the Leech lattice define a projective tight 5-design in $\mathbb{RP}^{23}$. The remaining strictly projective tight 5-design exists in the octonion projective plane $\mathbb{OP}^2$, which suggests that the Leech lattice might admit an octonion construction. Indeed, Hoggar conjectured that some connection between the tight 5-designs in $\mathbb{RP}^{23}$ and $\mathbb{OP}^2$, but could not yet provide one [**Hog82**].

Wilson constructs the Leech lattice using octonion vectors in $\mathbb{O}^3$, rather than $\mathbb{R}^{24}$ [**Wil09a**], [**Wil09b**]. Wilson's construction is not the first. Dixon provided a somewhat complicated construction in [**Dix95**], which is further updated in [**Dix10**]. Elkies and Gross also provide a Leech lattice construction using the exceptional Jordan algebra [**EG96**]. We will first discuss Wilson's construction, since it forms the template for our work in Chapter 5, and then describe the approach due to Elkies and Gross.

2.3.2. Wilson's Octonion Leech Lattice. Wilson's Leech lattice construction in [**Wil09a**] and [**Wil09b**] can be described in terms of integer octonion rings as follows. Let $\{i_t \mid t \in \mathrm{PL}(7) = \{\infty\} \cup \mathbb{F}_7\}$ be an orthonormal basis for the octonion algebra $\mathbb{O}$, as described in [**CS03**]. Let B denote the $\mathtt{E}_8/\sqrt{2}$ lattice with the following choice of 240 roots, for all $t \in \mathbb{F}_7$:

$$\pm 1, \quad \pm i_t, \quad \frac{1}{2}(\pm 1 \pm i_t \pm i_{t+1} \pm i_{t+3}), \quad \frac{1}{2}(\pm i_{t+2} \pm i_{t+4} \pm i_{t+5} \pm i_{t+6}).$$

This is a natural choice since $i_t i_{t+1} i_{t+3} = -1$ for all t in $\mathbb{F}_7$. The lattice B is not closed under octonion multiplication. But we obtain seven distinct octonion integer rings as follows:

$$A_t = \frac{1}{2}(1 - i_t)B(1 - i_t), \quad t \in \mathbb{F}_7.$$

The A_t octonion rings are each known as *Coxeter-Dickson integral octonions*. Finally, we can recover the standard coordinates for the $\mathtt{E}_8$ lattice in both the left-handed L and right-handed R form as follows:

$$L = (1 + i_t)A_t = B(1 - i_t), \qquad R = A_t(1 + i_t) = (1 - i_t)B.$$

The intersection $L \cap R$ is a standard copy of the $\mathtt{D}_8$ lattice, namely the span of all 112 $\mathtt{D}_8$ roots $\pm i_r \pm i_t$ for $r, t \in \mathrm{PL}(7)$. The remaining roots in L are the 128 vectors of the form $\frac{1}{2}(\pm 1 \pm i_0 \pm i_1 \cdots \pm i_6)$ with an odd number of minus signs. The remaining roots in R are the 128 vectors of this form but with an even number of minus signs instead. So the vector $s = \frac{1}{2}(-1 + i_0 + i_1 + i_2 + i_3 + i_4 + i_5 + i_6)$ is in L, while the vector $\overline{s}$ is in R.

Wilson's key observation in [**Wil09b**] is that, although they are not closed under multiplication, the lattices L, R, B nevertheless satisfy the following simple relations due to the octonion Moufang laws:

$$LR = 2B, \qquad BL = L, \qquad RB = R.$$

These identities simplify Wilson's proofs compared to Dixon's and allow Wilson to define the octonion Leech lattice as all row vectors (x, y, z) in $\mathbb{O}^3$ with norm,

$$N(x, y, z) = \frac{1}{2}(x, y, z)(x, y, z)^\dagger = \frac{1}{2}(x\overline{x} + y\overline{y} + z\overline{z}),$$

that satisfy,

(1) $x, y, z \in L$
(2) $x + y, y + z, x + z \in L\overline{s}$
(3) $x + y + z \in Ls$.

In what follows we denote as *Wilson's Leech lattice* this particular octonion construction.

In [**Wil09a**] and [**Wil11**], Wilson also shows how to construct $2 \cdot \mathrm{Co}_1$, the automorphism group of the Leech lattice, using 3×3 octonion matrices acting from the right on the row vectors of his Leech lattice. In particular, Wilson shows that right scalar multiplication by certain octonions preserves his Leech lattice as follows:

$$(x, y, z) \mapsto \frac{1}{2}\left((x, y, z)R_{1-i_0}\right)R_{1+i_t}, \quad t \in \mathbb{F}_7, \quad R_x = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}.$$

These nested matrix actions generate a $2 \cdot A_8$ group action on the Leech lattice. If we include all coordinate permutations, coordinate sign changes, and right multiplication by $\mathrm{diag}(1, i_t, i_t)$ for t in $\mathbb{F}_7$, then we obtain by right multiplication a $2^{3+12}(A_8 \times S_3)$ action on Wilson's Leech lattice. This group turns out to be a maximal subgroup of the Leech lattice automorphism group $2 \cdot \mathrm{Co}_1$. To obtain the full $2 \cdot \mathrm{Co}_1$ action, Wilson adjoins right multiplication by,

$$\frac{1}{2}\begin{pmatrix} 0 & \overline{s} & \overline{s} \\ s & -1 & 1 \\ s & 1 & -1 \end{pmatrix},$$

which is the negative of the reflection matrix of the vector $(s, 1, 1)$ [**Wil11**].

In [**Wil09a**] and [**Wil11**], Wilson uses subsets of these actions, and closely related ones, to exhibit the groups of the Suzuki chain of Co_1 subgroups acting on this octonion Leech lattice:

$$S_3 < S_4 < \mathrm{PSL}_2(7) < \mathrm{PSU}_3(3) < \mathrm{HJ} < G_2(4) < 3 \cdot \mathrm{Suz}.$$

Since the Suzuki chain subgroup are centralizers of alternating groups in Co_1, Wilson's construction via $2 \cdot A_8$ lends itself well to this task.

2.3.3. Baez and Egan. The octonion algebra, octonion integer rings, the exceptional Jordan algebra, the $\mathtt{E}_8$ lattice and the Leech lattice are all discussed together in an accessible form in a series of mathematics blog posts by Baez [**Bae14a**], who also authored a comprehensive review of the octonion algebra [**Bae02**]. In [**BE14b**], Baez and Egan describe how by restricting the exceptional Jordan algebra to octonion integer entries, we obtain a lattice containing a $(\mathtt{E}_8\mathtt{E}_8\mathtt{E}_8)/\sqrt{2}$ sublattice. They further explore ways to restrict to a Leech sublattice of $(\mathtt{E}_8\mathtt{E}_8\mathtt{E}_8)/\sqrt{2}$ and to ensure this lattice is closed under a quadruple Jordan product.

2.3.4. Elkies and Gross. Elkies and Gross define different inner products on an Albert algebra integer ring and octonion integer triples. These inner products respectively yield a lattice containing the 819 rank 1 elements with a $\mathrm{Gh}(2, 8)$ structure and the Leech lattice [**EG96**]. In this section we describe their constructions. We will also adopt notation similar to what we use in Chapters 5 and 6.

First, we construct the Albert algebra as matrices $\mathrm{Herm}(3, \mathbb{O})$ with the standard Jordan commutative product. The octonion algebra contains a maximal integer subring isometric to a scaled copy of E_8, which we denote $\mathsf{O} \subset \mathbb{O}$ (the subring in [**EG96**] is the 1-integer octavian ring of [**CS03**], denoted A_1 by Wilson, but any choice of A_t is equivalent up to automorphism). So we can restrict to a subring of the Albert algebra $\mathrm{Herm}(3, \mathsf{O})$, which is closed under twice the Jordan product, i.e. $2(x \circ y) = xy + yx$. We denote an element of $\mathrm{Herm}(3, \mathsf{O})$ as follows:

$$(d, e, f \mid D, E, F) = \begin{pmatrix} d & F & \overline{E} \\ \overline{F} & e & D \\ E & \overline{D} & f \end{pmatrix}, \quad d, e, f \in \mathbb{Z},\ D, E, F \in \mathsf{O}.$$

The Albert algebra is equipped with a determinant $\det(x)$ and corresponding symmetric trilinear form $\langle x, x, x \rangle = \det(x)$ [**SV00**, p. 120] (note that the convention in [**EG96**] is to use $6\langle x, y, z \rangle$, but we will use the definition of [**SV00**] in what follows). Elkies and Gross give the following explicit expression for the determinant [**EG96**]:

$$\det(d, e, f \mid D, E, F) = def + 2\mathrm{Re}(DEF) - dD\overline{D} - eE\overline{E} - fF\overline{F}.$$

Second, Elkies and Gross identity the following positive definite rank 3 elements in $\mathrm{Herm}(3, \mathsf{O})$ which have determinant 1:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 2 & \lambda & \overline{\lambda} \\ \overline{\lambda} & 2 & \lambda \\ \lambda & \overline{\lambda} & 2 \end{pmatrix}.$$

Here λ is any norm 2 element of O with $\mathrm{Re}(\lambda) = -\frac{1}{2}$ (we generalize from the λ given in [**EG96**], since any choice of λ is equivalent up to octonion integer automorphism). Let G be the group of all invertible linear transformations acting on $\mathrm{Herm}(3, \mathbb{O})$ that preserve the determinant $\det(x)$. Since G is transitive on positive-definite elements with determinate 1, there is a symmetry in G mapping I to E [**EG96**]. However, the stabilizer of $\mathrm{Herm}(3, \mathsf{O}) \subset \mathrm{Herm}(3, \mathbb{O})$ in G is not transitive on positive-definite determinant 1 elements, but has two orbits with I and E as representatives [**EG96**].

Given these two representatives, Elkies and Gross construct the inner products needed to exhibit the Leech lattice and the 819 point structure in the octonion projective plane. Specifically, for any positive-definite A with $\det(A) = 1$ they define the following inner product on $\mathrm{Herm}(3, \mathsf{O})$:

$$\langle x, y \rangle_A = 9\langle x, A, A \rangle \langle y, A, A \rangle - 6\langle x, y, A \rangle, \quad x, y \in \mathrm{Herm}(3, \mathsf{O}).$$

For $A = I$, $\mathrm{Herm}(3, \mathsf{O})$ has the standard Jordan inner product and the automorphisms preserving this inner product form a group isomorphic to $2^2 \cdot$

$\mathrm{O}_8^+(2) \cdot S_3$ [**Gro96**, p. 273]. For $A = E$, $\mathrm{Herm}(3, \mathsf{O})$ has inner product $\langle x, y\rangle_E$ and contains 819 elements that satisfy $\langle x, x\rangle_E = 4$ and $\langle x, E\rangle_E = 2$. These are called *integral Jordan roots* in [**EG96**], and they define the tight 5-design in the octonion projective plane, relative to $\langle x, y\rangle_E$. The lattice on $\mathrm{Herm}(3, \mathsf{O})$ relative to inner product $\langle x, y\rangle_E$ has automorphism group ${}^3D_4(2) \cdot 3$ [**Gro96**, p. 273]. This is the subgroup of G stabilizing the elements of $\mathrm{Herm}(3, \mathsf{O})$, the determinant, and E.

The authors also define the following inner product on row vectors $x, y \in \mathsf{O}^3$:

$$\{x, y\} = 2\mathrm{Re}(\overline{x} E^T y^T), \quad x, y \in \mathsf{O}^3.$$

Here E^T is the transpose of E in $\mathrm{Herm}(3, \mathsf{O})$ and $\overline{x}$ is the octonion conjugate transpose, a column vector. This inner product defines a Leech lattice on O^3. If we replace E with I, then this inner product instead yields a E_8^3 lattice on O^3.

2.3.5. Planned Contribution. Wilson's approach in [**Wil11**] makes use of a single octonion vector reflection, the reflection matrix of $(s, 1, 1)$. In our approach, described in Chapters 5 and 6, we will provide a construction of the sporadic simple group Co_1 using only reflection matrices as generators, and exhibit the Suzuki chain subgroup using subsets of these reflections as generators.

In Chapter 6 we will generalize Wilson's Leech lattice construction and explore Leech lattices that are sublattices of octonion integer triples. This will simplify our method of exhibiting Suzuki chain subgroups.

In Chapter 5 we will also connect the reflections generating $2 \cdot \mathrm{Co}_1$ and an octonion Leech lattice to a corresponding involution acting on $\mathbb{OP}^2$. This will permit us to provide a common construction of the two strictly projective tight 5-designs. Chapter 5 has been published elsewhere as [**Nas22**].

In Chapter 6 we approach many of the questions discussed by Baez and Egan in [**Bae14a**] and [**BE14b**] using a different approach. Instead of working out detailed examples in explicit coordinates, we work with the integral octonions evaluated modulo 2, namely with the ring $\mathsf{O}/2\mathsf{O}$. This reduces many questions about octonion integers to questions about the properties of a small strongly regular graph.

Finally, in Chapter 6, we will also describe the approach to the two tight projective 5-designs given by Elkies and Gross in [**EG96**] and [**EG01**] using the theory of Jordan isotopes and the quadratic map. Although Jordan isotopes are isomorphic as algebras, the restriction to an integer subring yields non-isomorphic isotope rings if we construct the isotope ring from an element in a distinct orbit among the Albert algebra integers. This provides an interesting avenue for future work exploring the connections between isotopic yet non-isomorphic Jordan algebra integer rings. A recent paper by Garibaldi, Petersson, and Racine also identifies Jordan isotopy as the main concept implicit in the work of Elkies and Gross [**GPR22**]. Our contribution is to examine the role of the orbit containing the squares in $\mathrm{Herm}(3, \mathsf{O})$ more closely.

CHAPTER 3

An Exceptional Combinatorial Sequence and Standard Model Particles

3.1. Introduction

We can succinctly describe many features of both Lie and Jordan structures in algebra and geometry using root systems. The following sequence of root systems has a number of exceptional properties:

$$(\star) \qquad \mathtt{E}_7 \to \mathtt{E}_6 \to \mathtt{D}_5 \to \mathtt{A}_4 \to \mathtt{A}_1 \times \mathtt{A}_2.$$

The final root system and nesting in this sequence, $\mathtt{A}_4 \to \mathtt{A}_1 \times \mathtt{A}_2$, corresponds to the Lie group of the standard model of particle physics: $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$. The third and fourth root systems correspond to two well-studied grand unification theories: the Spin(10) and SU(5) theories [**BH10**]. This chapter describes some special properties of this sequence of root systems and explains how it affords a natural representation of all three generations of standard model fermions.

3.2. Star-Closed Line Systems

Consider the three axes of a regular hexagon in $\mathbb{R}^2$. These lines have the special property that the angle between any two of the three is 60 degrees. That is, the three axes of a regular hexagon are a system of *equiangular lines*. It turns out that for any system of equiangular lines in $\mathbb{R}^d$, the number of lines n must satisfy $n \leq \binom{d+1}{2}$ [**GR01**, chap. 11]. The number $\binom{d+1}{2}$ is called the *absolute bound* on the number of equiangular lines in d-dimensions. The three axes of the hexagon meet this absolute bound in $d = 2$ dimensions. The only other known examples of equiangular lines at the absolute bound consist of the axes of an icosahedron in $d = 3$, a 28 line system for $d = 7$, and a 276 line system for $d = 23$ (Examples A.5, A.6, and A.7). Any further examples, if they exist, will occur in $d \geq 119$ [**BB09**, p. 1402].

In what follows we will refer to three lines at 60 degrees as a *star*. The star is the smallest system of equiangular lines at the absolute bound, and stars are responsible for an abundance of rich structures in algebra and combinatorics. Examples of structures that can be constructed from stars include root systems, root lattices, Lie algebras, Jordan grids, Jordan triple systems, Jordan algebras, and many interesting spherical and projective t-designs.

We will focus for the moment on line systems of type $(0, 1/2)$. A *line system of type* $(a_1, a_2, \ldots, a_n)$ is a finite set of lines through the origin of a real vector space (equivalently, points in a real projective space) such that the Euclidean inner product of any two unit vectors spanning distinct lines satisfies $|\cos\theta| \in \{a_1, a_2, \ldots, a_n\}$. Line systems of type $(0, 1/2)$ are studied in [**CGSS76**], while line systems of types $(0, 1/3)$ and $(0, 1/2, 1/4)$ are studied in [**SY80**]. In what follows we will refer to line systems of type $(0, 1/2)$ simply as line systems. That is, we will take a *line system* to be a set of lines in a real vector space such that any two lines in the system are either orthogonal or at 60 degrees.

Each pair of non-orthogonal lines in a line system defines a unique coplanar line that is at 60 degrees to both members of the pair. Three lines at 60 degrees form a *star*, and any two members of a star defines the third member. Using this concept, we can compute the *star-closure* of a line system by adding to the line system any missing third lines defined by any nonorthogonal pair. When a line system is equal to its own star-closure, it is a *star-closed line system*. When a line system cannot be partitioned into two mutually orthogonal subsets, it is an *indecomposable line system*. Finally, a *star-free line system* is a line system without stars, in which any three mutually non-orthogonal lines span a vector space of dimension three.

The indecomposable star-closed line systems are classified in [**CGSS76**]. The classification makes heavy use of the following lemma:

LEMMA 3.1. *Let L be a line system and let $S \subset L$ be a star. Then each line in $L \setminus S$ is orthogonal to either* 1 *or* 3 *members of S.*

That is, for line system L containing star S, we can partition the lines of L into S, lines orthogonal to S, and three sets of lines orthogonal to just one member of S. We may call this partition the *star-decomposition* of line system L with respect to star $S \subset L$. That is, for $S = \{a, b, c\}$ we can write $L = S \,\dot\cup\, A \,\dot\cup\, B \,\dot\cup\, C \,\dot\cup\, D$, where A is the set of lines in L orthogonal to just a, B orthogonal to just b, C orthogonal to just c, and D orthogonal to all three lines of S. We will see below that the physics concepts of particle *colour* and *generation* can be recovered from the combinatorial concept of line system star-decomposition.

When L is an indecomposable star-closed line system, we can say a number of helpful things about subsets of lines in the star-decomposition of L, as developed in [**GR01**, chap. 12]. First, L is the star-closure of $S \,\dot\cup\, A$. Second, the set A does not contain any stars and we can find a set of vectors spanning A with all non-negative inner products. Third, any pair of orthogonal lines in A belongs to a set of three mutually orthogonal lines in A, called a *triad*.

Fourth, the triads in A always form the "lines" of a generalized quadrangle. So the task of classifying indecomposable star-closed line systems is equivalent to the task of classifying the (possibly trivial) generalized quadrangle structures with "lines" of size 3 on the set A of the star-decomposition of that system.

A *generalized quadrangle* is a point-line incidence structure such that the bipartite incidence graph has diameter 4 and girth 8. We denote by $\mathrm{Gq}(s,t)$ a generlized quadrangle in which each "line" contains $s+1$ "points" and each "point" belongs to $t+1$ "lines". In terms of A, the "points" are the lines of A and the "lines" are the orthogonal triads of A. We will see below that the lines corresponding to a single generation of particles define a generalized quadrangle $\mathrm{Gq}(2,2)$ with automorphism group S_6 (the exceptional symmetric group, with a non-trivial outer automorphism).

We will say that the lines of A *represent* graph G if we can find a vector on each line of A such that the Gram matrix of these vectors, apart from the diagonal entries, is the adjacency matrix of G. In the case of star-free A, the graph G has the lines of A for vertices and two vertices adjacent if and only if they are non-orthogonal lines. The vertices of this graph and the maximal independent sets must form the "points" and "lines" of a generalized quadrangle, albeit a possibly trivial one. This restriction on the possible structure of A yields the classification of indecomposable star-closed line systems. For more details on the following theorem, see [**GR01**, chap. 12].

THEOREM 3.2. [**CGSS76**] *Every indecomposable star-closed line system is the star-closure of a system of lines $S \dot\cup A$, where S is a star and A is a star-free set of lines orthogonal to just one line in S, and where A represents graph G with maximal independent sets forming a generalized quadrangle:*

(a) $\overline{\mathtt{A}_n}$ *for G the complete graph K_{n-2},*
(b) $\overline{\mathtt{D}_n}$ *for G the cocktail party graph $CP(n-3)$ plus an isolated vertex,*
(c) $\overline{\mathtt{E}_6}$ *for G the unique* $\mathrm{srg}(9,4,1,2)$*,*
(d) $\overline{\mathtt{E}_7}$ *for G the unique* $\mathrm{srg}(15,8,4,4)$*,*
(e) $\overline{\mathtt{E}_8}$ *for G the unique* $\mathrm{srg}(27,16,10,8)$*.*

Here we denote by $\overline{\Phi}$ a star-closed line system and by Φ the set of length $\sqrt{2}$ vectors that span the individual lines of $\overline{\Phi}$. As the labels above suggest, the star-closed line systems are precisely the lines spanned by the roots of the more familiar *simply-laced root systems*, the root systems with all equal-length roots. Note that the standard terminology is such that an *indecomposable* line system $\overline{\Phi}$ corresponds to an *irreducible* root system Φ.

REMARK 3.3. Not all irreducible root systems are simply-laced. That is, there are irreducible root systems of types $\mathtt{B}_n$, $\mathtt{C}_n$, $\mathtt{G}_2$, and $\mathtt{F}_4$ that include roots of two different lengths. We can recover these systems from line systems via the root lattices of the corresponding simply-laced root systems. Put another way, every root lattice is also the root lattice of a simply-laced root system

[**CS13**, p. 99]. First, suppose that we have an $\overline{\mathtt{A}_1^n} = \overline{\mathtt{A}_1 \times \mathtt{A}_1 \times \cdots \times \mathtt{A}_1}$ star-closed system of lines. This is simply a set of n mutually orthogonal lines in $\mathbb{R}^n$. Take the vectors of length $\sqrt{2}$ spanning these lines. These vectors span the root lattice of type $\mathbb{Z}^n$. The second layer of that lattice (the lattice points at the second shortest distance to the origin with the roots forming the first layer) is a root system of type $\mathtt{D}_n$, containing roots of length 2. The sum of these $\mathtt{A}_1^n$ roots of length $\sqrt{2}$ and $\mathtt{D}_n$ roots of length 2 is a $\mathtt{B}_n$ root system. Second, suppose that we have a $\overline{\mathtt{D}_n}$ star-closed system of lines. Take the vectors of length $\sqrt{2}$ spanning these lines to obtain a $\mathtt{D}_n$ root system spanning a $\mathtt{D}_n$ root lattice. There exists in the second layer of the $\mathtt{D}_n$ lattice a subset of vectors that both spans an $\overline{\mathtt{A}_1^n}$ set of lines and identifies additional reflection symmetries of the underlying $\mathtt{D}_n$ system. If we include these vectors, we obtain a $\mathtt{C}_n$ root system. Finally, we obtain the $\mathtt{G}_2$ roots by taking the first two layers of the lattice defined by $\overline{\mathtt{A}_2}$, and the $\mathtt{F}_4$ roots by taking the first two layers of the lattice defined by $\overline{\mathtt{D}_4}$.

3.3. Nested Sequences of Binary Decompositions

We have seen that any indecomposable star-closed line system admits a star decomposition. Apart from $\overline{\mathtt{E}_8}$, it turns out that every indecomposable star-closed line system also admits at least one *binary decomposition*, namely a partition $\overline{\Phi} = \overline{\Phi}_0 \,\dot{\cup}\, \overline{\Phi}_1$ such that $\overline{\Phi}_0$ is *star-closed*, $\overline{\Phi}_1$ is *star-free*, and $\overline{\Phi}$ is the star-closure of the star-free component $\overline{\Phi}_1$. We can characterize binary decompositions in terms of 3-gradings of simply-laced root systems, since each star-closed line system corresponds to a simply-laced root system. Following [**LN04**, p. 168], we define a *3-grading on a root system* Φ as a partition,

$$\Phi = \Phi_{-1} \,\dot{\cup}\, \Phi_0 \,\dot{\cup}\, \Phi_1,$$

such that,

$$\Phi \cap (\Phi_a + \Phi_b) \subset \Phi_{a+b},$$

and also,

$$\Phi \cap (\Phi_1 - \Phi_1) = \Phi_0.$$

That is, if the difference between any two roots in Φ_1 is also a root, then it is a root in Φ_0. Also, every root in Φ_0 is the difference of some two roots in Φ_1. The following properties are described in [**LN04**, p. 168] for 3-gradings on root systems. Since every 3-grading corresponds to a homomorphism from the corresponding root lattice to the grading group $\mathbb{Z}$, we have $\Phi_{-1} = -\Phi_1$. This means that we can recover the entire root system from the Φ_1 piece alone, as linear combinations of roots in Φ_1. In particular, the 3-grading defined by Φ_1 defines a star-free set of lines $\overline{\Phi}_1$, spanned by the roots of Φ_1. Just as we can recover Φ from Φ_1 by familiar Weyl reflections, so we can also recover $\overline{\Phi}$ from $\overline{\Phi}_1$ by star-closure.

The *coweight* of a root system Φ is a vector q such that for each root α in Φ, the *Euclidean inner product* (α, q) is an integer. In general, a $\mathbb{Z}$-grading on a root system Φ can be identified with some coweight q as follows [**LN04**, p. 166]:

$$\Phi_i = \Phi_i(q) = \{\alpha \in \Phi \mid (\alpha, q) = i \in \mathbb{Z}\}.$$

The coweights responsible for 3-gradings are the *minuscule coweights* [**LN04**, p. 61] (described as glue vectors in Chapter 1 and [**CS13**]). That is, a minuscule coweight of Φ is a vector q such that $(\alpha, q) = -1, 0, 1$ for all roots α. These facts can be used to show that the possible 3-gradings on connected root systems are classified using the weighted Coxeter-Dynkin diagrams shown in Table 3.1. In each case, we obtain the 3-grading of an irreducible root system Φ by identifying the Φ_0 component as the root subsystem with its Coxeter-Dynkin diagram given by the dark vertices [**LN04**, p. 171].

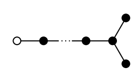
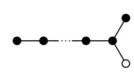
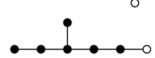
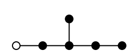
3-Grading Name	Diagram	$\Phi \xrightarrow{\lvert\Phi_1\rvert} \Phi_0$
rectangular		$\mathtt{A}_{p+q-1} \xrightarrow{pq} \mathtt{A}_{p-1} \times \mathtt{A}_{q-1}$
Hermitian		$\mathtt{C}_n \xrightarrow{\binom{n+1}{2}} \mathtt{A}_{n-1}$
odd quadratic		$\mathtt{B}_n \xrightarrow{2n-1} \mathtt{B}_{n-1}$
even quadratic		$\mathtt{D}_n \xrightarrow{2(n-1)} \mathtt{D}_{n-1}$
alternating		$\mathtt{D}_n \xrightarrow{\binom{n}{2}} \mathtt{A}_{n-1}$
Albert		$\mathtt{E}_7 \xrightarrow{27} \mathtt{E}_6$
bi-Cayley		$\mathtt{E}_6 \xrightarrow{16} \mathtt{D}_5$

TABLE 3.1. The 3-gradings on finite irreducible root systems.

We see from Table 3.1 that root systems of types $\mathtt{B}_n$, $\mathtt{C}_n$, $\mathtt{E}_6$, and $\mathtt{E}_7$ only admit one possible type of 3-grading. Root systems of types $\mathtt{A}_n$ and $\mathtt{D}_n$ admit multiple possible 3-gradings. In the case of $\mathtt{A}_n$ root systems, there are $\lfloor (n+1)/2 \rfloor$ possible rectangular 3-gradings. In the case of $\mathtt{D}_n$ root systems, there is a quadratic 3-grading and an alternating 3-grading. Root systems of types $\mathtt{E}_8$, $\mathtt{G}_2$, and $\mathtt{F}_4$ do not admit a 3-grading. In each case, we need only identify the Φ_0 component to identify the 3-grading.

We define a *sequence of nested* 3*-gradings* as a sequence of root systems $\Phi^{(n)} \subset \Phi^{(n+1)}$ such that $\Phi^{(n)} = \Phi_0^{(n+1)}$. We denote such a sequence using a diagram of the form,

$$\cdots \to \Phi^{(n+1)} \xrightarrow{|\Phi_1^{(n+1)}|} \Phi^{(n)} \xrightarrow{|\Phi_1^{(n)}|} \Phi^{(n-1)} \to \cdots$$

The weight of the arrow is the dimension of the 1-part of the 3-grading it represents. Figure 3.1 illustrates the structure of sequences of nested 3-gradings for the simply-laced root systems of rank 7 or less. Multiple sequences can pass through a single root system. For instance, from Figure 3.1 we see that both $\mathtt{D}_n \to \mathtt{D}_{n-1} \to \mathtt{A}_{n-2}$ and $\mathtt{D}_n \to \mathtt{A}_{n-1} \to \mathtt{A}_{n-2}$ represent possible nestings of 3-gradings containing both $\mathtt{D}_n$ and $\mathtt{A}_{n-2}$. The diagram could be extended to the upper-right by including higher rank root systems, adding the arrows $\mathtt{D}_8 \xrightarrow{14} \mathtt{D}_7$, $\mathtt{A}_8 \xrightarrow{8} \mathtt{A}_7$, $\mathtt{D}_8 \xrightarrow{28} \mathtt{A}_7$, and so on.

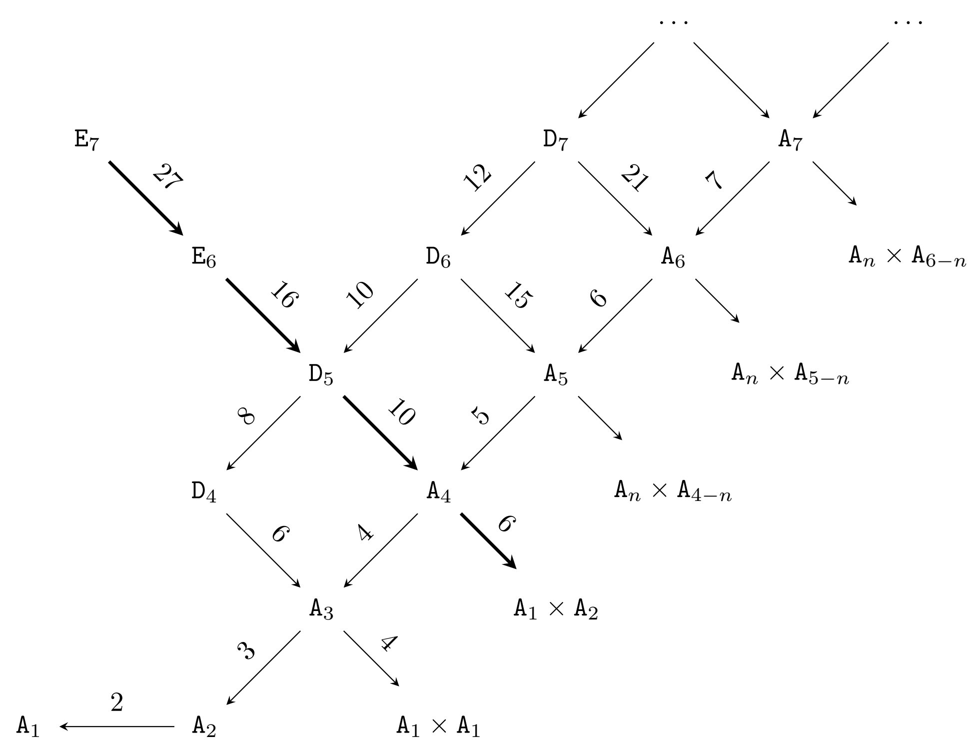


Figure 3.1. Nested 3-gradings of simply-laced root irreducible root systems in $\mathbb{R}^7$.

Sequences of nested 3-gradings on root systems correspond to sequences of nested binary decompositions on line systems, and vice versa. The mesh of available 3-gradings on irreducible root systems shown in Figure 3.1 also applies to indecomposable star-closed line systems to describe the available binary decompositions. By working with line systems, we can better appreciate that the combinatorial properties of Φ_1 and Φ_{-1} are equivalent. Indeed, the lines defined by Φ_1 are precisely the same lines defined by Φ_{-1}.

The exceptional sequence ($\star$), bolded in Figure 3.1, is one example of a sequence of nested 3-gradings of simply-laced root systems, or equivalently a sequence of nested binary decompositions of star-closed line systems.

3.4. The Exceptional Sequence

We now identify some special properties of sequence ($\star$), in comparison to all other possible sequences of nested binary decompositions, as illustrated in Figure 3.1.

First, sequence ($\star$) begins with $\mathtt{E}_7$, which is the only indecomposable star-closed line system (or irreducible simply-laced root system) that admits a binary decomposition but is not embedded in another line system as the zero-component of a binary decomposition. That is, any sequence of nested binary decompositions can be extended further to the left unless it begins with $\mathtt{E}_7$. So sequences that begin with $\mathtt{E}_7$ and end in either $\mathtt{A}_1$, $\mathtt{A}_1 \times \mathtt{A}_1$, or $\mathtt{A}_1 \times \mathtt{A}_2$ are unique in that they cannot be made any longer by being extended to the left or the right.

Second, sequence ($\star$) is a local sequence in the following sense. For a binary decomposition $\overline{\Phi} = \overline{\Phi}_1 \mathbin{\dot\cup} \overline{\Phi}_0$, we can define a *binary decomposition graph* G with the lines of $\overline{\Phi}_1$ for vertices and all pairs of nonorthogonal lines for edges. Using this definition, we can assign a graph to each binary decomposition, or arrow, in a nested sequence. The graph of a binary decomposition for $\overline{\Phi}$ indecomposable is always vertex-transitive. This means that there is a unique *local subgraph* of G, the induced subgraph on the neighbours of any given point. We will say that a sequence of nested binary decompositions is a *local sequence* when the binary decomposition graph of each arrow is isomorphic to the local subgraph of the binary decomposition graph in the preceding arrow. The possible local sequences beginning with indecomposable star-closed line systems are as follows:

$$\begin{aligned}
&\cdots \to \mathtt{A}_n \to \mathtt{A}_{n-1} \to \cdots \to \mathtt{A}_2 \to \mathtt{A}_1,\\
&\cdots \to \mathtt{D}_n \to \mathtt{D}_{n-1} \to \cdots \to \mathtt{D}_4 \to \mathtt{A}_3 \to \mathtt{A}_1 \times \mathtt{A}_1,\\
&\mathtt{D}_n \to \mathtt{A}_{n-1} \to \mathtt{A}_1 \times \mathtt{A}_{n-3},\\
&\mathtt{E}_7 \to \mathtt{E}_6 \to \mathtt{D}_5 \to \mathtt{A}_4 \to \mathtt{A}_1 \times \mathtt{A}_2.
\end{aligned}$$

If we restrict ourselves to local sequences that cannot be embedded in a longer sequence, then the exceptional sequence ($\star$) is the only one with this property, since it is the only local sequence that begins with $\mathtt{E}_7$.

Third, sequence ($\star$) is a maximal sequence in the following sense. We say that a sequence of nested binary decompositions is a *maximal sequence* when the path of the sequence through the possible binary decompositions, shown in Figure 3.1, is such that the largest $\overline{\Phi}_1$ component is chosen in each case.

That is, a maximal sequence always follows the highest weight arrows from a given starting point in Figure 3.1.

THEOREM 3.4. *The sequence* $(\star)$ *is the unique local and maximal sequence of nested 3-gradings (or binary decompositions) that cannot be embedded in a longer sequence.*

PROOF. Any sequence that cannot be embedded in a longer sequence begins with $\mathtt{E}_7$. The only local sequence beginning with $\mathtt{E}_7$ is the sequence $(\star)$. Likewise, the only maximal sequence beginning with $\mathtt{E}_7$ is the sequence $(\star)$. $\square$

REMARK 3.5. The minuscule coweights of $\mathtt{E}_7$ span the unique system of 28 equiangular lines in $\mathbb{R}^7$ that attain the absolute bound $\binom{7+1}{2}$ described earlier (Example A.6). By *acute minuscule coweights* we mean a set of minuscule coweights with positive pairwise inner product. Recall that $\overline{\mathtt{E}_6}$ is constructed by taking the lines of $\overline{\mathtt{E}_7}$ orthogonal to a single member of the 28 equiangular lines. Likewise, $\overline{\mathtt{D}_5}$, $\overline{\mathtt{A}_4}$, and $\overline{\mathtt{A}_1 \times \mathtt{A}_2}$ are constructed as the lines of $\overline{\mathtt{E}_7}$ orthogonal to a pair, triple, and quadruple of acute minuscule coweights, and the subset of the 28 equiangular lines they span. So we can also understand the sequence $(\star)$ by taking roots orthogonal to successively larger sets of acute minuscule coweights of $\mathtt{E}_7$.

3.5. Lie Algebras of Star-Closed Line Systems

Certain important Lie and Jordan structures correspond to star-closed line systems and binary decompositions. Indeed, all Jordan triple systems are constructed from 3-gradings on root systems, or equivalently from binary decompositions on line systems. In what follows we focus on Lie algebras, given their direct application to particle physics. Even so, many of the structures described below could be constructed using the Jordan triple systems corresponding to the 3-graded Lie algebra in question.

A *Lie algebra* is a vector space $\mathfrak{g}$ with product $[x, y]$ such that $[x, x] = 0$ and $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all vectors x, y, z. Lie algebras are non-associative in general, and we say that a Lie algebra is *abelian* when $[x, y] = 0$ for all x, y. We can construct certain important Lie algebras (the semi-simple ones) using root systems, including the simply-laced root systems corresponding to star-closed line systems.

THEOREM 3.6. [**Car72**, pp. 42-43] *Let* Φ *be an irreducible root system. Then there exists, up to Lie algebra isomorphism, a simple Lie algebra* $\mathfrak{g}$ *over* $\mathbb{C}$ *with a Chevalley basis.*

That is, given root system Φ, there is a Φ-graded Lie algebra of the form,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{r \in \Phi} \mathfrak{g}_r.$$

This is called the *Cartan grading* of Lie algebra $\mathfrak{g}$. Here $\mathfrak{h}$ is a *Cartan subalgebra* of $\mathfrak{g}$ and $\mathfrak{g}_r$ are the *root spaces* of the decomposition. The dimension of $\mathfrak{h}$ is equal to the dimension of the space $\mathbb{R}^n$ spanned by the roots Φ, whereas the dimension of each root space $\mathfrak{g}_r$ is 1. The *rank* of the Lie algebra is the dimension of $\mathfrak{h}$. The Cartan subalgebra $\mathfrak{h}$ has basis h_r, where r is each simple root of Φ (corresponding to the vertices of the Coxeter-Dynkin diagram of Φ). Each subalgebra $\mathfrak{g}_r$ is spanned by basis vector e_r, where r is a root in Φ. For any x not in Φ we have $\mathfrak{g}_x = 0$. The products involving the Cartan subalgebra $\mathfrak{h}$ are defined entirely in terms of the geometry of the roots r, s in Φ:

$$[h_r, h_s] = 0, \qquad [h_r, e_s] = \frac{2(r,s)}{(r,r)} e_s, \qquad [e_r, e_{-r}] = h_r.$$

Here (r, s) denotes the standard Euclidean inner product between vectors r, s in $\mathbb{R}^n$ (where $\dim_{\mathbb{C}}(\mathfrak{h}) = n$). Products of the root spaces of two linearly independent roots are defined by,

$$[e_r, e_s] = N_{r,s} e_{r+s}.$$

The structure constants $N_{r,s}$ can be fixed without loss of generality to define the Chevalley basis, as described in [**Car72**, pp. 56-57]. Theorem 3.6 applies to all irreducible root systems. In what follows we only make use of the cases involving simply-laced root systems, which are listed in Table 3.2 [**Car72**, p. 43].

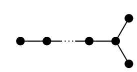
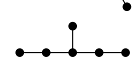
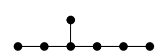
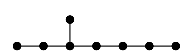
Type	$\mathfrak{g}$	dim $\mathfrak{g}$	rank $\mathfrak{g}$	$\lvert\Phi\rvert$	Dynkin diagram
$\mathtt{A}_n$ $(n \geq 1)$	$\mathfrak{sl}_{n+1}$	$n(n+2)$	n	$n(n+1)$	
$\mathtt{D}_n$ $(n \geq 4)$	$\mathfrak{so}_{2n}$	$n(2n-1)$	n	$2n(n-1)$	
$\mathtt{E}_6$	$\mathfrak{e}_6$	78	6	72	
$\mathtt{E}_7$	$\mathfrak{e}_7$	133	7	126	
$\mathtt{E}_8$	$\mathfrak{e}_8$	248	8	240	

TABLE 3.2. The Lie algebras of simply-laced root systems.

Suppose that indecomposable star-closed line system $\overline{\Phi}$ admits a binary grading, $\overline{\Phi} = \overline{\Phi}_1 \dot\cup \overline{\Phi}_0$. The lines of the star-free component $\overline{\Phi}_1$ can be spanned by roots with non-negative inner products. We denote these spanning roots by Φ_1 and define $\Phi_{-1} = -\Phi_1$ as the set of opposite roots, which also has all non-negative inner products. Then we have the following 3-grading on $\mathfrak{g}$ as a coarsening of the Cartan grading:

$$\mathfrak{g} = \left(\bigoplus_{r \in \Phi_{-1}} \mathfrak{g}_r \right) \oplus \left(\mathfrak{h} \oplus \bigoplus_{r \in \Phi_0} \mathfrak{g}_r \right) \oplus \left(\bigoplus_{r \in \Phi_1} \mathfrak{g}_r \right) = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1).$$

That is,

$$[\mathfrak{g}(i), \mathfrak{g}(j)] \subseteq \mathfrak{g}(i+j).$$

We see, then, that a 3-grading on a root system $\Phi \xrightarrow{n} \Phi_0$ defines abelian Lie subalgebras $\mathfrak{g}(-1)$ and $\mathfrak{g}(1)$ of dimension $n = |\Phi_{-1}| = |\Phi_1| = |\overline{\Phi}_1|$. The $\mathfrak{g}(0)$ Lie subalgebra acts on each of these abelian Lie subalgebras via $[\mathfrak{g}(0), \mathfrak{g}(\pm 1)] \subseteq \mathfrak{g}(\pm 1)$. Also, since the entire Cartan subalgebra $\mathfrak{h}$ is contained in $\mathfrak{g}(0)$, we see that $\mathfrak{g}(0)$ is not isomorphic to the Lie algebra constructed from root system Φ_0, but rather is the direct product of this algebra and the one-dimensional abelian Lie algebra:

$$\mathfrak{g}(0) = \mathbb{C} \oplus [\mathfrak{g}(0), \mathfrak{g}(0)].$$

That is, $\mathfrak{g}(0)$ contains $\mathfrak{h}$, the Cartan subalgebra of $\mathfrak{g}$. But $[\mathfrak{g}(0), \mathfrak{g}(0)]$ does not contain $\mathfrak{h}$. The Cartan subalgebra of $[\mathfrak{g}(0), \mathfrak{g}(0)]$ is a subalgebra of $\mathfrak{h}$ with one dimension less than $\mathfrak{h}$.

In particular, the binary decomposition $\mathtt{A}_4 \to \mathtt{A}_1 \times \mathtt{A}_2$ signifies the following Lie algebra 3-grading:

$$\mathfrak{sl}_5 = \mathfrak{sl}_5(-1) \oplus \mathfrak{sl}_5(0) \oplus \mathfrak{sl}_5(1),$$

where $\mathfrak{sl}_5(1)$ is six-dimensional and $[\mathfrak{sl}_5(0), \mathfrak{sl}_5(0)] = \mathfrak{sl}_2 \oplus \mathfrak{sl}_3$. This means that the 0-piece of this 3-grading is,

$$\mathfrak{sl}_5(0) = \mathbb{C} \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_3.$$

This Lie algebra—the 0-piece of the 3-grading due to the $\mathtt{A}_4 \to \mathtt{A}_1 \times \mathtt{A}_2$ binary decomposition—is in fact the Lie algebra of the standard model of particle physics.

3.6. Connection to the Standard Model

The exceptional sequence ($\star$) corresponds to the following sequence of nested Lie algebra 3-gradings:

$$\mathfrak{e}_7 \xrightarrow{27} \mathfrak{e}_6 \xrightarrow{16} \mathfrak{so}_{10} \xrightarrow{10} \mathfrak{sl}_5 \xrightarrow{6} \mathfrak{sl}_2 \oplus \mathfrak{sl}_3.$$

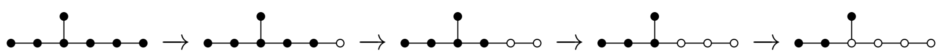
The final arrow, $\mathtt{A}_4 \to \mathtt{A}_1 \times \mathtt{A}_2$, corresponding to the diagram $\bullet - \bullet - \circ - \bullet$, yields the Lie algebra of the standard model of particle physics as the 0-piece of the 3-grading.

$$\mathfrak{g}_{SM} = \mathbb{C} \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_3.$$

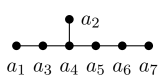
Our next step is to determine the action of $\mathfrak{g}_{SM}$ on the rest of $\mathfrak{e}_7$, so that we can identify certain root spaces with familiar standard model particles.

Each root in $\mathtt{E}_7$ indexes a one-dimensional root space $\mathfrak{g}_r$, spanned by vector e_r, in the Lie algebra $\mathfrak{e}_7$ described above. By construction, each e_r is an eigenvector of each h in the Cartan subalgebra $\mathfrak{h}$, since we have $[h_s, e_r] = 2(s,r)/(s,s)e_r = (s,r)e_r$. Recall that (in the Chevalley basis) $\mathfrak{g}_{SM}$ contains the Cartan subalgebra of $\mathfrak{sl}_5$, which has dimension 4 and is itself a subalgebra of $\mathfrak{h}$, the Cartan subalgebra of $\mathtt{E}_7$. In order to find the correspondence between root spaces $\mathfrak{g}_r$ and particles, we need to find a well-chosen basis of $\mathfrak{h} \cap \mathfrak{g}_{SM}$ (the Cartan subalgebra of $\mathfrak{sl}_5$). The four simultaneous eigenvalues with respect to this basis give us the familiar *hypercharge*, *isospin*, and *colour* of each particle (where colour signifies a pair of eigenvalues). Since $\mathfrak{h}$ is seven dimensional, there are three possible remaining simultaneous eigenvalues. We can use two of these to assign a *generation* to each root space $\mathfrak{g}_r$ and the remaining eigenvalue to distinguish particles of the standard model from additional particles.

For specificity, we will denote the exceptional sequence ($\star$) in terms of Coxeter-Dynkin diagrams as follows:

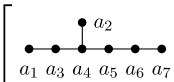


We may write vectors in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{e}_7$ using Dynkin diagrams, e.g.:



$$= \sum_{i=1}^{7} a_i h_{s_i} \in \mathfrak{h},$$

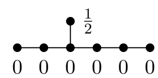
where s_i are a set of simple roots of $\mathtt{E}_7$. This means that we compute the eigenvalues of the action of a vector in $\mathfrak{h}$ on a root space as follows:



$$\left[\;\cdot\;, \mathfrak{g}_r\right] = \left(\sum_{i=1}^{7} a_i (s_i, r)\right) \mathfrak{g}_r.$$

We define the *isospin* of each root space $\mathfrak{g}_r$ as its eigenvalue for multiplication by the following vector in $\mathfrak{h} \cap \mathfrak{g}_{SM}$:

$$W_0 =$$



The vector $2W_0$ is a coweight of $\mathtt{E}_7$ (also a coroot) and defines an isospin 5-grading on $\mathtt{E}_7$. Root spaces with isospin 0 correspond to *right-handed* particles (left-handed anti-particles). Root spaces with isospin $\pm\frac{1}{2}$ correspond to *left-handed* particles (right-handed anti-particles). The unique root spaces with isospins ± 1 correspond to the $W^{\pm}$ bosons. Specifically, W_0 and $W^{\pm}$ span the $\mathfrak{sl}_2$ (i.e., $\mathtt{A}_1$), component of the standard model Lie algebra $\mathfrak{g}_{SM}$.

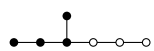
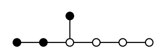
We define the *colour* of each root space as the pair of eigenvalues of the following vectors in $\mathfrak{h} \cap \mathfrak{g}_{SM}$:

$$\lambda_3 = \begin{smallmatrix} & & 0 & & & \\ 1 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}, \qquad \sqrt{3}\lambda_8 = \begin{smallmatrix} & & 0 & & & \\ 1 & 2 & 0 & 0 & 0 & 0 \end{smallmatrix}.$$

The vectors λ_3, $\sqrt{3}\lambda_8$ in $\mathfrak{h}$ as well as the six unique root spaces $\mathfrak{g}_r$ with eigenvalues $\pm(2,0)$, $\pm(-1,3)$, $\pm(-1,-3)$ form the $\mathfrak{sl}_3$ (i.e., $\mathtt{A}_2$) component of the standard model Lie algebra $\mathfrak{g}_{SM}$. These eight-dimensions of $\mathfrak{e}_7$ represent the eight *gluons*, the bosons responsible for the strong force. The corresponding $\mathtt{A}_2$ root system defines a star-decomposition of $\mathtt{E}_7$ that allows us to assign particle colour. Specifically, we will call *blue* the fifteen root spaces with $\lambda_3, \sqrt{3}\lambda_8$ eigenvalues $(0,2)$, their opposite root spaces are called *anti-blue*. Likewise, eigenvalues $(-1,-1)$ signify *red* and eigenvalues $(1,-1)$ signifies *green*. The opposite eigenvalues signify *anti-red* and *anti-green*. Finally, the 30 root spaces with eigenvalues $(0,0)$ are called *colourless*. Root spaces outside of $\mathfrak{g}_{SM}$ that are red, green, or blue correspond to *quarks* whereas those that are colourless correspond to *leptons*.

We define the *hypercharge* of each root space as the eigenvalue of the following operator:

$$B = \begin{smallmatrix} & & 1 & & & \\ \frac{2}{3} & \frac{4}{3} & 2 & 0 & 0 & 0 \end{smallmatrix}.$$

The 3-grading defined by  $\to$  also defines a unique line perpendicular to the $\mathbb{R}^3$ of $\mathtt{A}_1 \times \mathtt{A}_2$ (spanned by W_0, λ_3, λ_8) but also within the $\mathbb{R}^4$ spanned by the coroots of $\mathtt{A}_4$. This unique line is spanned by the hypercharge operator B. Although B is not a coweight, there is a 13-grading defined by coweight $3B$. This means that the eigenvalues of B are in the set $\left\{0, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm\frac{4}{3}, \pm\frac{5}{3}, \pm 2\right\}$. All of these values correspond to known and observed physical particles except for $\pm\frac{5}{3}$, which are the eigenvalues of the root spaces of the roots in $\mathtt{A}_4 \setminus (\mathtt{A}_1 \times \mathtt{A}_2)$. These $B = \pm\frac{5}{3}$ particles are not observed within the standard model but emerge when the standard model is embedded within an $\mathtt{A}_4$ Lie algebra, as in the original SU(5) grand unification theory of Georgi and Glashow [**GG74**].

Using the four simultaneous eigenvalues of $B, W_0, \lambda_3, \lambda_8$, we can assign a standard particle name to each of the root spaces $\mathfrak{g}_r$ with roots in $\mathtt{E}_7 \setminus \mathtt{A}_4$, as shown in Table 3.3. Here we label particles according to the eigenvalues for hypercharge and isospin given in [**BH10**], while the three colour labels (red, green, blue) are treated as conventional.

Remark 3.7. *Anti-particles* correspond to roots with opposite eigenvalues of the partner particle. Just as each root describes a particle or anti-particle, each line in the corresponding line system describes a particle/anti-particle

Name	Symbol	B	W_0	λ_3	$\sqrt{3}\lambda_8$
Right-handed neutrino	ν_R	0	0	0	0
Right-handed electron	e_R^-	-2	0	0	0
Right-handed red up quark	u_R^r	$\frac{4}{3}$	0	-1	-1
Right-handed green up quark	u_R^g	$\frac{4}{3}$	0	1	-1
Right-handed blue up quark	u_R^b	$\frac{4}{3}$	0	0	2
Right-handed red down quark	d_R^r	$-\frac{2}{3}$	0	-1	-1
Right-handed green down quark	d_R^g	$-\frac{2}{3}$	0	1	-1
Right-handed blue down quark	d_R^b	$-\frac{2}{3}$	0	0	2
Left-handed neutrino	ν_L	-1	$\frac{1}{2}$	0	0
Left-handed electron	e_L^-	-1	$-\frac{1}{2}$	0	0
Left-handed red up quark	u_L^r	$\frac{1}{3}$	$\frac{1}{2}$	-1	-1
Left-handed green up quark	u_L^g	$\frac{1}{3}$	$\frac{1}{2}$	1	-1
Left-handed blue up quark	u_L^b	$\frac{1}{3}$	$\frac{1}{2}$	0	2
Left-handed red down quark	d_L^r	$\frac{1}{3}$	$-\frac{1}{2}$	-1	-1
Left-handed green down quark	d_L^g	$\frac{1}{3}$	$-\frac{1}{2}$	1	-1
Left-handed blue down quark	d_L^b	$\frac{1}{3}$	$-\frac{1}{2}$	0	2

TABLE 3.3. Fermion particle nomenclature.

pair. Whether we choose to work with Lie structures (roots) or Jordan structures (lines) largely corresponds to whether we choose to work with particles or with particle/anti-particle pairs.

The next task is to sort the particles into generations, and to identify any additional particles beyond those given in the standard model. To do so, we note that the Lie centralizer of the standard model Lie algebra in $\mathfrak{e}_7$ has the form,

$$C_{\mathfrak{e}_7}(\mathfrak{g}_{SM}) = \mathbb{C}^2 \oplus \mathfrak{sl}_3.$$

The $\mathfrak{sl}_3$ component is generated by the root spaces $\mathfrak{g}_r$ corresponding to the unique six roots in $\mathtt{E}_7$ perpendicular to each root in $\mathtt{A}_4$. These six root are unique in $\mathtt{E}_7$ in that their root spaces have null hypercharge, isospin, and are colourless. For this reason, we call them *right-handed neutrinos* (and left-handed anti-neutrinos)—the undetectable partners to left-handed neutrinos (and right-handed anti-neutrinos). These six $\mathfrak{g}_r$ root spaces in the centralizer of $\mathfrak{g}_{SM}$ serve the same role as the six coloured gluons in $\mathfrak{g}_{SM}$. Just as the six coloured gluons define the star-decomposition of $\mathtt{E}_7$ that gives us particle colour, the three right-handed neutrinos and their anti-particles can be used to define a second star-decomposition of $\mathtt{E}_7$ that gives us particle *generation*. We assign particle generation to each root space $\mathfrak{g}_r$ using the eigenvalue pair of the following two operators:

$$\rho_3 = \begin{smallmatrix} & & 0 & & & \\ 0 & 0 & 0 & 0 & 1 & 0 \end{smallmatrix}, \qquad \sqrt{3}\rho_8 = \begin{smallmatrix} & & 0 & & & \\ 0 & 0 & 0 & 0 & 1 & 2 \end{smallmatrix}.$$

Specifically, we may call the thirty root spaces with $\rho_3, \sqrt{3}\rho_8$ eigenvalues $\pm(0, 2)$ the *first generation*, the thirty with eigenvalues $\pm(1, 1)$ the *second generation*, and the thirty with eigenvalues $\pm(1, -1)$ the *third generation*. Each generation consists of fifteen particles with the eigenvalues given in Table 3.3 and the corresponding fifteen anti-particles. Any root spaces with eigenvalues $(0, 0)$ do not belong to any generation. These include the boson root spaces of $\mathfrak{g}_{SM}$ and 22 additional root spaces.

So far we have defined an orthogonal basis $\{\rho_3, \rho_8, B, W_0, \lambda_3, \lambda_8\}$ for a $\mathbb{C}^6$ subspace of $\mathfrak{h}$, and can use the simultaneous eigenvalues of this basis to partition $\mathfrak{e}_7$ into the familiar standard model bosons $\mathfrak{g}_{SM}$, a right-handed neutrino $\mathfrak{sl}_3$, three generations of fifteen particles and their anti-particles, plus 22 additional root spaces and one remaining dimension of $\mathfrak{h}$ perpendicular to this $\mathbb{C}^6$. We can use this remaining dimension to distinguish familiar particles from potentially new and unobserved ones. That is, we define a seventh vector in $\mathfrak{h}$ perpendicular to $\mathbb{C}^6$:

$$H = \begin{smallmatrix} & & \frac{3}{2} & & & \\ 1 & 2 & 3 & \frac{5}{2} & \frac{5}{3} & \frac{5}{6} \end{smallmatrix}$$

The vector $3H$ is a coweight of $\mathtt{E}_7$ and defines a 7-grading, so the eigenvalues of H are in the set $\left\{0, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1\right\}$.

It turns out that the three generations of particles are precisely the root spaces $\mathfrak{g}_r$ with H eigenvalues $\pm\frac{1}{3}$ and $\pm\frac{2}{3}$. Furthermore the particles with H eigenvalue 0 consist of the bosons of $\mathfrak{g}_{SM}$, the right-handed neutrino $\mathfrak{sl}_3$, and the particles with hypercharge $\pm\frac{5}{3}$ (corresponding to root spaces $\mathfrak{g}_r$ with r in $\mathtt{A}_4 \setminus \mathtt{A}_1 \times \mathtt{A}_2$).

To summarize, we can trim out the unobserved particles of $\mathfrak{e}_7$ by making ± 1 a forbidden eigenvalue of H and $\pm\frac{5}{3}$ a forbidden eigenvalue of B. All other root spaces correspond to familiar bosons and the three generations of fermions.

Remark 3.8. The fifteen particle/anti-particle pairs of a single generation correspond to a generalized quadrangle structure in the following way. If we take the corresponding 15 roots in $\mathtt{E}_7$, then these span a star-free line system representing the unique graph $\mathrm{srg}(15, 8, 4, 4)$. This graph has precisely 15 maximal independent sets, all of size 3, representing triads of orthogonal lines. These triads serve as the "lines" of a generalized quadrangle $\mathrm{Gq}(2, 2)$. In terms of the nomenclature given in Table 3.3, the particle triads are:

$$\begin{array}{lll}\{\nu_L, u^r_L, u^r_R\}, & \{\nu_L, u^g_L, u^g_R\}, & \{\nu_L, u^b_L, u^b_R\},\\ \{e^-_L, d^r_L, u^r_R\}, & \{e^-_L, d^g_L, u^g_R\}, & \{e^-_L, d^b_L, u^b_R\},\\ \{e^-_R, u^r_R, d^r_R\}, & \{e^-_R, u^g_R, d^g_R\}, & \{e^-_R, u^b_R, d^b_R\},\\ \{u^r_L, d^g_L, d^b_R\}, & \{u^r_L, d^b_L, d^g_R\}, & \{u^g_L, d^r_L, d^b_R\},\\ \{u^b_L, d^r_L, d^g_R\}, & \{u^g_L, d^b_L, d^r_R\}, & \{u^b_L, d^g_L, d^r_R\}.\end{array}$$

Of these fifteen triads, six have the property that they do not contain a lepton. These six are also the only six where the eigenvalues of $B, W_0, \lambda_3, \lambda_8$ each add to zero over the triad:

$$\begin{array}{lll}\{u^r_L, d^g_L, d^b_R\}, & \{u^r_L, d^b_L, d^g_R\}, & \{u^g_L, d^r_L, d^b_R\},\\ \{u^b_L, d^r_L, d^g_R\}, & \{u^g_L, d^b_L, d^r_R\}, & \{u^b_L, d^g_L, d^r_R\}.\end{array}$$

In fact, this subset of six triads forms a smaller generalized quadrangle $\mathrm{Gq}(2,1)$ on nine points. The roots corresponding to these particle root spaces have the following interesting property. The roots of the $\mathrm{Gq}(2,1)$ particles all have non-negative inner product, as do the roots of the $\mathrm{Gq}(2,2)\setminus\mathrm{Gq}(2,1)$ particles. However, the inner products between a root from each of the two sets is always non-positive. The fact that a generation of 15 particles does not correspond to a set of roots with all non-negative inner products, but rather describes an embedding of $\mathrm{Gq}(2,1)$ within $\mathrm{Gq}(2,2)$, leaves a tempting combinatorial clue regarding abundance of matter and the dearth of antimatter in the physical universe. Specifically, a generation of fermions splits into matter and antimatter components in such a way that a $\mathrm{Gq}(2,1)$ substructure emerges from a larger $\mathrm{Gq}(2,2)$.

3.7. Discussion

This chapter does not attempt to account for the Higgs mechanism, the embedding of electromagnetism within the electroweak force, or particle spin. Neither does it speculate on a role for the 22 additional root-spaces within $\mathfrak{e}_7$ that do not correspond to familiar particles of the standard model. Rather, this chapter converts certain questions about the accidental properties of particle physics into corresponding questions about exceptional mathematical objects. To the question of why we have this particular standard model Lie algebra $\mathfrak{g}_{SM}$ and not another, perhaps we could answer that this is the Lie algebra in which the exceptional sequence terminates. To the question of why there are three generations of fifteen particles that represent this Lie algebra (or sixteen with the right-handed neutrino), perhaps we could answer that the exceptional sequence defines an action of $\mathfrak{g}_{SM}$ on $\mathfrak{e}_7$ and that star-decomposition explains the existence of three generations. Most remarkably, questions about physical symmetries and structures can perhaps be answered in terms of systems of equiangular lines at the absolute bound, beginning with 3 line stars and the 28 lines spanned by the minuscule coweights of $\mathtt{E}_7$.

CHAPTER 4

Rational Angles and Tight T-Designs

4.1. Introduction

Combinatorial t-designs were generalized to spherical t-designs in [**DGS77**] and to projective spaces in [**Neu81**] (see also [**Sei90**]). Given a finite subset X of a sphere or projective space we can evaluate both the angle set $A(X)$ and strength t of that subset. For a given strength t there exists an absolute lower bound on the cardinality $|X|$ such that X is a t-design. Likewise, for a given cardinality s, there is an absolute upper bound on the cardinality $|X|$ such that $|A(X)| = s$. Furthermore, t is bounded by s according to the inequality $t \leq 2s - \varepsilon$ where $\varepsilon = |A \cap \{0\}|$. These three bounds are satisfied simultaneously if any one of them is met. When a set X meets these absolute bounds, X is called a *tight t-design*.

The full classification of tight t-designs is incomplete, but various theorems place upper bounds on the value of t for different geometries (e.g., [**Hog89**], [**BH89**]). In the case of projective geometries, many of these theorems constraining t depend on a result given in [**Hog84**] that, except for the real projective line, the angle set $A(X)$ must be rational. However, a counter-example exists in the case of the complex projective line: a subset corresponding to the vertices of an icosahedron. This counter-example is examined in [**Lyu09**], which attempts to repair the defective proof in [**Hog84**]. Unfortunately, the repair in [**Lyu09**] is restricted to the real, complex, and quaternion projective cases. It neglects the octonion projective case and the spherical cases. The aim of this chapter is to complete the repair in [**Lyu09**] by including the remaining octonion and spherical cases. This also generalizes the attempted proof in [**Hog84**] to the full family of spherical cases (a recent independent proof of the spherical cases is available in [**BNS22**]). In order to treat all possible cases at once, we will work with the primitive idempotents of simple Euclidean Jordan algebras. This allows us to treat the spherical, projective, and octonion cases in a unified way.

4.2. Jordan Algebras and T-Designs

This section reviews simple Euclidean Jordan algebras and the concepts required to identify and describe tight t-designs. In addition to the real numbers $\mathbb{R}$, the classification of simple Euclidean Jordan algebras consists of four infinite families and one exception. The first infinite family has rank $\rho = 2$ and degree $d \geq 1$. The second, third, and fourth family respectively have degree $d = 1, 2, 4$ and rank $\rho \geq 3$. The exceptional Euclidean Jordan algebra has rank $\rho = 3$ and degree $d = 8$. Each Euclidean Jordan algebra has a well defined trace that we can use to define a Euclidean inner product,

$$\langle x, y \rangle = \mathrm{Tr}(x \circ y).$$

Here $\circ$ denotes the Jordan product. Let V be a simple Euclidean Jordan algebra of rank ρ and degree d. We denote by $\mathcal{J}(V)$ the manifold of primitive idempotents of V. The rank $\rho = 2$ family has manifolds of primitive idempotents isometric to spheres. The degrees $d = 1, 2, 4$ families have manifolds of primitive idempotents respectively isometric to real, complex, and quaternionic projective spaces. Finally, the rank $\rho = 3$, degree $d = 8$ exceptional case has a manifold of primitive idempotents isometric to the octonion projective plane. This means we can use simple Euclidean Jordan algebras and their manifolds of primitive idempotents to model the following geometries for $d \geq 1$ and $\rho \geq 3$:

$$\Omega_{d+1}, \quad \mathbb{RP}^{\rho-1}, \quad \mathbb{CP}^{\rho-1}, \quad \mathbb{HP}^{\rho-1}, \quad \mathbb{OP}^2.$$

More details about simple Euclidean Jordan algebras can be found in [**FK94**].

Let X be a finite subset of $\mathcal{J}(V)$, the manifold of primitive idempotents of simple Euclidean Jordan algebra V. Then we call X an *A-code*, namely a set of points where all inner products of distinct elements belong in set A. We define the *angle set* $A(X)$ as,

$$A(X) = \{\langle x, y \rangle \mid x \neq y \in X \subset \mathcal{J}(V)\}.$$

Subset X is also a *t-design* where t is the largest integer such that X satisfies [**DGS77**], [**Hog82**],

$$\sum_{x \in X} \sum_{y \in X} Q_k^0(\langle x, y \rangle) = 0, \quad k = 1, 2, \ldots, t.$$

Here we use *renormalized Jacobi functions* $Q_k^\varepsilon(x)$ with $\varepsilon = 0, 1$ in terms of the rank ρ and degree d of V as follows (recall that Ω_{d+1} has rank 2 and degree d):

$$Q_k^\varepsilon(x) = \left(\frac{\frac{1}{2}\rho d + 2k + \varepsilon - 1}{\frac{1}{2}\rho d + k + \varepsilon - 1} \right) \frac{(\frac{1}{2}\rho d)_{k+\varepsilon}}{(\frac{1}{2}d)_{k+\varepsilon}} P_k^{(\frac{1}{2}d(\rho-1)-1, \frac{1}{2}d-1+\varepsilon)}(2x-1).$$

Here and below, $P_k^{(\alpha,\beta)}(x)$ denotes Jacobi polynomials as defined in [**AS72**, 22.2.1]. We use the *Pochhammer symbol* $(x)_n$ for non-negative integer n, which can also be defined in terms of the usual Γ function:

$$(x)_n = x(x+1)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}.$$

The expression for $Q_k^\varepsilon(x)$ given here generalizes both the spherical and projective cases described respectively in [**DGS77**] and [**Hog82**]. Appendix B verifies that this generalization is the correct one.

Some of the polynomials that we will use are given below, with $N = \frac{1}{2}\rho d$ and $m = \frac{1}{2}d$. The N, m notation is more common in the literature about projective designs.

$$\begin{aligned}
Q_0^0(x) &= 1, \\
Q_0^1(x) &= \frac{N}{m}, \\
Q_1^0(x) &= (N+1)\left(\frac{N}{m}x - 1\right), \\
Q_2^0(x) &= \left(\frac{N(N+3)}{2m(m+1)}\right)\left((N(N+3)+2)x^2\right. \\
&\quad \left. -2(N+1)(m+1)x + m(m+1)\right).
\end{aligned}$$

Next we construct the *annihilator polynomial* of X [**Hog82**, 242]:

$$\operatorname{ann}(x) = \frac{|X|}{\prod_{\alpha \in A}(1-\alpha)} \prod_{\alpha \in A}(x - \alpha).$$

By construction, we have $\operatorname{ann}(1) = |X|$ and $\operatorname{ann}(\alpha) = 0$ for each angle $\alpha \in A$. The polynomial $\operatorname{ann}(x)$ has degree $|A| = s$, and can be written as a linear combination of our renormalized Jacobi functions (which depend on the rank and degree of the Jordan algebra containing X):

$$\operatorname{ann}(x) = \sum_{i=0}^{s} a_i Q_i^0(x).$$

The coefficients $a_0, a_1, \ldots, a_s$ are known as the *indicator coefficients* of X. To summarize, given a finite subset $X \subset \mathcal{J}(V)$ we can determine the values of A and t needed to describe X as an A-code and t-design. We can also compute the annihilator polynomial $\operatorname{ann}(x)$ and indicator coefficients $a_0, a_1, \ldots, a_s$.

A *tight* $(2s - \varepsilon)$*-design* is a finite subset $X \subset \mathcal{J}(V)$ where the annihilator polynomial, as defined above, obtains the following value [**DGS77**], [**Hog82**]:

$$\operatorname{ann}(x) = x^\varepsilon R_{s-\varepsilon}^\varepsilon(x), \quad R_{s-\varepsilon}^\varepsilon(x) = \sum_{i=0}^{s-\varepsilon} Q_i^\varepsilon(x).$$

Appendix B verifies that $R_{s-\varepsilon}^\varepsilon(x)$ is proportional to a single Jacobi polynomial with different indices. As the inner products of primitive idempotents in a Euclidean Jordan algebra, the elements of $\alpha \in A(X)$ are all real-valued in the range $0 \le \alpha < 1$. We are interested in whether a tight t-design will have only *rational* elements in angle set $A(X)$. This chapter proves the following theorem, which generalizes the theorems of [**Hog84**] and [**Lyu09**]:

THEOREM 4.1. *Let V be a simple Euclidean Jordan algebra of rank ρ and degree d with manifold of primitive idempotents $\mathcal{J}(V)$. Let X be a finite subset of $\mathcal{J}(V)$ forming a tight $(2s-\varepsilon)$-design, namely with* $\mathrm{ann}(x) = x^\varepsilon R^\varepsilon_{s-\varepsilon}(x)$. *Then the roots of* $\mathrm{ann}(x)$, *which form the angle set $A(x)$, are rational with exceptions when $(\rho, d) = (2, 1)$ with $t \neq 1, 2, 3, 5$ and when $(\rho, d) = (2, 2)$ with $t = 5$.*

4.3. Bose-Mesner Algebras and the Idempotent Basis

We now review the faulty proof given in [**Hog84**], which was intended for the degree $d = 1, 2, 4, 8$ cases only (i.e., the projective cases). The notation here is not necessarily the same as that in [**Hog84**] or [**Lyu09**]. The problem with the faulty proof in [**Hog84**] is that the matrices E_i identified in that paper are not in fact the idempotent basis for the Bose-Mesner algebra that they are assumed to be. The burden of [**Lyu09**] is to replace E_i with the correct idempotents L_i and complete the remainder of the proof, in the case of degrees $d = 1, 2, 4$. We do the same here for any rank and degree.

First, a tight $(2s-\varepsilon)$-design has the property that $t \geq 2s-2$, which ensures that X defines an *association scheme*. We can describe an association scheme in terms of the Gram matrix of the elements of X with respect to the Jordan inner product $\langle x, y\rangle = \mathrm{Tr}(x \circ y)$ given above. That is, the elements of G, a $|X| \times |X|$ matrix, are given by:

$$(G)_{x,y} = \langle x, y\rangle.$$

We can write this Gram matrix as a linear combination of adjacency matrices as follows:

$$G = I + \sum_{\alpha \in A(X)} \alpha D_\alpha.$$

Here D_α is the adjacency matrix of the graph on X where an edge exists between any x, y in X with $\langle x, y\rangle = \alpha$. In this notation, we can write $I = D_1$ since $\langle x, x\rangle = 1$ for all x in X. Specifically, for $t \geq 2s - 2$ (which is satisfied for tight t-designs), the D_α for α in $A(X)$ define the $s = |A(X)|$ classes of an association scheme.

The matrices D_α and I form the basis for a commutative matrix algebra of dimension $s + 1$ known as the *Bose-Mesner algebra* of X (or rather of the association scheme defined on X via its Gram matrix). The Bose-Mesner algebra consists of all $\mathbb{C}$-linear combinations of the commuting basis given by the adjacency matrices and the identity matrix,

$$\{D_\alpha \mid \alpha \in A(X)\} \cup \{I\}$$

The simultaneous eigenvectors of these commuting symmetric matrices can be used to construct a unique orthogonal idempotent basis for the Bose-Mesner algebra [**CVL91**, pp. 201-204]:

$$\{L_i \mid i = 0, 1, \dots, s\}, \quad L_i L_j = \delta_{i,j} L_i.$$

We denote by $q_i(\alpha)/|X|$ the coefficients of the L_i elements in the D_α basis, such that:

$$|X| L_i = q_i(1) I + \sum_{\alpha \in A(X)} q_i(\alpha) D_\alpha.$$

That is, we define the entries of L_i as follows:

$$(L_i)_{x,y} = \frac{1}{|X|} q_i(\langle x, y \rangle), \quad i = 0, 1, \dots, s.$$

The faulty proof in [**Hog84**] assumes that $q_i(\alpha) = Q_i^0(\alpha)$ for X any tight $(2s - \varepsilon)$-design. Indeed, [**Hog84**] uses matrices E_i instead of L_i:

$$(E_i)_{x,y} = \frac{1}{|X|} Q_i^0(\langle x, y \rangle), \quad i = 0, 1, \dots, s.$$

The matrices $E_0, E_1, \dots, E_{s-\varepsilon}$ are orthogonal idempotents (for $\varepsilon = 0$, E_s is idempotent). The problem, as described in [**Lyu09**], is that for $\varepsilon = 1$ the matrix E_s is not necessarily idempotent, so we cannot assume that $L_s = E_s$, where $s = |A(X)|$. To see why, note that the orthogonal basis of $s + 1$ idempotents $L_0, L_1, \dots, L_s$ must satisfy,

$$I = \sum_{i=0}^{s} L_i.$$

The components of this matrix equation are given by,

$$\delta_{x,y} = \frac{1}{|X|} \sum_{i=0}^{s} q_i(\langle x, y \rangle).$$

For a tight t-design we also have,

$$\delta_{x,y} = \frac{1}{|X|} \operatorname{ann}(\langle x, y \rangle) = \frac{1}{|X|} \langle x, y \rangle^\varepsilon R_{s-\varepsilon}^\varepsilon(\langle x, y \rangle) = \frac{1}{|X|} \langle x, y \rangle^\varepsilon \sum_{i=0}^{s-\varepsilon} Q_i^\varepsilon(\langle x, y \rangle).$$

This means that we require,

$$\sum_{i=0}^{s} q_i(\langle x, y \rangle) = \langle x, y \rangle^\varepsilon \sum_{i=0}^{s-\varepsilon} Q_i^\varepsilon(\langle x, y \rangle).$$

When $\varepsilon = 0$, this constraint is satisfied by setting $q_i(\langle x, y \rangle) = Q_i^0(\langle x, y \rangle)$. When $\varepsilon = 1$ we need to select $q_s(\langle x, y \rangle)$ more carefully.

To find L_s we begin with,

$$L_s = I - \sum_{i=0}^{s-1} L_i.$$

Since the E_i are idempotent for $i \neq s$ we set $L_i = E_i$ for $i \neq s$. This yields the following components of L_s:

$$(L_s)_{x,y} = \delta_{x,y} - \frac{1}{|X|} \sum_{i=0}^{s-1} Q_i^0(\langle x, y \rangle).$$

The first term is equal to $\operatorname{ann}(\langle x, y\rangle)/|X|$ and the sum in the second term is equal to $R_{s-1}^0(\langle x, y\rangle)$. This provides us with a general expression for L_s, regardless of whether ε equals 0 or 1:

$$(L_s)_{x,y} = \frac{1}{|X|} \left(\operatorname{ann}(\langle x, y\rangle) - R_{s-1}^0(\langle x, y\rangle) \right).$$

When $\varepsilon = 0$ we have $\operatorname{ann}(\langle x, y\rangle) = R_s^0(\langle x, y\rangle)$ which ensures that $L_s = E_s$. However, when for $\varepsilon = 1$ we have $L_s \neq E_s$.

4.4. Idempotent Ranks and Complex Automorphisms

Having replaced the faulty E_i with a proper L_i idempotent basis, as described in [**Lyu09**], we return to the proof in [**Hog84**]. Hoggar's proof involves the so-called wild automorphisms of $\mathbb{C}$. The identity and the complex conjugation map are the automorphisms of $\mathbb{C}$ that fix $\mathbb{R}$. The other field automorphisms of $\mathbb{C}$ are constructed using the axiom of choice, and are called *wild automorphisms* of $\mathbb{C}$. These wild automorphisms map $\mathbb{R}$ to a dense subset of $\mathbb{C}$, leaving $\mathbb{Q}$ fixed [**Yal66**]. Indeed,

Lemma 4.2. [**Yal66, Cai13**] *Let $x \in \mathbb{R}$ be fixed by all wild automorphisms of $\mathbb{C}$. Then x is rational.*

We use this property of $\mathbb{Q}$ immediately below in Lemma 4.3, assuming the axiom of choice. Let σ be an automorphism of $\mathbb{C}$, potentially among the wild automorphisms. The map σ acts as an automorphism of the the Bose-Mesner algebra by acting on all matrix coefficients. Even so, σ leaves the basis matrices I and D_α fixed, since they only have 0 and 1 for entries. Since each L_i is a $\mathbb{C}$-linear combination of the D_α matrices, and since the orthogonal idempotent basis is unique because we are working over a commutative ring, the action of σ on $\{L_0, L_1, \ldots, L_s\}$ must permute these idempotent matrices. It must also preserve matrix rank, so that $\operatorname{rank} L_i = \operatorname{rank} \sigma(L_i)$.

Lemma 4.3. [**Hog84**] *If the idempotent matrices $\{L_0, L_1, \ldots, L_s\}$ have distinct ranks then $\langle x, y\rangle$ is rational for all x, y in tight t-design X.*

PROOF. If the matrices $\{L_0, L_1, \ldots, L_s\}$ have distinct ranks then any field automorphism of $\mathbb{C}$ must fix these matrices, so that $\sigma(L_i) = L_i$. This ensures that the L_i are matrices with rational entries. Specifically, each L_i is of the form $|X|(L_i)_{x,y} = q_i(\langle x, y\rangle)$. If $\sigma(L_i) = L_i$ then we also have $\sigma q_i(\langle x, y\rangle) = q_i(\langle x, y\rangle)$. If $q_i(\langle x, y\rangle)$ is fixed by all σ then it is rational, by Lemma 4.2. In the case of $i = 1$ we have,

$$(L_1)_{x,y} = Q_1^0(\langle x, y\rangle) = \left(\frac{1}{2}\rho d + 1\right)(\rho\langle x, y\rangle - 1),$$

This means that $\langle x, y\rangle$ is rational.

$\square$

The next task is to compute the ranks of the L_i idempotent matrices. The ranks found here are the same as those calculated in [**Lyu09**], but presented in a slightly different form.

LEMMA 4.4. *Let X be a tight $(2s - \varepsilon)$-design. Then the orthogonal idempotents of the Bose-Mesner algebra have the following ranks:*

$$\text{rank } L_i = \begin{cases} Q_i^0(1), & i = 0, 1, \ldots, s-1 \\ R_{s-\varepsilon}^{\varepsilon}(1) - R_{s-1}^0(1), & i = s \end{cases}$$

Here we have,

$$Q_i^0(1) = \left(\frac{\frac{1}{2}\rho d + 2i - 1}{\frac{1}{2}\rho d + i - 1}\right)\frac{(\frac{1}{2}\rho d)_i(\frac{1}{2}\rho d - \frac{1}{2}d)_i}{(\frac{1}{2}d)_i i!}.$$

For $\varepsilon = 0$ we have $R_s^0(1) - R_{s-1}^0(1) = Q_s^0(1)$. For $\varepsilon = 1$ we have,

$$R_{s-1}^1(1) - R_{s-1}^0(1) = \frac{s}{\frac{1}{2}\rho d + 2s - 1}Q_s^0(1).$$

PROOF. The rank of an idempotent matrix is equal to its trace. For L_i with $i \neq s$ we have,

$$\text{rank } L_i = \text{Tr } L_i = \sum_{x\in X}(L_i)_{x,x} = \sum_{x\in X}\frac{Q_i^0(1)}{|X|} = Q_i^0(1).$$

For L_s we have,

$$\text{rank } L_s = \text{Tr } L_s = \sum_{x\in X}(L_s)_{x,x} = \sum_{x\in X}\frac{1}{|X|}\left(\text{ann}(1) - R_{s-1}^0(1)\right).$$

Since $\text{ann}(1) = 1^{\varepsilon}R_{s-\varepsilon}^{\varepsilon}(1)$ we have,

$$\text{rank } L_s = R_{s-\varepsilon}^{\varepsilon}(1) - R_{s-1}^0(1).$$

The specific expression for $Q_i^0(1)$ given above is obtained from the expression for $Q_k^\varepsilon(x)$ given earlier and the property $P_k^{(\alpha,\beta)}(1) = \binom{\alpha+k}{k} = \frac{(\alpha+1)_k}{k!}$. In order to evaluate $R_{s-\varepsilon}^\varepsilon(1)$ we use,

$$R_{s-\varepsilon}^\varepsilon(x) = \frac{(\frac{1}{2}\rho d)_s}{(\frac{1}{2}d)_s} P_{s-\varepsilon}^{(\frac{1}{2}d(\rho-1),\frac{1}{2}d-1+\varepsilon)}(2x-1).$$

This expression is given in [**Lyu09**] and verified in Appendix B. Therefore,

$$R_{s-\varepsilon}^\varepsilon(1) = \frac{(\frac{1}{2}\rho d)_s(\frac{1}{2}\rho d - \frac{1}{2}d + 1)_{s-\varepsilon}}{(\frac{1}{2}d)_s(s-\varepsilon)!} = \frac{(\frac{1}{2}\rho d)_s}{(\frac{1}{2}d)_s}\binom{\frac{1}{2}d(\rho-1)+s-\varepsilon}{s-\varepsilon}$$

By construction $R_s^0(1) - R_{s-1}^0(1) = Q_s^0(1)$. When $\varepsilon = 1$ we instead have,

$$R_{s-1}^1(1) - R_{s-1}^0(1) = \left(\frac{(\frac{1}{2}\rho d)_s}{(\frac{1}{2}d)_s} - \frac{(\frac{1}{2}\rho d)_{s-1}}{(\frac{1}{2}d)_{s-1}}\right)\binom{\frac{1}{2}d(\rho-1)+s-1}{s-1}.$$

Using $\binom{a}{b-1} = \frac{b}{a-b+1}\binom{a}{b}$ and then $\binom{a+b}{b} = \frac{(a+1)_b}{b!}$ we have,

$$\begin{aligned}
R_{s-1}^1(1)-&R_{s-1}^0(1)\\
&= \frac{(\frac{1}{2}\rho d)_s}{(\frac{1}{2}d)_s}\left(1 - \frac{\frac{1}{2}d+s-1}{\frac{1}{2}\rho d+s-1}\right)\frac{s}{\frac{1}{2}d(\rho-1)}\binom{\frac{1}{2}d(\rho-1)+s-1}{s}\\
&= \frac{s}{\frac{1}{2}\rho d+s-1}\frac{(\frac{1}{2}\rho d)_s(\frac{1}{2}\rho d-\frac{1}{2}d)_s}{(\frac{1}{2}d)_s s!}\\
&= \frac{s}{\frac{1}{2}\rho d+2s-1}Q_s^0(1).
\end{aligned}$$

□

The exceptional case of the unit circle, $\Omega_2 \cong \mathbb{RP}^1$, deserves specific attention. We examine it before proceeding with the remainder of the proof.

LEMMA 4.5. *When rank* $\rho = 2$ *and degree* $d = 1$*, the case of* $\Omega_2 \cong \mathbb{RP}^1$*, we have*

$$\operatorname{rank} L_0 = 1,\ \operatorname{rank} L_i = 2, \quad i = 1, \ldots, s-\varepsilon.$$

When $\varepsilon = 1$*, we have* $\operatorname{rank} L_s = 1$.

PROOF. We evaluate $Q_i^0(1)$ for $\rho = 2$ and $d = 1$. First, $Q_0^0(x) = 1$ so we have $\operatorname{rank} L_0 = Q_0^0(1) = 1$. In what follows we assume $i > 0$. The following expression simplifies to 2:

$$Q_i^0(1) = \left(\frac{1+2i-1}{1+i-1}\right)\frac{(1)_i(1-\frac{1}{2})_i}{(\frac{1}{2})_i i!} = 2.$$

When $\varepsilon = 1$ we have,

$$\text{rank } L_s = R^1_{s-1}(1) - R^0_{s-1}(1) = \frac{s}{1+2s-1} Q^0_s(1) = \frac{2}{2} = 1.$$

□

The remaining cases satisfy the following lemma:

LEMMA 4.6. [**Lyu09**] *For any tight $(2s-\varepsilon)$-design, not in $\Omega_2 \cong \mathbb{RP}^1$, the idempotent basis matrices satisfy,*

$$\text{rank } L_0 < \text{rank } L_1 < \cdots < \text{rank } L_{s-\varepsilon}.$$

PROOF. This is equivalent to $Q^0_0(1) < Q^0_1(1) < \cdots < Q^0_{s-\varepsilon}(1)$. We can compute the following expression:

$$Q^0_{i+1}(1) = \left(\frac{\frac{1}{2}\rho d + 2i + 1}{\frac{1}{2}\rho d + 2i - 1}\right) \left(\frac{\frac{1}{2}d(\rho-1) + i}{\frac{1}{2}d + i}\right) \left(\frac{\frac{1}{2}\rho d + i - 1}{i+1}\right) Q^0_i(1)$$

The first factor is always greater than one. The second factor is always greater than or equal to one. The third factor is greater than or equal to one when $\rho d \geq 4$. This leaves the $(\rho, d) = (3, 1)$ case to check, which yields $Q^0_{i+1}(1) = Q^0_i(1)\left(2i + \frac{5}{2}\right) / \left(2i + \frac{1}{2}\right)$. Therefore, for $(\rho, d) \neq (2, 1)$ we have $Q^0_i(1) < Q^0_{i+1}(1)$ for all $i \geq 0$. □

4.5. Completing the Proof

To complete the proof of Theorem 4.1 we need to apply Lemma 4.3 to all the relevant cases. However, since $\Omega_2 \cong \mathbb{RP}^1$ generally involves idempotent basis matrices of equal rank, we cannot use Lemma 4.3 for this exceptional case. This case is examined carefully for completeness in [**Lyu09**] and we address it in the following theorem.

THEOREM 4.7. [**Lyu09**] *A tight t-design in $\Omega_2 \cong \mathbb{RP}^1$ has a rational angle set if and only if $t = 1, 2, 3, 5$.*

PROOF. A tight t-design in $\Omega_2 \cong \mathbb{RP}^1$ always exists and is given by the corners of a regular $(t+1)$-gon. The angle between any pair of design points on the unit circle is $\theta = 2m\pi/(t+1)$ for some integer m. The corresponding Jordan inner product is $\langle x, y\rangle = \cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2} + \frac{1}{2}\cos\theta$. This means that $\langle x, y\rangle$ is rational if and only if $\cos\theta$ is rational. The only values of t for which $\cos\theta$ is rational are known to be $t = 1, 2, 3, 5$. □

We now examine the remaining cases with $(\rho, d) \neq (2, 1)$, i.e., distinct from the unit circle. The simplest to deal with, using Lemma 4.3, is the case where $t = 2s$ is even.

THEOREM 4.8. *A tight $(2s)$-design, not in $\Omega_2 \cong \mathbb{RP}^1$, has a rational angle set.*

PROOF. According to Lemma 4.6, the idempotent basis matrices L_i each have distinct rank. Therefore, by Lemma 4.3, the angle set is rational. $\square$

We now examine cases with odd $t = 2s - 1$.

LEMMA 4.9. *Let $(\rho, d) \neq (2, 1)$. If L_s and L_1 have distinct ranks then the angle set is rational.*

PROOF. According to Lemma 4.6, the ranks of all L_i are distinct except possibly for L_s when $\varepsilon = 1$. Therefore no field automorphism of $\mathbb{C}$ interchanges L_1 with any L_i other than possibly L_s. If L_s and L_1 have distinct ranks, then L_1 is fixed by all field automorphisms of $\mathbb{C}$ and therefore is a matrix with rational entries $Q_1^0(\alpha)/|X|$. As described in the proof of Lemma 4.3, it follows from the rationality of $Q_1^0(\alpha)$ that α is rational. $\square$

The simplest odd $t = 2s - 1$ case is for $s = 1$, corresponding to a tight 1-design. A tight 1-design exists in each Ω_{d+1} and $\mathbb{FP}^{\rho-1}$ and is also known as a *Jordan frame*, or full rank set of orthogonal primitive idempotents. The annihilator polynomial is $\mathrm{ann}(x) = \rho x$, so the angle set is $A = \{0\}$, which is clearly rational. Since this case is fully understood, we will assume $s > 1$ in what follows.

We now address the remaining spherical cases.

THEOREM 4.10. *A tight $(2s-1)$-design in Ω_{d+1} with $d > 1$ has a rational angle set except when $d = 2$ and $s = 3$.*

PROOF. In the spherical cases we have $\rho = 2$ and can simplify $Q_i^0(1)$ to the following:

$$Q_i^0(1) = \left(\frac{d+2i-1}{d+i-1}\right)\frac{(d)_i}{i!}.$$

This means that,

$$\mathrm{rank}\ L_1 = Q_1^0(1) = d + 1.$$

Likewise,

$$\mathrm{rank}\ L_s = R_{s-1}^1(1) - R_{s-1}^0(1) = \frac{s}{d+2s-1} Q_s^0(1) = \binom{d+s-2}{s-1}.$$

For $s = 2$, we always have $\mathrm{rank}\ L_s - \mathrm{rank}\ L_1 = -1$. By Lemma 4.3, a tight spherical 3-design must therefore have a rational angle set. For $s > 2$, $\mathrm{rank}\ L_s - \mathrm{rank}\ L_1 \geq 0$ with the equality achieved only when $d = 2$ and $s = 3$. Aside from this exception, Lemmas 4.3 and 4.9 ensure a rational angle set. Therefore, the only case where $\mathrm{rank}\ L_s = \mathrm{rank}\ L_1$ is for $d = 2$ and $s = 3$. A tight 5-design exists in this case and is known to be the vertices of a regular icosahedron in Ω_3. $\square$

REMARK 4.11. In contrast to Lyubich, who only deals with projective cases of degree $d = 1, 2, 4$, we have dealt here with all spherical ($\rho = 2$) cases simultaneously. This provides a slightly different approach to identifying Lyubich's exception in $\Omega_3 \cong \mathbb{CP}^1$, namely as spherical design rather than as a projective design.

We will call a design with $\rho > 2$ a *strictly projective design*. All remaining cases are strictly projective. In what follows we will therefore require that $\rho > 2$.

THEOREM 4.12. *A strictly projective tight* $(2s-1)$*-design, i.e., with* $\rho > 2$*, has a rational angle set.*

PROOF. The proof method shown here is equivalent to the method used in [**Lyu09**], but the parameters are allowed to extend to the octonion case ($d = 8$) yet restricted to the strictly projective $\rho > 2$ cases. We need to verify that rank $L_s \neq$ rank L_1. In the cases below we use that fact that,

$$\text{rank } L_s \geq \frac{2}{\frac{1}{2}\rho d + 3} Q_2^0(1)$$

Specifically, as a function of s the expression $sQ_s^0(1)/(\frac{1}{2}\rho d + 2s - 1)$ decreases as the value of s decreases. We can confirm this fact using the expression given in the proof of Lemma 4.6. We will therefore assume that rank L_s is greater than or equal to the same expression evaluated at $s = 2$. This ensures that,

$$\text{rank } L_s - \text{rank } L_1 \geq \frac{2}{\frac{1}{2}\rho d + 3} Q_2^0(1) - Q_1^0(1).$$

Writing the expression on the right hand side explicitly, in simplified form, we have,

$$\text{rank } L_s - \text{rank } L_1 \geq \frac{1}{2}\left(\frac{d(\rho-1)}{d(d+2)}\right)\left(\rho^2 d^2 - 2\rho d^2 - 2d - 4\right)$$

Whenever the right hand side of the inequality is greater than zero, we have confirmed by Lemmas 4.3 and 4.9 that rank $L_s \neq$ rank L_1 and therefore that the angle set is rational. We define,

$$f_d(\rho) = \rho^2 d^2 - 2\rho d^2 - 2d - 4.$$

If $f_d(\rho) > 0$ then the corresponding tight projective $(2s-1)$-design has a rational angle set. Therefore, we only need to consider the cases where $f_d(\rho) \leq 0$ as potential examples of an irrational angle set.

Real Projective Case. Set $d = 1$ and $\rho > 2$. Then we have,

$$f_1(\rho) = \rho^2 - 2\rho - 6.$$

The only integer value of $\rho > 2$ with $f_1(\rho) \leq 0$ is $\rho = 3$. This means that the real projective case of $\rho = 3$ and $d = 1$ is a possible case for rank $L_s =$ rank L_1. We must examine this possibility more closely.

Let $d = 1$ and $\rho = 3$. If $s = 2$, for a tight 3-design, then we know that rank $L_s \neq$ rank L_1 by the fact that $f_1(3) \neq 0$, as shown above. We need to also ensure rank $L_s \neq$ rank L_1 for $s > 2$. To do so, we repeat the argument given above except instead we use,

$$\text{rank } L_s \geq \frac{3}{\frac{1}{2}\rho d + 5} Q_3^0(1).$$

For $d = 1$ and $\rho = 3$ we have,

$$\text{rank } L_s - \text{rank } L_1 \geq \frac{6}{13} Q_3^0(1) - Q_1^0(1) = 1.$$

Therefore each real tight strictly projective designs has a rational angle set.

Complex Projective Cases. Let $\rho \geq 3$ and $d = 2$. Then the second factor in the inequality above simplifies to,

$$f_2(\rho) = 4\rho^2 - 8\rho - 8.$$

All integer values of $\rho > 2$ satisfy $f_2(\rho) > 0$. This means that rank $L_s >$ rank L_1 for all of the complex projective cases.

Quaternion Projective Cases. Let $\rho \geq 3$ and $d = 4$. Then the second factor in the inequality above simplifies to,

$$f_4(\rho) = 16\rho^2 - 32\rho - 12.$$

All integer values of $\rho > 2$ satisfy $f_4(\rho) > 0$. This means that rank $L_s >$ rank L_1 for all of the quaternion projective cases.

Exceptional Octonion Projective Case. Let $\rho = 3$ and $d = 8$. Then the second factor in the inequality above simplifies to,

$$f_8(3) = 172.$$

This means that rank $L_s >$ rank L_1 for the exceptional case. Having checked all the cases, we have distinct ranks for all idempotent basis matrices L_i and by Lemma 4.3 the angle sets of a strictly projective tight $(2s-1)$-design must be rational. □

4.6. Conclusion

We have extended the result of [**Hog84**] for $d = 1, 2, 4, 8$, which is corrected by [**Lyu09**] for $d = 1, 2, 4$, to the full range of possible values of rank ρ and degree d, proving Theorem 4.1. This clarifies the full conditions under which a tight t-design—whether spherical or projective—has a rational angle set. The only examples of irrational angles sets exist on the unit circle $\Omega_2 \cong \mathbb{RP}^1$ for $t \neq 2, 3, 4, 5$ and on the unit sphere $\Omega_3 \cong \mathbb{CP}^1$ for $t = 5$.

A recent independent paper provides a complementary proof in the case of spherical t-designs only [**BNS22**]. That paper applies to all spherical designs with $t \geq 2s - 2$. A comparison suggests that it may be possible to extend the results of this chapter to all spherical and projective designs with $t \geq 2s - 2$. Since a t-design with $t \geq 2s - 2$ also defines an association scheme, many of the techniques used in this chapter would still apply. Computing the explicit ranks of L_i might be difficult since these would depend on the angle set A, which would not be fixed by tightness.

A comparison with [**BNS22**] also suggests that the use of wild automorphisms of $\mathbb{C}$ and the axiom of choice may not be essential for the proof of this chapter. Instead, we could follow [**BNS22**] and use some severe constraints on the properties of association schemes given in [**Suz98**] and the properties of the splitting field of the design, namely the smallest field extension $\mathbb{F}$ of $\mathbb{Q}$ containing all entries of the L_i matrices. Of course, assuming the axiom of choice, any automorphism of $\mathbb{F}$ extends to a wild automorphism of $\mathbb{C}$. Yet an alternate proof of the results of this chapter is likely possible without use of wild automorphisms of $\mathbb{C}$.

CHAPTER 5

Octonions and the Two Strictly Projective Tight 5-Designs

5.1. Introduction

A spherical t-design is a finite subset of points on the unit sphere in a real vector space with the following special property: the average value of any polynomial of degree at most t over the sphere is equal to the average value of the polynomial evaluated at the points of the t-design [**DGS77**]. Every t-design is also an A-code, where A is the set of angles between distinct points in the t-design. A projective t-design (and A-code) generalizes this concept from spheres to projective spaces [**Neu81, Hog82**]. Taken together, the spheres and infinite projective spaces constitute the compact symmetric spaces of rank 1 [**Hog92**], the compact and connected two-point homogeneous spaces [**Wan52**], and also the manifolds of primitive idempotents for the simple Euclidean Jordan algebras [**FK94**]. That is, for V a simple Euclidean Jordan algebra of rank ρ and degree d, the manifold of primitive idempotents $\mathcal{J}(V)$ is given by,

$$\Omega_{d+1}, \quad \mathbb{RP}^{\rho-1}, \quad \mathbb{CP}^{\rho-1}, \quad \mathbb{HP}^{\rho-1}, \quad \mathbb{OP}^2, \quad d \geq 1, \ \rho \geq 3.$$

There are no repetitions on this list, but the projective lines (not listed) are isomorphic to the following spheres:

$$\Omega_2 \cong \mathbb{RP}^1, \quad \Omega_3 \cong \mathbb{CP}^1, \quad \Omega_5 \cong \mathbb{HP}^1, \quad \Omega_9 \cong \mathbb{OP}^1.$$

A t-design is *spherical* when it is a subset of sphere Ω_{d+1} (with $\rho = 2$), and *projective* when it is a subset of projective space $\mathbb{FP}^{\rho-1}$ (with $d = [\mathbb{F} : \mathbb{R}] = 1, 2, 4, 8$). We will call a t-design *strictly projective* when it has $\rho \geq 3$ and is therefore not also spherical. Both spherical and projective t-designs are interesting objects in part because of their connections to real, complex, and quaternion reflection groups, as well as other sporadic simple groups [**ST54, Coh76, Coh80, Hog82, BB09**].

A *tight* t-design (whether spherical or projective) simultaneously meets a *lower bound*, given its value of t, and an *upper bound*, given its value of A. While t-designs are common, tight t-designs are rare and continue to elude full classification. Even so, a projective tight t-design with $\mathbb{FP}^{\rho-1} \neq \mathbb{RP}^1$

has $t \leq 5$ [**Hog84, Hog89**] and there are precisely four projective tight 5-designs. Two of these are also spherical, the hexagon in $\Omega_2 \cong \mathbb{RP}^1$ and the icosahedron in $\Omega_3 \cong \mathbb{CP}^1$ [**Lyu09**]. The remaining two tight 5-designs are strictly projective: one in $\mathbb{RP}^{23}$ consisting of the 98280 lines spanned by the Leech lattice short vectors and the other in $\mathbb{OP}^2$ realizing the 819 lines of the unique generalized hexagon $\mathrm{Gh}(2,8)$ [**Hog89**].

Hoggar conjectured that the two strictly projective tight 5-designs are closely related, since the first has cardinality $98280 = 120 \cdot 819$, the second has cardinality 819, and there are 120 pairs of opposite octonion integer units. Hoggar reports that this conjecture was initially met with skepticism [**Hog82**]. This chapter provides the missing common construction using certain octonion involutionary matrices. These matrices can act on both the octonion vector space $\mathbb{O}^3 \cong \mathbb{R}^{24}$ and the octonion projective plane $\mathbb{OP}^2$ to produce the two strictly projective tight 5-designs. This common construction also serves as a new connection between the generalized hexagon $\mathrm{Gh}(2,8)$ and the Leech lattice.

5.2. Tight t-Designs

This section reviews tight t-designs and their partial classification in order to provide context for the common construction that follows.

5.2.1. Definitions. Let V be a simple Euclidean Jordan algebra of rank ρ and degree d and let $\mathcal{J}(V)$ be the manifold of primitive idempotents. We can use the following *renormalized Jacobi polynomials* to describe both the spherical ($\rho = 2$) and projective ($d = 1, 2, 4, 8$) cases:

$$Q_k^{\varepsilon}(x) = \left(\frac{1}{2}\rho d + 2k + \varepsilon - 1\right) \frac{(\frac{1}{2}\rho d)_{k+\varepsilon-1}}{(\frac{1}{2}d)_{k+\varepsilon}} P_k^{(\frac{1}{2}d(\rho-1)-1, \frac{1}{2}d-1+\varepsilon)}(2x-1).$$

Here we have k a non-negative integer, $\varepsilon = 0$ or 1, Pochhammer symbol $(a)_i = a(a+1)(a+2)\cdots(a+i-1)$, and Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ as defined in [**AS72**, 22.2.1]. Let X be a finite subset of $\mathcal{J}(V)$ and let $\langle x, y\rangle = \mathrm{Tr}(x \circ y)$ be the Jordan inner product. We can compute *angle set* A as follows:

$$A(X) = \{\langle x, y\rangle \mid x, y \in X \subset \mathcal{J}(V), x \neq y\}.$$

We can also compute *strength* t as the maximum non-negative integer t that satisfies

$$\sum_{x\in X}\sum_{y\in X} Q_k^0(\langle x, y\rangle) = 0, \quad k = 1, 2, \ldots, t.$$

With these values we call X both an *A-code* and *t-design*.

Remark 5.1. In the spherical $\rho = 2$ case, the Jordan inner product has the form $\langle x, y\rangle = \frac{1}{2} + \frac{1}{2}\cos\theta$, where θ is the angle between the two points x, y on the sphere. This ensures that antipodal points on the sphere are orthogonal relative to the Jordan inner product. In general, for primitive idempotents $x \neq y$, we have $0 \leq \langle x, y\rangle < 1$ and $\langle x, x\rangle = 1$.

The *annihilator polynomial* of $X \subset \mathcal{J}(V)$, denoted $\operatorname{ann}(x)$, is the unique degree $|A|$ polynomial constructed to ensure that

$$|X| = \operatorname{ann}(1), \quad A(X) = \{\alpha \in \mathbb{R} \mid \operatorname{ann}(\alpha) = 0\}.$$

Given finite X, we can compute t, A, and $\operatorname{ann}(x)$ directly. The more difficult task is to specify t, A, or $\operatorname{ann}(x)$ and then find a finite subset $X \subset \mathcal{J}(V)$ that realizes them. In general a t-design (and A-code) satisfies the inequality $t \leq 2s - \varepsilon$, where $s = |A(X)|$ and $\varepsilon = |A(X) \cap \{0\}|$. A tight t-design achieves $t = 2s - \varepsilon$. It also simultaneously achieves the *lowest* possible cardinality $|X|$ for the given t value and the *highest* possible cardinality $|X|$ for the given $s = |A|$ value [**Hog82, BH85**]. We can equivalently define a *tight t-design* as a finite subset $X \subset \mathcal{J}(V)$ with the annihilator polynomial $\operatorname{ann}(x) = x^{\varepsilon} R^{\varepsilon}_{s-\varepsilon}(x)$, where we have

$$R^{\varepsilon}_{s-\varepsilon}(x) = Q^{\varepsilon}_0(x) + Q^{\varepsilon}_1(x) + \cdots + Q^{\varepsilon}_{s-\varepsilon}(x).$$

That ensures that a tight $(2s - \varepsilon)$-design has cardinality $|X| = R^{\varepsilon}_{s-\varepsilon}(1)$ and an angle set $A(X)$ given by the roots of polynomial $x^{\varepsilon} R^{\varepsilon}_{s-\varepsilon}(x)$.

5.2.2. Partial Classification. Tight t-designs are partially classified as follows (see also Appendix A). In the case of the circle $\Omega_2 \cong \mathbb{FP}^1$ ($\rho = 2$, $d = 1$), a tight t-design exists for all positive integer t values, corresponding to the vertices of a $(t + 1)$-gon.

As described in [**CD07, BB09**], in the remaining spherical cases ($\rho = 2$, $d \geq 2$) a tight t-design must have $t = 1, 2, 3, 4, 5, 7, 11$. On the sphere Ω_{d+1}, a tight 1-design is a pair of antipodal points, a tight 2-design is a simplex of $d + 2$ points, and a tight 3-design is a cross polytope of $2d + 2$ points. Tight spherical 4- and 5-designs are in one-to-one correspondence and the search for tight spherical 5-designs is still open. The only known examples outside of Ω_2 are sets of vectors spanning equiangular lines in Ω_3, Ω_7, and Ω_{23}. If any further example exists, it will have $d \geq 118$ [**BMV05, BB09**]. Likewise, the search for spherical tight 7-designs is open with examples known in Ω_8 and Ω_{23}. If any further spherical tight 7-designs exist, they will have $d \geq 103$ [**BMV05, BB09**]. Finally, there is precisely one spherical tight 11-design, the points defined by the short vectors of the Leech lattice in Ω_{24}.

As described in [**Hog84, Hog89, Lyu09**], the remaining projective cases ($\rho \geq 3$, $d = 1, 2, 4, 8$) have $t = 1, 2, 3, 5$. A projective tight 1-design is a Jordan frame, namely ρ orthogonal points. A Jordan frame exists in each projective space and generalizes antipodal points on the sphere for $\rho = 2$. All remaining examples have $t \geq 2$. A real projective tight t-design ($d = 1$, $\rho \geq 3$), corresponds to a spherical tight $(2t+1)$-design, so the search for spherical tight

5- and 7-designs is equivalent to the search for real projective tight 2- and 3-designs. There is just one real projective tight 5-design: the lines spanned by the short vectors of the Leech lattice. The remaining complex or quaternion projective tight t-designs ($d = 2, 4,\ \rho \geq 3$) must have $t = 2, 3$. The search for complex and quaternion projective tight 2- and 3-designs remains open. As described in [**CKM16**], there are many known examples of tight 2-designs in $\mathbb{CP}^{\rho-1}$, but surprisingly few have been found in $\mathbb{HP}^{\rho-1}$. Finally, the remaining tight t-designs in the octonion case ($d = 8,\ \rho = 3$) must have $t = 2, 5$. The tight 2-design in $\mathbb{OP}^2$ has been proven to exist without an explicit construction in [**CKM16**]. The tight 5-design in $\mathbb{OP}^2$ was constructed in [**Coh83**].

To summarize, the classification of tight t-designs will remain open until the tight 2- and 3-designs in $\mathbb{RP}^{\rho-1}$, $\mathbb{CP}^{\rho-1}$, and $\mathbb{HP}^{\rho-1}$ are all identified. In contrast, the projective tight 5-designs are fully classified.

THEOREM 5.2. *A projective tight* 5*-design* $X \subset \mathcal{J}(V)$ *is either,*

(1) *The vertices of a regular hexagon in* $\Omega_2 \cong \mathbb{RP}^1$*;*
(2) *The vertices of a regular icosahedron in* $\Omega_3 \cong \mathbb{CP}^1$*;*
(3) *The lines spanned by the short vectors of the Leech lattice in* $\mathbb{RP}^{23}$*; or*
(4) *The unique realization of* $\mathrm{Gh}(2, 8)$ *in* $\mathbb{OP}^2$.

REMARK 5.3. Two of the four projective tight 5-designs in Theorem 5.2 are also spherical tight 5-designs constructed from systems of equiangular lines in $\mathbb{R}^2$ and $\mathbb{R}^3$. The other two examples, in $\mathbb{RP}^{23}$ and $\mathbb{OP}^2$, are not spherical and constitute the only two *strictly projective tight* 5*-designs*. This chapter will identify a common construction for the two unique strictly projective 5-designs of Theorem 5.2.

REMARK 5.4. The proof in [**Hog84, Hog89**] that $t = 1, 2, 3, 5$ for projective tight t-designs ($d = 1, 2, 4, 8,\ \rho \geq 2$) other than $\mathbb{RP}^1$ ($d = 1,\ \rho = 2$) rests on a faulty lemma. Specifically, Hoggar attempts to prove that for a projective tight t-design X, the angle set $A(X)$ must be rational [**Hog84**], and the proofs of various restrictions on t depend on this result. However, as described in [**Lyu09**], the icosahedron vertices in $\mathbb{CP}^1 \cong \Omega_3$ serve as a counter-example since the angle set of that projective tight 5-design is irrational. Lyubich repairs the faulty lemma in [**Hog84**] for $d = 1, 2, 4$, accounting for the exceptions in $\mathbb{RP}^1 \cong \Omega_2$ and $\mathbb{CP}^1 \cong \Omega_3$, but ignores the octonion $d = 8$ cases [**Lyu09**]. The repair in [**Lyu09**] involves correctly identifying the idempotent basis of the Bose–Mesner algebra of a tight t-design, which was incorrectly chosen in [**Hog84**]. This can also be done for the remaining spherical ($\rho = 2$) and octonion ($d = 8$) cases in the same way as outlined in [**Lyu09**]. No new exceptions exist beyond those captured in [**Lyu09**]. As a result, the results about possible t values for tight t-designs in [**Hog84, Hog89**] still hold true. Furthermore, aside from the known exceptions in $\mathbb{RP}^1 \cong \Omega_2$ and $\mathbb{CP}^1 \cong \Omega_3$, the angle set of a tight t-design is indeed rational. Chapter 4 provides this general proof.

5.3. Octonions and Isometries

This section describes how certain involutionary isometries of vector space $\mathbb{F}^\rho$ and projective space $\mathbb{FP}^{\rho-1}$, with associative $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, generalize to the non-associative octonion case where $\mathbb{F} = \mathbb{O}$.

5.3.1. Definitions. The division composition algebras over the real numbers are precisely the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. A standard basis for the octonions is $\{i_t \mid t \in \mathrm{PL}(7) = \{\infty\} \cup \mathbb{F}_7\}$, with $1 = i_\infty$ the identity and

$$i_t^2 = -1, \quad i_t = i_{t+1} i_{t+3} = -i_{t+3} i_{t+1}, \quad t \in \mathbb{F}_7.$$

Octonion conjugation is the $\mathbb{R}$-linear involution defined by $\overline{1} = 1$ and $\overline{i}_t = -i_t$ for $t \in \mathbb{F}_7$. The real-valued norm is given by $N(x) = x\overline{x} = \overline{x}x$. The subalgebra of $\mathbb{O}$ generated by a single octonion is commutative (isomorphic to $\mathbb{R}$ or $\mathbb{C}$) and the subalgebra generated by any two octonions is associative (isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$). Many further details about this non-associative algebra are available in [**SV00, Bae02, CS03, Sch17**].

As described above, simple Euclidean Jordan algebras of rank $\rho \geq 3$ can be described as Hermitian matrices relative to octonion conjugation, which we denote $\mathrm{Herm}(\rho, \mathbb{F})$, with the *Jordan product* defined in terms of the usual matrix product xy as follows:

$$x \circ y = \frac{1}{2}(xy + yx).$$

Here we have $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (in the octonion case, we must have $\rho = 3$ and the underlying matrix product xy is non-associative). In addition to the Jordan product, each Jordan algebra element defines an endomorphism $P(x)$, known as the *quadratic representation* [**FK94**, II.3]:

$$P(x) : y \mapsto 2x \circ (x \circ y) - x^2 \circ y.$$

When the Jordan product is constructed from an *underlying associative algebra* (as in all but the $\rho = 3$ and $d = 8$ octonion case) then $P(x)$ simplifies to $P(x)y = xyx$, with xyx computed using that underlying associative product. Many further details about Euclidean Jordan algebras are available in [**FK94, SV00, Bae02, Sch17**].

Let $x = (x_1, x_2, \ldots, x_\rho)$ be a row vector in $\mathbb{F}^\rho$ and let $x^\dagger$ be the conjugate transpose, a column vector. We will call a vector x in $\mathbb{F}^\rho$ *commutative* when the coefficients $\{x_1, x_2, \ldots, x_\rho\}$ generate a real or complex subalgebra of $\mathbb{O}$ and *associative* when the coefficients generate a real, complex, or quaternion subalgebra of $\mathbb{O}$. We can extend the norm on $\mathbb{F}$ described above to vectors in $\mathbb{F}^\rho$ as follows:

$$N(x) = xx^\dagger = x_1\overline{x}_1 + \cdots + x_\rho\overline{x}_\rho = N(x_1) + \cdots + N(x_\rho).$$

This norm is real-valued and serves as a Euclidean norm for the corresponding real vector space $\mathbb{R}^{\rho d} \cong \mathbb{F}^\rho$ with $d = [\mathbb{F} : \mathbb{R}]$. The inner product is constructed from the norm in the standard way:

$$(x, y) = \frac{1}{2}\left(N(x+y) - N(x) - N(y)\right) = \frac{1}{2}(xy^\dagger + yx^\dagger) = \mathrm{Re}(xy^\dagger) = \mathrm{Re}(yx^\dagger).$$

If vector x is associative, then $[x] = x^\dagger x/N(x)$ is also a primitive idempotent in $\mathbb{FP}^{\rho-1} \subset \mathrm{Herm}(\rho, \mathbb{F})$. Indeed, any primitive idempotent $[x]$ in $\mathbb{FP}^{\rho-1}$ can be constructed this way for some (non-unique) associative vector x in $\mathbb{F}^\rho$.

5.3.2. Isometries. Given our real-valued inner products defined on both $\mathbb{F}^\rho$ and $\mathbb{FP}^{\rho-1}$, we now want to construct isometries. An important property of the quadratic representation is that when $w \circ w = I_\rho$ the map $P(w)$ is an involutionary automorphism of the Jordan algebra and an isometry of the manifold of primitive idempotents $\mathcal{J}(V)$ relative to $\langle x, y\rangle = \mathrm{Tr}(x \circ y)$.

For associative $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the following pairs of maps defined by associative vector r are involutionary isometries of $\mathbb{F}^\rho$ and $\mathbb{FP}^{\rho-1}$ respectively:

$$x \mapsto x(I_\rho - 2[r]), \qquad [x] \mapsto (I_\rho - 2[r])[x](I_\rho - 2[r]). \tag{5.1}$$

Matrices of the form $W(r) = I_\rho - 2[r]$ satisfy $W(r)^\dagger W(r) = I_\rho$ and therefore belong to the matrix groups $\mathrm{O}(\rho)$, $\mathrm{U}(\rho)$, or $\mathrm{Sp}(\rho)$ respectively when $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. In the non-associative case, with $\mathbb{F} = \mathbb{O}$ and $\rho = 3$, we can ensure that the maps of Eq. (5.1) are isometries by selecting a *commutative* vector r in $\mathbb{O}^3$.

LEMMA 5.5. *Let r be a commutative vector in $\mathbb{F}^\rho$. Then the maps of Eq.* (5.1) *are respectively isometries of $\mathbb{F}^\rho \cong \mathbb{R}^{\rho d}$ with inner product $(x, y) = \mathrm{Re}(xy^\dagger)$ and of $\mathbb{FP}^{\rho-1}$ with inner product $\langle x, y\rangle = \mathrm{Tr}(x \circ y)$.*

PROOF. Consider the map $x \mapsto xW(r)$ with $W(r) = I_\rho - 2[r]$. In the associative cases with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the matrix $W(r)$ belongs to one of the matrix groups $\mathrm{O}(\rho)$, $\mathrm{U}(\rho)$, or $\mathrm{Sp}(\rho)$ since $W(r)^2 = I_\rho$. This ensures that the map above is an isometry for $\mathbb{F}$ associative [**Ada96**, pp. 1-2]. It remains to check the non-associative case with $\mathbb{F} = \mathbb{O}$ and $\rho = 3$, which includes $\rho = 2$ when we restrict to the appropriate subspace. Any linear transformation acting on $\mathbb{O}^3$ preserving the norm $N(x)$ will also preserve the inner product (x, y). Since our map is linear we need to show that $N(x) = N(xW(r))$. To do so, let $x = a + b + c$ with $a = (A, 0, 0)$, $b = (0, B, 0)$, $c = (0, 0, C)$ and with

A, B, C in $\mathbb{O}$. Writing $W = W(r)$, we have

$$N(xW) = N(aW) + N(bW) + N(cW) + 2(aW, bW) \\ + 2(bW, cW) + 2(cW, aW).$$

The first term satisfies $N(aW) = N(a)$ since the coefficients of a and W belong to a common quaternion subalgebra. Likewise we have $N(bW) = N(b)$ and $N(cW) = N(c)$. Since $N(x) = N(a) + N(b) + N(c)$, it remains to show that the cross terms of the form (aW, bW) vanish. To do so we write primitive idempotent matrix $[r]$ in the form

$$[r] = \begin{pmatrix} d & F & \overline{E} \\ \overline{F} & e & D \\ E & \overline{D} & f \end{pmatrix}, \; d, e, f \in \mathbb{R}, \; D, E, F \in \mathbb{O}.$$

The following inner product evaluates to

$$\frac{1}{4}(aW, bW) = \mathrm{Re}((A\overline{E})(\overline{D}\,\overline{B})) + \left(e - \frac{1}{2}\right) \mathrm{Re}((AF)\overline{B}) \\ + \left(d - \frac{1}{2}\right) \mathrm{Re}(A(F\overline{B})).$$

In general, $\mathrm{Re}(A(F\overline{B})) = \mathrm{Re}((AF)\overline{B})$ for any octonions A, B, F [**Wil09a**, p. 145]. Likewise, $\mathrm{Re}((A\overline{E})(\overline{D}\,\overline{B})) = \mathrm{Re}(((A\overline{E})\overline{D})\overline{B})$. We can also use the primitive idempotent relations $e + d - 1 = -f$ and $fF = \overline{E}\,\overline{D}$ [**Wil09a**, p. 157]. Finally, by construction E and D belong to a common complex subalgebra of $\mathbb{O}$, since r is a commutative vector. This ensures that $(A\overline{E})\overline{D} = A(\overline{E}\,\overline{D})$. Taken together, our expression simplifies to zero:

$$\begin{aligned} \frac{1}{4}(aW, bW) &= \mathrm{Re}(((A\overline{E})\overline{D})\overline{B}) + (d + e - 1)\,\mathrm{Re}((AF)\overline{B}) \\ &= \mathrm{Re}(((A\overline{E})\overline{D})\overline{B}) - \mathrm{Re}((A(fF))\overline{B}) \\ &= \mathrm{Re}(((A\overline{E})\overline{D} - A(\overline{E}\,\overline{D}))\overline{B}) \\ &= 0. \end{aligned}$$

A similar calculation cycling $a \mapsto b \mapsto c \mapsto a$, $d \mapsto e \mapsto f \mapsto d$, and $D \mapsto E \mapsto F \mapsto D$ verifies that $(bW, cW) = (cW, aW) = 0$. This confirms that $x \mapsto xW(r)$ is an isometry of $\mathbb{O}^3$ when r is a commutative vector.

In the associative cases, with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the map $[x] \mapsto W(r)[x]W(r) = P(W(r))[x]$ is a known isometry of $\mathbb{FP}^{\rho-1}$. In the non-associative case with $\mathbb{F} = \mathbb{O}$, we verify that $[x] \mapsto W(r)[x]W(r)$ is an isometry of $\mathbb{OP}^2$ by beginning with the known isometry $P(W(r))[x]$ given by the quadratic map. We can write $[x] = a + b + c + A + B + C$ for a, b, c real-valued matrices corresponding to the diagonal entries of $[x]$ and A, B, C octonion-valued Hermitian matrices corresponding to the octonion valued off-diagonal entries of $[x]$.

$$P(W(r))[x] = P(W(r))a + \cdots + P(W(r))C.$$

Each term in this expansion contains matrix entries in the factors that share a common quaternion subalgebra, so we can use the simplification of $P(x)y = xyx$ available in associative cases for each term:

$$P(W(r))[x] = W(r)aW(r) + \cdots + W(r)CW(r) = W(r)[x]W(r).$$

This confirms that for r a commutative vector, the map $[x] \mapsto W(r)[x]W(r)$ is an isometry of $\mathbb{OP}^2$. $\square$

5.4. The Common Construction

This section defines a common construction for pairs of t-designs, provides a familiar example, and then applies the common construction to the two strictly projective tight 5-designs of Theorem 5.2.

DEFINITION 5.6 (Common construction). Let $r_1, r_2, \ldots, r_n$ be commutative row vectors in $\mathbb{F}^\rho$, with $d = [\mathbb{F} : \mathbb{R}]$, and let $[r_1], [r_2], \ldots, [r_n]$ be the corresponding primitive idempotents in projective space $\mathbb{FP}^{\rho-1}$. Let G be the group acting on $\mathbb{F}^\rho$ generated by the following isometries under composition:

$$x \mapsto x\,(I_\rho - 2[r_i])\,,\ i = 1, 2, \ldots, n.$$

Let H be the group acting on $\mathbb{FP}^{\rho-1}$ generated by the following isometries under composition:

$$[x] \mapsto (I_\rho - 2[r_i])\,[x]\,(I_\rho - 2[r_i])\,,\ i = 1, 2, \ldots, n.$$

If G is finite, then the orbit of $r_1, r_2 \ldots, r_n$ defines a spherical design in $\Omega_{\rho d}$ using $\mathbb{R}^{\rho d} \cong \mathbb{F}^\rho$. The lines spanned by the points of this spherical design define a real projective design in $\mathbb{R}^{\rho d - 1}$. When H is finite, the orbit of $[r_1], [r_2], \ldots, [r_n]$ defines a projective design in $\mathbb{FP}^{\rho-1}$.

REMARK 5.7. This common construction definition relies on Lemma 5.5 to ensure that the needed maps are indeed isometries when $r_1, r_2, \ldots, r_n$ are each commutative. When $\mathbb{F} \neq \mathbb{O}$, we can relax the commutative requirement of Definition 5.6, since for any r in $\mathbb{F}^\rho$ with $\mathbb{F}$ associative, the matrix $W(r)$ will belong to the isometry group of $\mathbb{F}^\rho$ (either $\mathrm{O}(\rho)$, $\mathrm{U}(\rho)$ or $\mathrm{Sp}(\rho)$). In general, a selection $r_1, r_2, \ldots, r_n$ of commutative vectors must be chosen carefully in order for the isometries given in Definition 5.6 to generate finite groups. More details about the classification of finite reflection groups are available in [**ST54, Coh76, Coh80, Wil09a**].

EXAMPLE 5.8. Let $r_1, r_2, \ldots, r_6 \in \mathbb{C}^6$ be the rows of the following matrix, where ω is a complex cube root of unity:

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & \omega & \omega & 1 & 0 & 0 \\ \omega & 1 & \omega & 0 & 1 & 0 \\ \omega & \omega & 1 & 0 & 0 & 1 \end{bmatrix}.$$

The common construction of Definition 5.6 yields finite groups G and H. The group G acting on $\mathbb{C}^6$ is the complex reflection group $W(K_6) = (6.\mathrm{PSU}_4(3)) : 2$. The orbit of $r_1, r_2, \ldots, r_6$ under the action of G form the 756 shortest vectors of the K_{12} integral lattice, defining a 5-design in Ω_{12}. This corresponds to the projective 2-design in $\mathbb{RP}^{11}$ consisting of the 378 lines spanned by the K_{12} short vectors. The orbit of $[r_1], [r_2], \ldots, [r_6]$ under the action of $H = \mathrm{PSU}_4(3) : 2$ form the 126 points of a tight 3-design in $\mathbb{CP}^5$. More details about this example are available in [**CS13**, pp. 127-129].

We may now introduce the main result of this chapter.

THEOREM 5.9. *There exists a set of commutative vectors $r_1, r_2, \ldots, r_n$ in $\mathbb{O}^3$ that yield the two strictly projective tight 5-designs under the common construction of Definition 5.6. The vectors $r_1, r_2, \ldots, r_n$ are not unique and can be given, for example, by the rows of the following matrix, for any $t \in \mathbb{F}_7$ and with $s = \frac{1}{2}(-1 + i_0 + i_1 + i_2 + i_3 + i_4 + i_5 + i_6)$:*

$$\begin{bmatrix} 2 & 2 & 0 \\ 2s & 0 & 0 \\ s^2 & s & s \\ 2 & 2i_t & 0 \\ 2 & 2i_{t+1} & 0 \\ 2 & 2i_{t+3} & 0 \end{bmatrix}.$$

PROOF. The examples given in Theorem 5.9 were found and checked using the software GAP [**Gro22**]. Setting $t = 0$, the computation begins by first applying the isometries of Definition 5.6 respectively to $x \in \{r_1, \ldots, r_6\}$ and $[x] \in \{[r_1], \ldots, [r_6]\}$. Any new elements in $\mathbb{O}^3$ and $\mathbb{OP}^2$ are added to the respective sets. This process is repeated and each application of the isometries either provides new elements or permutes the elements of the set. Once all six isometries permute the appropriate set, without providing new elements, those permutations are used to generate the groups G and H. Permutation group tools in GAP identify G and H as $2 \cdot G_2(4)$ and ${}^3D_4(2)$ respectively. The properties of the tight 5-designs given by the full orbits of $\{r_1, \ldots, r_6\}$ and $\{[r_1], \ldots, [r_6]\}$ are verified directly using the definitions above. Since $i_t \mapsto i_{t+1}$

is an automorphism of $\mathbb{O}$, the result is also true for $0 \neq t \in \mathbb{F}_7$. For more details on this construction, see Remark 5.10 below. $\square$

REMARK 5.10. If we apply the common construction of Definition 5.6 to the vectors given in Theorem 5.9 then we obtain an isometry group $G = 2 \cdot G_2(4) \subset \mathrm{O}(24)$ acting on $\mathbb{R}^{24} \cong \mathbb{O}^3$. The orbit of $r_1, \ldots, r_6$ under the action of G forms a $\sqrt{2} : 1$ scale copy of the short vectors of the Leech lattice, which define the unique tight 11-design on Ω_{24} with cardinality 196560. The lines spanned by the spherical tight 11-design vectors form the corresponding tight 5-design in $\mathbb{RP}^{23}$ with cardinality 98280. The common construction also yields isometry group ${}^3D_4(2) \subset F_4$ acting on $\mathbb{OP}^2$. The orbit of $[r_1], [r_2], \ldots, [r_6]$ under the action of H form a copy of the unique tight 5-design on $\mathbb{OP}^2$ of cardinality 819.

5.5. Leech Lattice Symmetries and the Octonion Projective Plane

In light of Theorem 5.9, we can use the involutionary isometries from our common construction to generate certain symmetries of the Leech lattice and exhibit their relation to the octonion projective plane. This section outlines a construction of the Suzuki chain of Leech lattice symmetries acting on $\mathbb{O}^3$ and describes their corresponding action on $\mathbb{OP}^2$ where possible.

The vectors in $\mathbb{O}^3 \cong \mathbb{R}^{24}$ of the spherical tight 11-design in Theorem 5.9 are precisely the short vectors of Wilson's octonion Leech lattice construction [**Wil09b, Wil09a, Wil11**]. In fact, a GAP computation confirms that any choice of $t \in \mathbb{F}_7$ in Theorem 5.9 yields the same orbit in $\mathbb{O}^3$ but distinct orbits in $\mathbb{OP}^2$ and distinct groups of type $2 \cdot G_2(4)$ acting on $\mathbb{O}^3$. The union of these seven distinct $2 \cdot G_2(4)$ groups generates the full Leech lattice automorphism group, Conway's group $\mathrm{Co}_0 = 2 \cdot \mathrm{Co}_1$. The corresponding permutation group acting on the 98280 lines defining the tight 5-design in $\mathbb{RP}^{23}$ is the sporadic simple group Co_1.

The group Co_1 contains the alternating group A_9 as a subgroup, with symmetric group S_3 centralizing A_9 in Co_1. The *Suzuki chain* is a chain of centralizers in Co_1 of the corresponding chain of alternating groups $A_9 > A_8 > \cdots > A_3$ [**Wil09a**, p. 219]:

$$S_3 < S_4 < \mathrm{PSL}_2(7) < \mathrm{PSU}_3(3) < \mathrm{HJ} < G_2(4) < 3 \cdot \mathrm{Suz}.$$

Here HJ and Suz are respectively the Hall–Janko and Suzuki sporadic simple groups. Wilson uses scalar octonion multiplication acting on a Leech lattice in $\mathbb{O}^3$ to construct the isometries needed to generate a chain of double covers of the alternating groups used in the Suzuki chain. Wilson then also includes known coordinate symmetries of the octonion triples (coordinate permutations and sign changes) to construct a maximal subgroup of $2 \cdot \mathrm{Co}_1$. By appending the single reflection $x \mapsto xW(r)$, with $r = (s, 1, 1)$, Wilson is able to recover the entire Leech lattice automorphism group Co_0. The Suzuki chain subgroups

$\{r_1, \ldots, r_n\}$	$G/\{\pm 1\} \subset \mathrm{Co}_1$	$H \subset F_4$
V_∞	S_4	S_4
S	$\mathrm{PSL}_2(7)$	$\mathrm{PSL}_2(7)$
$S \cup V_t$	$\mathrm{PSU}_3(3)$	$\mathrm{PSU}_3(3)$
$S \cup V_t \cup V_{t'}$	HJ	${}^3D_4(2)$
$S \cup V_t \cup V_{t+1} \cup V_{t+3}$	$G_2(4)$	${}^3D_4(2)$
$S \cup V_{t+2} \cup V_{t+5} \cup V_{t+6} \cup V_{t+7}$	$3 \cdot \mathrm{Suz}$	
$S \cup V_\infty \cup V_0 \cup V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6$	Co_1	

TABLE 5.1. The common construction applied to orbits of combinations of S, V_∞, and V_t for $t \neq t' \in \mathbb{F}_7$.

are described in the context of these isometries. More details about Wilson's construction and the group theory involved are available in [**Wil09b, Wil09a, Wil11**].

Using the computational results of the construction in Theorem 5.9, we can provide an alternative construction of Leech lattice automorphisms and the Suzuki chain. The benefit of this alternative construction is that all the generators involved are involutionary isometries of $\mathbb{O}^3$ with corresponding isometries on $\mathbb{OP}^2$ via the common construction of Definition 5.6.

DEFINITION 5.11. Let $t \in \mathbb{F}_7$, let $s = \frac{1}{2}(-1 + i_0 + i_1 + i_2 + i_3 + i_4 + i_5 + i_6)$, and let V_∞, V_t, and S be, respectively, the sets of vectors of the form $(2, 2, 0)$, $(2, 2i_t, 0)$, and (s^2, s, s), under all coordinate permutations and sign changes.

EXAMPLE 5.12. As depicted in Table 5.1, under the common construction of Definition 5.6, different choices of commutative vectors yield different finite group actions $G/\{\pm 1\}$ acting on $\mathbb{O}^3/\mathbb{R} \cong \mathbb{RP}^{23}$ and H acting on $\mathbb{OP}^2$. Here we have $t, t' \in \mathbb{F}_7$ and $t \neq t'$.

REMARK 5.13. The groups described in Example 5.12 (Table 5.1) are computed using GAP in the same manner as described in the proof of Theorem 5.9. Computation time is saved by using the octonion automorphism $i_t \mapsto i_{t+1}$ to reduce the number of cases to check.

REMARK 5.14. In Table 5.1, the bottom three rows yield the full tight 5-design in $\mathbb{RP}^{23}$. Both the fourth and fifth rows yield the tight 5-design in $\mathbb{OP}^2$. The fifth row yields both tight 5-designs and corresponds to the example in Theorem 5.9.

REMARK 5.15. The first three rows of the table in Table 5.1 involve initial vectors $\{r_1, \ldots, r_n\}$ with coefficients belonging to a common associative subalgebra of $\mathbb{O}$. Accordingly, the isometries generating G and H also generate matrix groups, which accounts for the agreement between $G/\{\pm 1\}$ and H. The bottom four rows in the table involve sets $\{r_1, \ldots, r_n\}$ with coefficients that generate the full non-associative octonion algebra. This means that the groups G and H generated by the initial isometries are no longer related to a matrix group generated by matrices $W(r_i) = I_3 - 2[r_i]$. Indeed, the matrices $W(r_i)$ instead generate a non-associative octonion matrix loop rather than a matrix group. This partly explains the divergence between the properties of the groups G and H in the bottom four rows of the table. In the bottom two rows of the table, the initial isometries generate a finite group G acting on $\mathbb{O}^3$ but do not seem to generate a corresponding finite group acting on $\mathbb{OP}^2$. An open question is whether they generate the Lie group F_4, the full isometry group of $\mathbb{OP}^2$.

Remark 5.16. As described above, Theorem 5.9 yields the same tight 5-design in $\mathbb{RP}^{23}$ but distinct tight 5-designs in $\mathbb{OP}^2$ for distinct $t \in \mathbb{F}_7$. In contrast, if we take the conjugate $s \mapsto \overline{s}$ in Theorem 5.9, then for distinct $t \in \mathbb{F}_7$ the common construction will instead yield distinct tight 5-designs in $\mathbb{RP}^{23}$ but just one common tight 5-design in $\mathbb{OP}^2$.

Remark 5.17. Possible variations on initial vectors in Theorem 5.9 include using carefully selected norm 2 integral octonions to construct the initial vectors $\{r_1, \ldots, r_n\}$ of the common construction of Definition 5.6. Chapter 6 will explore constructions of this form and how they can be used to exhibit Suzuki chain symmetries of the Leech lattice.

5.6. Conclusion

We have seen in Theorem 5.9 and Table 5.1 that vectors $\{r_1, \ldots, r_n\} = S \cup V_t \cup V_{t+1} \cup V_{t+3}$, with $t \in \mathbb{F}_7$, define isometries of $\mathbb{O}^3$ and $\mathbb{OP}^2$ according to the common construction of Definition 5.6. These isometries generate a finite group $G_2(4)$ acting on $\mathbb{O}^3/\mathbb{R} \cong \mathbb{RP}^{23}$ and a finite group ${}^3D_4(2)$ acting on $\mathbb{OP}^2$ that yield the two strictly projective tight 5-designs as orbits. Specifically, the two tight 5-designs are, respectively, the orbits of the initial vectors $\{r_1, \ldots, r_n\}$ and of the primitive idempotents $\{[r_1], \ldots, [r_n]\}$. This common construction accounts for the previously conjectured connection between these two tight 5-designs in [**Hog82**].

Hoggar remarks in [**Hog82**] that his conjectured connection was met with skepticism. "Against this, the referee remarks: the automorphism group of the unique $(2, 8)$ hexagon has index 3 subgroup ${}^3D_4(2)$, which has no irreducible projective representation of degree ≤ 24. Furthermore, ${}^3D_4(2)$ has no proper subgroups acting transitively on the 819 points" [**Hog82**]. In our common construction of Definition 5.6, the link between our two designs is the initial commutative vectors $\{r_1, r_2, \ldots, r_n\}$ rather than the group ${}^3D_4(2)$, its subgroups, or its representations. Indeed the pair of groups, $G_2(4)$ acting on $\mathbb{RP}^{23}$ and ${}^3D_4(2)$ acting on $\mathbb{OP}^2$, have relative cardinality 25/21 so that one cannot be a subgroup of the other. The non-associativity of the octonion algebra permits the common construction to yield these seemingly unrelated groups and the tight 5-designs given by their orbits.

CHAPTER 6

Octonion Integers and Tight 5-Designs

6.1. Introduction

Let $\mathbb{F}$ be a real division composition algebra—one of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. A previous paper [**Nas22**] (Chapter 5 of this thesis) introduced a common construction for t-designs in $\mathbb{RP}^{\rho d-1}$ and $\mathbb{FP}^{\rho-1}$ and showed how this common construction can yield the only two strictly projective tight 5-designs, which exist in $\mathbb{RP}^{23}$ and $\mathbb{OP}^2$. The example used in that paper was obtained by computation and given to establish the existence of the common construction of these two tight 5-designs. That example matches the coordinates used in Wilson's octonion Leech lattice construction [**Wil09b, Wil11**]. This chapter instead examines the common construction of these two tight 5-designs using octonion integers and their symmetries.

We begin with a review of [**Nas22**]. In what follows, let rank ρ and degree d be positive integers with ρ greater than 1 and $d = [\mathbb{F} : \mathbb{R}]$ when ρ is greater than 2. We will call vector $x = (x_1, x_2, \ldots, x_\rho)$ in $\mathbb{F}^\rho$ respectively *commutative* or *associative* when the coefficients $x_1, x_2, \ldots, x_\rho$ belong to a common commutative or associative subalgebra of $\mathbb{F}$. The norm $N(x)$ is defined as $N(x) = xx^\dagger$, where $x^\dagger$ is the conjugate transposed of row vector x. When x is associative we can define the *projector* $[x] = x^\dagger x / N(x)$, which is a primitive idempotent Hermitian matrix in $\mathrm{Herm}(\rho, \mathbb{F})$. Projectors of this form belong to the projective space $\mathbb{FP}^{\rho-1}$, represented as primitive idempotent Hermitian matrices, as described in [**Hog82**]. If r in $\mathbb{F}^\rho$ is commutative or $\mathbb{F}$ is associative then we can define the following pair of *reflection isometries* acting on $\mathbb{F}^\rho$ and $\mathbb{FP}^{\rho-1}$:

$$\mathtt{W}_r : x \mapsto x(I_\rho - 2[r]), \quad [x] \mapsto (I_\rho - 2[r])[x](I_\rho - 2[r]).$$

That both actions of $\mathtt{W}_r$ are in fact respectively isometries of $\mathbb{F}^\rho$ and $\mathbb{FP}^{\rho-1}$ is verified in [**Nas22**]. We can use these actions to construct spherical and projective t-designs, or simply designs. A *spherical design* X is a finite subset of points $X \subset \Omega_{\rho d}$ where $\Omega_{\rho d}$ is a sphere. A *projective design* is a finite subset of points $X \subset \mathbb{FP}^{\rho-1}$ where $\mathbb{FP}^{\rho-1}$ is a projective space. A design X has certain properties such as tightness, angle set A, and strength t which are reviewed in [**Nas22**]. Of note, the only two strictly projective tight 5-designs are shown to have a common construction in [**Nas22**], where the *common construction* is defined as follows:

Definition 6.1 (Common Construction). Let $r_1, r_2, \ldots, r_n$ be commutative vectors in $\mathbb{F}^\rho$ (or let $\mathbb{F}$ be associative). Let G and H be the group of isometries generated by $\mathtt{W}_{r_1}, \mathtt{W}_{r_2}, \ldots, \mathtt{W}_{r_n}$ respectively acting on $\mathbb{F}^\rho$ and $\mathbb{FP}^{\rho-1}$. If G is finite then the union of orbits of $r_1, r_2, \ldots, r_n$ defines a spherical design on $\Omega_{\rho d}$, since $\mathbb{F}^\rho \cong \mathbb{R}^{\rho d}$, with a corresponding projective design on $\mathbb{RP}^{\rho d-1}$. If H is finite then the union of orbits of $[r_1], [r_2], \ldots, [r_n]$ defines a projective design on $\mathbb{FP}^{\rho-1}$.

Suitable vectors $r_1, \ldots, r_6$ exist such that the common construction above yields the two strictly projective tight 5-designs [**Nas22**]. The coordinates given in [**Nas22**] ensure that the tight 5-design on $\mathbb{RP}^{23}$ is given by the lines spanned by the short vectors of Wilson's octonion Leech lattice construction [**Wil09b**]. In this chapter we will instead work with octonion integers to describe the common construction of these two tight 5-designs. We will also clarify Leech lattice constructions using octonion integer triples and describe the Suzuki chain subgroups of Leech lattice automorphisms as octonion reflection groups. Finally, we will compare the common construction described here to treatment of the two tight strictly projective 5-designs given in [**EG96**].

6.2. Octonion Integers

This section reviews octonion algebra and octonion integer ring concepts in order to provide a complete description of the common construction of [**Nas22**] in terms of octonion integers. Many important properties of octonions and their integer rings are described in [**SV00**], [**Bae02**] [**CS03**].

A *composition algebra* is an algebra over some field equipped with a nondegenerate quadratic form N that satisfies the composition rule $N(xy) = N(x)N(y)$. A composition algebra is *unital* if it also includes an identity element. A division composition algebra lacks isotropic vectors, i.e. $N(x) = 0$ only for $x = 0$. A major theorem due to Hurwitz confirms that there are precisely four unital division composition algebras $\mathbb{F}$ over the real numbers: the real field $\mathbb{R}$, the complex field $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. The octonion algebra $\mathbb{O}$ contains the others as subalgebras.

The corresponding *inner product* $\langle x, y\rangle = N(x+y) - N(x) - N(y)$ is twice the standard Euclidean inner product. This means that $N(x) = \frac{1}{2}\langle x, x\rangle$, and the standard Euclidean inner product is $\frac{1}{2}\langle x, y\rangle$.

Remark 6.2. Many authors prefer to define the inner product as $\frac{1}{2}(N(x+y) - N(x) - N(y))$ so that it matches the standard Euclidean inner product. In our definition of inner product $\langle x, y\rangle$ we omit the factor of $\frac{1}{2}$ for convenience when working with octonion integers modulo 2 below. This also follows the convention in [**SV00**].

The *real component* of an octonion x is the component projected onto the identity: $\mathrm{Re}(x) = \frac{1}{2}\langle 1, x\rangle$. The difference $\mathrm{Im}(x) = x - \mathrm{Re}(x)$ is the *imaginary component* of x. The *octonion conjugate* is defined as $\overline{x} = 2\mathrm{Re}(x) - x$. The octonion product ensures that $N(x) = x\overline{x} = \overline{x}x$ and that every octonion satisfies the following characteristic equation:

$$x^2 - 2\mathrm{Re}(x)x + N(x) = 0.$$

In what follows we will denote by $\mathbb{R}(x_1, x_2, \dots, x_n)$ the subalgebra of $\mathbb{O}$ generated by the products and $\mathbb{R}$-linear combinations of octonions $1, x_1, x_2, \dots, x_n$. Any octonion x not contained in the real subalgebra, so that $x \neq \mathrm{Re}(x)$, generates a complex subalgebra $\mathbb{R}(x) \cong \mathbb{C}$. Any two octonions x, y generating distinct complex subalgebras $\mathbb{R}(x) \neq \mathbb{R}(y)$ will generate a quaternion subalgebra $\mathbb{R}(x, y) \cong \mathbb{H}$. Finally, any three octonions x, y, z that pairwise generate distinct quaternion subalgebras $\mathbb{R}(x, y)$, $\mathbb{R}(y, z)$, $\mathbb{R}(x, z)$ will generate the full octonion algebra $\mathbb{R}(x, y, z) = \mathbb{O}$. Every octonion belongs to some complex and quaternion subalgebra of $\mathbb{O}$. The complex numbers form a commutative and associative algebra, while the quaternion algebra is not commutative but still associative. The octonions are neither commutative nor associative but still have many special properties and symmetries.

An *octonion algebra automorphism* is an invertible $\mathbb{R}$-linear map $\sigma : \mathbb{O} \to \mathbb{O}$ that also preserves the octonion product, so that $\sigma(xy) = \sigma(x)\sigma(y)$. The group $\mathrm{Aut}(\mathbb{O})$ of all such octonion algebra automorphisms is a Lie group of type G_2. The group $\mathrm{Aut}(\mathbb{O})$ is transitive on *imaginary units*, namely octonions i such that $\mathrm{Re}(i) = 0$ and $N(i) = 1$. It follows that $\mathrm{Aut}(\mathbb{O})$ is transitive on the complex subalgebras of $\mathbb{O}$, which each have the form $\mathbb{C} \cong \mathbb{R}(i)$ for some imaginary unit i. Likewise, $\mathrm{Aut}(\mathbb{O})$ is transitive on ordered pairs of orthogonal imaginary units, namely all (i, j) such that $\langle i, j\rangle = 0$. This ensures that $\mathrm{Aut}(\mathbb{O})$ is transitive on quaternion subalgebras of $\mathbb{O}$ since these all have the form $\mathbb{H} \cong \mathbb{R}(i, j)$ for some pair of orthogonal imaginary units (i, j). Finally, a *basic triple* is an ordered triple of orthogonal imaginary units (i, j, l), where i, j, l pairwise generate distinct quaternion subalgebras. The group $\mathrm{Aut}(\mathbb{O})$ is also transitive on basic triples [**Bae02**, p. 185], so we can select any basic triple (i, j, l) in $\mathbb{O}$ without loss of generality.

Remark 6.3. For any quaternion subalgebra $\mathbb{H} \subset \mathbb{O}$, the group $\mathrm{Aut}(\mathbb{O})$ contains an involution fixing $\mathbb{H}$ and multiplying the orthogonal component by -1. That is, for any basic triple (i, j, l), the map $(i, j, l) \mapsto (i, j, -l)$ defines an octonion algebra automorphism fixing $\mathbb{H} = \mathbb{R}(i, j)$.

As described in [**CS03**], we now define the more familiar octonion *standard basis* $\{i_t \mid t \in \mathrm{PL}(7)\}$, indexed by the projective line $\mathrm{PL}(7) = \{\infty\} \cup \mathbb{F}_7$, in terms of some basic triple (i, j, l):

$$\begin{array}{llll} i_\infty = 1, & i_0 = -(ij)l, & i_1 = il, & i_2 = i, \\ i_3 = j, & i_4 = l, & i_5 = ij, & i_6 = jl. \end{array}$$

The units in this basis multiply as expected according to their properties in the quaternion subalgebras they pairwise define. That is, for all $t \neq \infty$, the set $\{1, i_t, i_{t+1}, i_{t+3}\}$ forms a standard quaternion basis with $i_t i_{t+1} i_{t+3} = -1$. The PL(7) indexing of the standard basis vectors exhibits certain helpful octonion algebra automorphisms, namely those defined on the basis by $i_t \mapsto i_{t+1}$ and $i_t \mapsto i_{2t}$ (here $1 = i_\infty$ is fixed and the remaining indices are computed modulo 7).

We will call $\mathbb{Z}$ the *rational integers* to distinguish $\mathbb{Z}$ from octonion integer rings in what follows. A *Gravesian integer ring* is all $\mathbb{Z}$-linear combinations of some standard basis [**CS03**, p. 100]. Equivalently, any basic triple (i, j, l) defines a Gravesian integer ring $\mathbb{Z}(i, j, l)$, consisting of all $\mathbb{Z}$-linear combinations of $1, i, j, l$ and their products, which include the standard basis vectors. Since a Gravesian integer ring contains a basic triple, and since $\mathrm{Aut}(\mathbb{O})$ is transitive on basic triples, the octonion automorphism group is transitive on Gravesian integer rings. This means we can select a representative and speak of *the* Gravesian integer ring without loss of generality.

The Gravesian integers are an example of an octonion *order*: a subring of the octonion algebra where each subring element x has $\mathbb{Z}$-valued $2\mathrm{Re}(x)$ and $N(x)$ [**CS03**, p. 100]. If an octonion order contains basic triple (i, j, l) it must also contain the corresponding standard basis $\{i_t \mid t \in \mathrm{PL}(7)\}$ defined above. Therefore, any octonion order containing a basic triple also contains the corresponding Gravesian integer ring. An *octonion arithmetic* is a maximal order. It is known that there are precisely seven octonion arithmetics containing any given Gravesian integer ring, and that these form a single orbit under the octonion algebra automorphism subgroup generated by $i_t \mapsto i_{t+1}$ [**CS03**, p. 100]. It follows that any octonion arithmetic containing a basic triple is one of seven containing the Gravesian integer ring defined by that basic triple.

We want to distinguish one of the seven octonion arithmetics containing the Gravesian integers as canonical. We will do so using properties of some underlying basic triple. Basic triple (i, j, l) defines a unique pair of opposite imaginary units in the standard basis corresponding to all possible triple products of i, j, l, namely $\{i_0, -i_0\}$ (recall that $i_0 = -(ij)l = l(ij)$, etc.). We will select a canonical octonion arithmetic for basic triple (i, j, l) that distinguishes i_0 from the other units in the standard basis defined by that basic triple. To do so, we note that there are three quaternion double bases in $\{\pm i_t \mid \mathrm{PL}(7)\}$ containing $\pm i_0$, namely $\{\pm 1, \pm i_0, \pm i_r, \pm i_{3r}\}$ for $r = 1, 2, 4$ (indices computed modulo 7). The rings $\mathbb{Z}(i_0, \omega_r)$ with $\omega_r = \frac{1}{2}(-1 + i_0 + i_r + i_{3r})$, for r one of $1, 2, 4$, are isomorphic to the *Hurwitz integer ring* H [**CS03**, p. 57]. The three

Hurwitz integer rings $\mathbb{Z}(i_0, \omega_1)$, $\mathbb{Z}(i_0, \omega_2)$, $\mathbb{Z}(i_0, \omega_4)$ together generate one of the seven arithmetics containing the basic triple (i, j, l). Indeed, a brief computation confirms that this arithmetic has the form $\mathbb{Z}(\omega_1, \omega_2, \omega_4)$. Accordingly, we define the *canonical arithmetic* O containing basic triple (i, j, l) as the ring,

$$\mathsf{O} = \mathsf{O}_{i,j,l} = \mathbb{Z}(\omega_1, \omega_2, \omega_4), \quad \omega_r = \frac{1}{2}\left(-1 + i_0 + i_r + i_{3r}\right).$$

The ring O is called the *octavian integers* in [**CS03**, p. 99] but we will call it the *octonion integer ring.* The remaining six octonion arithmetics containing (i, j, l) are given by the orbit of O under the automorphism defined on the basis by $i_t \mapsto i_{t+1}$.

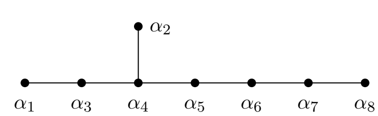
We now verify that any octonion arithmetic containing a basic triple is isomorphic to the canonical arithmetic defined above.

LEMMA 6.4. *The octonion algebra automorphism group is transitive on octonion arithmetics containing at least one basic triple.*

PROOF. Let O be an octonion arithmetic containing a basic triple (i, j, l). It follows that O contains the Gravesian integer ring defined by (i, j, l) and is one of the seven isomorphic arithmetics containing that order. Likewise, let O' be an octonion arithmetic containing a basic triple (i', j', l'), one of seven isomorphic arithmetics containing this basic triple. The image of O' under any octonion automorphism is also an isomorphic arithmetic containing the image of basic triple (i', j', l'). The octonion algebra automorphism group is transitive on basic triples, which means that there exists an automorphism σ such that $\sigma(i') = i$, $\sigma(j') = j$, $\sigma(l') = l$. Since $\sigma(\mathsf{O}')$ is also an arithmetic and contains (i, j, l), it must be one of the seven arithmetics containing this basic triple. Therefore O and $\sigma(\mathsf{O}')$ are in the same orbit of the automorphism subgroup generated by cycle $i_t \mapsto i_{t+1}$. Therefore an automorphism exists mapping O' to O. $\square$

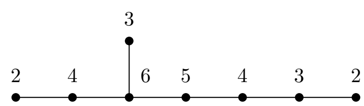
Using the *double* Euclidean inner product $\langle x, y\rangle = N(x+y) - N(x) - N(y)$, the canonical arithmetic O of basic triple (i, j, l) has the geometry of an $\mathtt{E}_8$ lattice. Since we have $N(x) = \frac{1}{2}\langle x, x\rangle$, the units of O correspond to $\mathtt{E}_8$ roots and we can select a simple root system as a basis.

REMARK 6.5. For the canonical arithmetic O defined above, the following Coxeter-Dynkin diagram provides a simple $\mathtt{E}_8$ root system:



$$\alpha_1 = \frac{1}{2}(-i_1 + i_5 + i_6 + i_0), \qquad \alpha_2 = \frac{1}{2}(-i_1 - i_2 - i_4 - i_0),$$
$$\alpha_3 = \frac{1}{2}(i_2 + i_3 - i_5 - i_0), \qquad \alpha_4 = \frac{1}{2}(i_1 - i_3 + i_4 + i_5),$$
$$\alpha_5 = \frac{1}{2}(-i_2 + i_3 - i_5 + i_0), \qquad \alpha_6 = \frac{1}{2}(i_2 - i_4 + i_5 - i_6),$$
$$\alpha_7 = \frac{1}{2}(-i_1 - i_3 + i_4 - i_5), \qquad \alpha_8 = \frac{1}{2}(-1 + i_1 - i_4 + i_6).$$

These particular simple roots have been chosen so that the highest root β is opposite the identity,

$\beta =$  $= -1,$

and also so that $\alpha_1, \ldots, \alpha_7$ are the simple roots of an $\mathtt{E}_7$ sublattice of purely imaginary octonion integers. This choice of simple roots is not uniquely defined by these properties.

The ring automorphism group $\mathrm{Aut}(\mathsf{O})$ has type $G_2(2) \cong \mathrm{PSU}_3(3) : 2$ and order 12096. We can represent $\mathrm{Aut}(\mathsf{O})$ as a finite subgroup of $\mathrm{Aut}(\mathbb{O})$, the Lie group of automorphisms.

Lemma 6.6. *Every element of* $\mathrm{Aut}(\mathsf{O})$ *is a restriction of a unique element of* $\mathrm{Aut}(\mathbb{O})$ *to* $\mathsf{O} \subset \mathbb{O}$.

Proof. The group $\mathrm{Aut}(\mathsf{O})$ is known to be of type $G_2(2) \cong \mathrm{PSU}_3(3) : 2$, with cardinality 12096 [**Wil09a**, pp. 132-134]. A ring automorphism group preserves units and we can represent $\mathrm{Aut}(\mathsf{O})$ faithfully using a permutation representation acting on the 240 units of O. Any map acting on $\mathbb{O}$ of the form $x \mapsto a^{-1}xa$ with $\mathrm{Re}(a^3) = a^3$ is an octonion algebra automorphism [**CS03**, p. 98]. Computation using GAP on a canonical copy of O verifies that the 56 units ω in O of order 3 define permutations on the units of O of the form $x \mapsto \omega^{-1}x\omega$, and these permutations generate a ring automorphism group of order 6048 and type $\mathrm{PSU}_3(3)$. Suppose that O contains basic triple (i, j, l) as described above. There exists an octonion algebra automorphism such that $(i, j, l) \mapsto (i, j, -l)$, since $(i, j, -l)$ is also a basic triple. This octonion algebra automorphism fixes the subalgebra generated by i and j, while negating the perpendicular component of $\mathbb{O}$. Computation confirms that this algebra automorphism also permutes the units in O and is not contained in the $\mathrm{PSU}_3(3)$ ring automorphism subgroup given above. Therefore the ring automorphism group $\mathrm{Aut}(\mathsf{O})$ has a permutation representation on the 240 units of O and a matrix representation as endomorphisms of $\mathbb{O}$ in $\mathrm{Aut}(\mathbb{O})$. That is, we can characterize each element of $\mathrm{Aut}(\mathsf{O})$ as an automorphism in $\mathrm{Aut}(\mathbb{O})$ restricted to O. Finally, we show uniqueness. Let O be the canonical arithmetic of basic triple (i,j,l). The orbit of ordered triple (i,j,l) under the action of $\mathrm{Aut}(\mathsf{O})$ has length 12096. That is, the only ring automorphism in $\mathrm{Aut}(\mathsf{O})$ fixing (i,j,l) is the identity. Let σ be an octonion automorphism fixing basic triple (i,j,l). It follows that σ also fixes the standard basis defined by this basic triple. Since σ is an automorphism of an $\mathbb{R}$-algebra that fixes a basis it must be the identity automorphism. It follows that the only algebra automorphism that restricts to the ring identity automorphism is the identity. Therefore, each ring automorphism is a restriction of a unique algebra automorphism. □

Remark 6.7. We can also describe $\mathrm{Aut}(\mathsf{O})$ as the stabilizer of O in $\mathrm{Aut}(\mathbb{O})$. That is, we can take the subgroup of octonion automorphisms mapping basic triple (i,j,l) to (i',j',l') such that $\mathsf{O}_{i,j,l} = \mathsf{O}_{i',j',l'}$, i.e. such that the canonical arithmetics defined by the two triples are equivalent. This includes the maps $(i,j,l) \mapsto (\omega^{-1}i\omega, \omega^{-1}j\omega, \omega^{-1}l\omega)$ for any ω in O with $\omega^3 = 1$ and $(i,j,l) \mapsto (i,j,-l)$. Under the $\mathrm{Aut}(\mathsf{O})$ action generated by these maps, the basic triple (i,j,l) belongs to an orbit of 12096 basic triples for which O is the canonical arithmetic.

Octonion integers O form a non-associative ring and have the property that every ideal is a two-sided principle ideal of the form $n\mathsf{O}$, for n a rational integer [**CS03**, pp. 109-110]. The quotient ring $\mathsf{O}/2\mathsf{O}$ is a finite simple non-associative ring. In fact, the ring $\mathsf{O}/2\mathsf{O}$ is precisely the unique finite octonion algebra over the field with two elements $\mathbb{F}_2$ [**SV00**, pp. 19-22]. The automorphism group of the ring $\mathsf{O}/2\mathsf{O}$ is the automorphism group $G_2(2)$ of this ring as an $\mathbb{F}_2$-algebra.

The residue classes of $\mathsf{O}/2\mathsf{O}$ (of the form $x+2\mathsf{O}$) each have representatives x of minimal norm either 0, 1, or 2. With respect to the *standard Euclidean inner product* $\frac{1}{2}\langle x,y\rangle$, O is isomorphic to the scaled $\mathtt{E}_8/\sqrt{2}$ lattice, with 240 norm 1 and 2160 norm 2 elements:

$$|\mathsf{O}/2\mathsf{O}| = 1 + \frac{240}{2} + \frac{2160}{16}.$$

Following [**CS03**, p. 136] we will call the 16 norm 2 representatives of a residue class a *frame*. Each frame is geometrically eight orthogonal pairs of opposite norm 2 vectors. The norm 1 and 2 elements of O form the following orbits under the action of $\mathrm{Aut}(\mathsf{O})$:

(1) The zero of $x^2 - 2x + 1$, the identity 1.
(2) The zero of $x^2 + 2x + 1$, the element -1.
(3) The 126 zeros of $x^2 + 1$, order 4 imaginary units denoted i.
(4) The 56 zeros of $x^2 + x + 1$, order 3 units denoted ω.
(5) The 56 zeros of $x^2 - x + 1$, order 6 units $-\omega$.
(6) The 126 zeros of $x^2 - 2x + 2$, norm 2 elements $1 + i$.
(7) The 126 zeros of $x^2 + 2x + 2$, norm 2 elements $-1 - i$.

(8) The 576 zeros of $x^2 - x + 2$, norm 2 elements denoted $-\lambda$.
(9) The 576 zeros of $x^2 + x + 2$, norm 2 elements λ.
(10) The 756 zeros of $x^2 + 2$, norm 2 elements denoted $i + j$.

To summarize, when working with the octonion integers we can use the arithmetic O without loss of generality since an arithmetic containing a standard basis (indeed a basic triple) is unique up to automorphism. Furthermore, we can select a root λ of $x^2 + x + 2$ in O without loss of generality since any choice is unique up to automorphism. This will allow us to construct octonion Leech lattices using properties of octonion integers but without reference to a particular choice of coordinates. In what follows we will also make use of the finite ring $\mathsf{O}/2\mathsf{O}$ to simplify computations and proofs.

6.3. Octonion Integer Leech Lattices

In what follows we will make use of certain theorems described in [**LM82**] to identify $\mathtt{E}_8$ sublattices of O and Leech sublattices of O^3. First we describe a method to identify the $\mathtt{E}_8$ and Leech lattices using the classification of unimodular lattices in low dimensions.

THEOREM 6.8. *The Gosset lattice* $\mathtt{E}_8$ *is the unique unimodular lattice in* $\mathbb{R}^8$ *with minimal norm at least* 2. *The Leech lattice* Λ_{24} *is the unique unimodular lattice in* $\mathbb{R}^{24}$ *with minimal norm at least* 4.

PROOF. Every unimodular lattice is either odd (type I) or even (type II). A unimodular lattice with a norm 1 vector is of the form $\mathbb{Z} \oplus L$ for some other unimodular lattice L. The only unimodular lattice in $\mathbb{R}^8$, with minimal norm 2, is the Gosset lattice. So any unimodular lattice in $\mathbb{R}^8$ without vectors of norm 1 has minimal norm at least 2 and must be the Gosset lattice. The unimodular lattices in dimension $d \leq 24$ with minimal norm at least 2 are classified in [**CS13**, chaps. 16-18]. There are 24 even and 156 odd unimodular lattices in $\mathbb{R}^{24}$ with minimal norm at least 2. The only odd unimodular lattice in $\mathbb{R}^{24}$ with minimal norm at least 3 is the *odd Leech lattice* O_{24}, which contains vectors with minimal norm 3. The only even unimodular lattice in $\mathbb{R}^{24}$ with minimal norm at least 3 is the Leech lattice Λ_{24}, which contains vectors with minimal norm 4. This classification confirms that any unimodular lattice (whether odd or even) in $\mathbb{R}^{24}$ with minimal norm at least 4 is the Leech lattice. □

In order to understand the main result of [**LM82**] we need to describe the concept of a totally isotropic subspace of quotient $L/2L$ for L an even unimodular lattice. The quotient $L/2L$ has the structure of a $\mathbb{F}_2$-vector space with the residue classes modulo $2L$ (i.e., the additive cosets $x + 2L$) acting as points. The corresponding inner product on $L/2L$ has value 0 or 1 in $\mathbb{F}_2$ according to whether any given representatives in L have even or odd Euclidean inner product. A *totally isotropic subspace* of $L/2L$ is the image, modulo $2L$, of a sublattice of L which contains $2L$ and for which all inner products are even. Since L is even unimodular, dim $L/2L$ is also even. A *maximal totally isotropic subspace* $M/2L$ of $L/2L$ is a totally isotropic subspace with dimension dim $M/2L = \frac{1}{2}$dim $L/2L$. We are interested in the sublattice preimage $M \subset L$ of maximal totally isotropic subspace $M/2L \subset L/2L$.

THEOREM 6.9. [**LM82**] *Let L be an even unimodular lattice with M a sublattice satisfying,*

$$2L \subset M \subset L.$$

The lattice $M/\sqrt{2}$ is an even unimodular lattice if and only if $M/2L$ is a maximal totally isotropic subspace of $L/2L$.

As described in [**LM82**], the requirement that $M/2L$ is totally isotropic ensures that $M/\sqrt{2}$ is an even lattice and the condition that $M/2L$ is *maximal* as a totally isotropic subspace ensures that $M/\sqrt{2}$ is also unimodular. We verify the maximal condition by simply checking that dim $M/2L = \frac{1}{2}$dim $L/2L$.

We will now adapt this theorem to the case where $L = \mathsf{O}^n$. To do so, we extend the octonion norm N to octonion vectors x in O^n such that $N(x) = xx^\dagger$. The double Euclidean inner product on octonion vectors is,

$$\langle x, y\rangle = N(x+y) - N(x) - N(y) = xy^\dagger + yx^\dagger.$$

For any pair of residue classes $x+2\mathsf{O}^n$, $y+2\mathsf{O}^n$ in the $\mathbb{F}_2$-vector space $\mathsf{O}^n/2\mathsf{O}^n$, we define the inner product as $(x+2\mathsf{O}^n, y+2\mathsf{O}^n) \mapsto \langle x, y\rangle \pmod 2$. The following theorem adapts Theorem 6.9 so that it identifies even unimodular sublattices of O^n with respect to $\frac{1}{2}\langle x, y\rangle$:

THEOREM 6.10. *Let N be a sublattice of O^n that satisfies,*

$$2\mathsf{O}^n \subset \mathsf{N} \subset \mathsf{O}^n.$$

The lattice N is an even unimodular lattice with respect to Euclidean inner product $\frac{1}{2}\langle x, y\rangle$ if and only if $\mathsf{N}/2\mathsf{O}^n$ is a maximal totally isotropic subspace of $\mathsf{O}^n/2\mathsf{O}^n$ with respect to inner product $\langle x, y\rangle \pmod 2$.

PROOF. With respect to inner product $\langle x, y\rangle$, the lattice O^n is an $\mathtt{E}_8^n$ lattice. It is therefore an even unimodular lattice relative to $\langle x, y\rangle$ and we can apply Theorem 6.9 to the sublattice N. According to that theorem, $\mathsf{N}/\sqrt{2}$ is an even unimodular lattice with respect to $\langle x, y\rangle$ if and only if $\mathsf{N}/2\mathsf{O}^n$ is a maximal totally isotropic subspace of $\mathsf{O}^n/2\mathsf{O}^n$. The condition that $\mathsf{N}/\sqrt{2}$ is an even unimodular lattice with respect to $\langle x, y\rangle$ is equivalent to the condition that N is an even unimodular lattice with respect to $\frac{1}{2}\langle x, y\rangle$. The condition that $\mathsf{N}/2\mathsf{O}^n$ is a totally isotropic subspace is equivalent to the condition that for any x, y in N, inner product $\langle x, y\rangle$ is an even integer. Therefore, this theorem is a special case of Theorem 6.9. □

EXAMPLE 6.11. Let $n = 1$. Then $\mathsf{O}/2\mathsf{O}$ is an 8-dimensional $\mathbb{F}_2$-vector space. This vector space contains a zero vector, 120 norm 1 vectors and 135 non-zero isotropic vectors (with $N(x) = 0$). A totally isotropic subspace is spanned by isotropic vectors with representatives satisfying $\langle x, y\rangle \pmod 2 = 0$ for any $x + 2\mathsf{O} \neq y + 2\mathsf{O}$ in the subspace. If we construct a graph on the 135 isotropic vectors, assigning an edge when $\langle x, y\rangle \pmod 2 = 0$, we obtain a strongly regular graph $\mathrm{srg}(135, 70, 37, 35)$. This graph has 270 maximal cliques, each corresponding to the non-zero vectors of a four-dimensional totally isotropic subspace of $\mathsf{O}/2\mathsf{O}$. Since these totally isotropic subspaces have dimension equal to half the dimension of $\mathsf{O}/2\mathsf{O}$, they are maximal totally isotropic subspaces. These subspaces also form a single orbit under the action of the graph automorphism group. Each clique defines a maximal totally isotropic subspace $\mathsf{N}/2\mathsf{O}$, and the corresponding pre-image $\mathsf{N} \subset \mathsf{O}$ is an even unimodular lattice with respect to Euclidean inner product $\frac{1}{2}\langle x, y\rangle$. Each clique has sixteen vectors, forming a finite $\mathbb{F}_2$-vector space of dimension 4. Since even unimodular lattice N has minimal norm 2, it must be the $\mathtt{E}_8$ lattice. Therefore, there are precisely 270 $\mathtt{E}_8$ sublattices of O that contain $2\mathsf{O}$.

REMARK 6.12. The fact that $\mathtt{E}_8$ contains at least 270 sublattices isometric to $\sqrt{2}\mathtt{E}_8$ is discussed in a mathematics blog post [**BE14a**]. The use of $\mathsf{O}/2\mathsf{O}$ structure and Theorem 6.10 makes it possible to determine this fact by examining the properties of a strongly regular graph on 135 points, the isotropic vectors of $\mathsf{O}/2\mathsf{O}$.

We now describe some further properties of the $\mathtt{E}_8$ sublattices of O that contain $2\mathsf{O}$. In the following lemmas, let s, s' denote norm 2 elements in O and recall that λ denotes any zero of $x^2 + x + 2$ in O, so that $\mathrm{Re}(\lambda) = -\frac{1}{2}$ and $N(\lambda) = 2$.

LEMMA 6.13. *Two $\mathtt{E}_8$ sublattices of O containing $2\mathsf{O}$ of the form $\mathsf{O}s$, $\mathsf{O}s'$ have equal images modulo $2\mathsf{O}$ if and only if $s \equiv s' \pmod{2\mathsf{O}}$. The same is true of sublattices $s\mathsf{O}$ and $s'\mathsf{O}$.*

PROOF. Left or right octonion multiplication is a conformal mapping, meaning that the $\langle sx, sy\rangle = N(s)\langle x, y\rangle = \langle xs, ys\rangle$ [**SV00**, p. 5]. Since $N(s) = 2$, this ensures that $\mathsf{O}s$ and $s\mathsf{O}$ are both $\mathtt{E}_8$ sublattices of O, relative to inner product $\frac{1}{2}\langle x, y\rangle$. Since mapping modulo $2\mathsf{O}$ is a ring homomorphism, $\mathsf{O}s$ and $\mathsf{O}s'$ have the same image if $s \equiv s' \pmod{2\mathsf{O}}$. A brief computation confirms that $\mathsf{O}s$ and $\mathsf{O}s'$ have the different images modulo $2\mathsf{O}$ if instead $s \not\equiv s' \pmod{2\mathsf{O}}$. The same argument applies for $s\mathsf{O}$ and $s'\mathsf{O}$. $\square$

REMARK 6.14. Lemma 6.13 ensures that each frame in O defines a pair of $\mathtt{E}_8$ sublattices of the form $\mathsf{O}s$ and $s\mathsf{O}$ for s any norm 2 frame representative. For $s \not\equiv s' \pmod 2\,\mathsf{O}$, the lattices $\mathsf{O}s$ and $\mathsf{O}s'$ are distinct since they have distinct images modulo $2\mathsf{O}$.

LEMMA 6.15. *Each* $\mathtt{E}_8$ *sublattice of* O *containing* $2\mathsf{O}$ *has the form of either* $\mathsf{O}s$ *or* $s\mathsf{O}$*, where* s *is any norm* 2 *representative of* $s + 2\mathsf{O}$*.*

PROOF. We can construct 135 lattices of the form $\mathsf{O}s$ and 135 more lattices of the form $s\mathsf{O}$, yielding a total of 270 $\mathtt{E}_8$ sublattices of O. By taking the images modulo $2\mathsf{O}$ we can verify that there are no repetitions, so that the preimages are $135 + 135$ distinct sublattices of O. By Example 6.11 there are only 270 $\mathtt{E}_8$ sublattices of O containing $2\mathsf{O}$ so we have found them all. □

LEMMA 6.16. *Let* $s \not\equiv s' \pmod{2\mathsf{O}}$*. Then* $\mathsf{O}s \cap s'\mathsf{O} \neq 2\mathsf{O}$*. Also,* $\mathsf{O}s \cap \mathsf{O}s' = 2\mathsf{O}$ *if and only if* $N(s + s')$ *is odd.*

PROOF. We can verify this lemma efficiently by computation using the ring $\mathsf{O}/2\mathsf{O}$. That is, $s \not\equiv s' \pmod{2\mathsf{O}}$ ensures that $\mathsf{O}s$ and $\mathsf{O}s'$ have distinct images modulo $2\mathsf{O}$ and therefore represent distinct lattices (likewise for left multiplication by s, s'). Both $\mathsf{O}s$ and $s\mathsf{O}$ contain $2\mathsf{O}$. The properties of the images modulo $2\mathsf{O}$ determine the properties of the intersections given above and can be quickly verified by computation. □

COROLLARY 6.16.1. *Since* $N(\lambda + \overline{\lambda}) = 1$ *we have* $\mathsf{O}\lambda \cap \mathsf{O}\overline{\lambda} = 2\mathsf{O}$*.*

The following can be verified by computation using a canonical copy of O and is also proven in [**CS03**].

LEMMA 6.17. [**CS03**, pp. 138-141] *The stabilizer in* $\mathrm{Aut}(\mathsf{O})$ *of residue class* $\lambda + 2\mathsf{O}$ *is a subgroup of type* $\mathrm{PSL}_2(7)$*, which also stabilizes* $\overline{\lambda} + 2\mathsf{O}$*.*

We can now describe how to construct a Leech sublattice of octonion integer triples in O^3, adapting the techniques described in [**LM82**] for octonion integers. We begin by selecting any two $\mathtt{E}_8$ sublattices of O with respect to $\frac{1}{2}\langle x, y\rangle$, call them Φ and Ψ, that satisfy the following requirements:

$$\Phi + \Psi = \mathsf{O}, \quad \Phi \cap \Psi = 2\mathsf{O}.$$

Such lattices exist, as established by Example 6.11 and Corollary 6.16.1. By Theorem 6.10, the images of Φ and Ψ modulo $2\mathsf{O}$ are both maximal totally isotropic subspaces of $\mathsf{O}/2\mathsf{O}$ that only intersect in zero, the $0 + 2\mathsf{O}$ residue class.

DEFINITION 6.18. Let Φ and Ψ be any two $\mathtt{E}_8$ sublattices of O, with respect to $\frac{1}{2}\langle x, y\rangle$, that satisfy $\Phi + \Psi = \mathsf{O}$ and $\Phi \cap \Psi = 2\mathsf{O}$. Let $\Lambda(\Phi, \Psi)$ be the following sublattice of O^3:

$$\Lambda(\Phi, \Psi) = \big\{(a, b, c) \subset \mathsf{O}^3 \mid a + b, b + c, a + c \in \Phi, a + b + c \in \Psi\big\}$$

Equivalently,

$$\Lambda(\Phi, \Psi) = \{(x_1 + z, x_2 + z, x_3 + z) \mid x_i \in \Phi, x_1 + x_2 + x_3, z \in \Psi\}.$$

and also,

$$\Lambda(\Phi, \Psi) = \{(x + y + z, x + z, y + z) \mid x, y \in \Phi, z \in \Psi\}.$$

Remark 6.19. To verify that the three definitions of $\Lambda(\Phi,\Psi)$ are equivalent, it suffices to verify that the second definition is a sublattice of the first, the third a sublattice of the second, and the first a sublattice of the third. To show that the second is a sublattice of the first, we use $a+b = x_1+x_2+2z$, which ensures that $a+b$ is in Φ because x_1 and x_2 are. We also know that $a+b+c = (x_1+x_2+x_3+2z)+z$ is in Ψ since the component in brackets is in $\Psi\cap\Phi = 2\mathsf{O}$ and z is in Ψ. To show that the third is a sublattice of the second we use $x_2 = x$, $x_3 = y$, and $x_1 = x+y$. These three elements are in Φ since x and y are. We know that $x_1+x_2+x_3 = 2(x+y)$ is in Ψ since it is in $2\mathsf{O} = \Phi\cap\Psi$. To show that the first is a sublattice of the third we use $y = a+b-2x-2z$, which is in Φ because $a+b$ is. We also use $x = b+c-y-2z$, which is in Φ because $b+c$ and y are. We also use $z = (a+b+c)-2(x+y+z)$ which is in Ψ because $a+b+c$ is.

Theorem 6.20. *The lattice $\Lambda = \Lambda(\Phi,\Psi)$ of Definition 6.18, with inner product $\frac{1}{2}\langle x,y\rangle$, is a Leech lattice.*

Proof. We need to apply Theorem 6.10 to show that Λ is an even unimodular lattice with respect to $\frac{1}{2}\langle x,y\rangle$. The following sketch parallels a proof given in [**LM82**], which does not involve octonions. Since Φ and Ψ are sublattices of O we know that $\Lambda\subset\mathsf{O}^3$. Since $2\mathsf{O}$ is in both Φ and Ψ, we know that Λ contains $(2a,0,0)$ for any a in O. The same is true for the other coordinate positions. Therefore we also have $2\mathsf{O}^3\subset\Lambda$. Relative to inner product $\langle x,y\rangle \pmod 2$ on pre-image vectors, the $\mathbb{F}_2$-vector space $\Lambda/2\mathsf{O}^3$ consists of three mutually orthogonal totally isotropic subspaces. Each of these subspaces has dimension 4. These are the three subspaces spanned respectively by vectors with pre-images of the form $(x,x,0)$, $(y,0,y)$, and (z,z,z) (for x,y in Φ and z in Ψ). Therefore totally isotropic subspace $\Lambda/2\mathsf{O}^3$ has dimension 12, which is maximal for a totally isotropic subspace in $\mathsf{O}^3/2\mathsf{O}^3$ since it is half of 24. Therefore it is a maximal totally isotropic subspace of $\mathsf{O}^3/2\mathsf{O}^3$. This ensures that Λ is an even unimodular lattice relative to the standard Euclidean inner product $\frac{1}{2}\langle x,y\rangle$. Finally, the proof in [**LM82**, Theorem 2.2] verifies that Λ lacks any norm 2 vectors. By Theorem 6.8, it is a Leech lattice. $\square$

We can describe a family of octonion Leech lattices with respect to certain norm 2 octonions as follows.

Definition 6.21. Let s,s' in O be norm 2 octonion integers with $\mathsf{O}s\cap\mathsf{O}s' = 2\mathsf{O}$. We define $\Lambda(s,s')$ to be the Leech lattice $\Lambda(\mathsf{O}s,\mathsf{O}s')$.

Remark 6.22. In what follows we restrict ourselves to describing Leech lattices of the form $\Lambda(\mathsf{O}s,\mathsf{O}s')$, rather than $\Lambda(s\mathsf{O},s'\mathsf{O})$, since the latter lattices can easily be obtained from the former by octonion conjugation.

Lemma 6.23. *The lattices $\Lambda(s,s')$ and $\Lambda(t,t')$ are equal if and only if $s\equiv t \pmod{2\mathsf{O}}$ and $s'\equiv t' \pmod{2\mathsf{O}}$.*

Proof. First, by Lemma 6.13, Lemma 6.15, since $\Lambda(s, s') = \Lambda(\mathsf{O}s, \mathsf{O}s')$, and since $\Lambda(t, t') = \Lambda(\mathsf{O}t, \mathsf{O}t')$, if $\mathsf{O}s = \mathsf{O}t$ and $\mathsf{O}s' = \mathsf{O}t'$ then $\Lambda(s, s') = \Lambda(t, t')$. Suppose instead that either $s \not\equiv t \pmod{2\mathsf{O}}$ or $s' \not\equiv t' \pmod{2\mathsf{O}}$. It follows by Lemma 6.13 that either $\mathsf{O}s \neq \mathsf{O}t$ or $\mathsf{O}s' \neq \mathsf{O}t'$. Indeed, either $\mathsf{O}s/2\mathsf{O} \neq \mathsf{O}t/2\mathsf{O}$ or $\mathsf{O}s'/2\mathsf{O} \neq \mathsf{O}t'/2\mathsf{O}$. It follows that $\Lambda(s, s')$ and $\Lambda(t, t')$ have distinct images modulo $2\mathsf{O}^3$, given the construction of that totally isotropic subspace, and therefore cannot be the same lattice. □

Remark 6.24. The octonion integer units $\alpha_1, \ldots, \alpha_8$ given in Remark 6.5 form a basis for O, since they correspond to simple roots. This means that the lattice $\Lambda(s, s')$ has the following basis,

$$(\alpha_i s, \alpha_i s, 0),\ (0, \alpha_i s, \alpha_i s),\ (\alpha_i s', \alpha_i s', \alpha_i s'), \quad i = 1, 2, \ldots, 8.$$

Definition 6.25. The following *translation* or *multiplication maps* acting on O are defined as in [**CS03**]:

$$\mathtt{L}_x : y \mapsto xy, \quad \mathtt{R}_x : y \mapsto yx, \quad \mathtt{B}_x : y \mapsto xyx.$$

We can extend these translations to a diagonal action on O^3 as follows.

Definition 6.26. For $\mathtt{X}$ one of $\mathtt{L}, \mathtt{R}, \mathtt{B}$ we define the following diagonal action on $(x, y, z) \subset \mathsf{O}^3$:

$$\mathtt{X}_u(x, y, z) = (\mathtt{X}_u(x), \mathtt{X}_u(y), \mathtt{X}_u(z)).$$

For any unit u in O, the translation maps $\mathtt{L}_u$, $\mathtt{R}_u$, and $\mathtt{B}_u$ are isometries of both O and O^3. This means that given Leech lattice $\Lambda(s, s')$ we can obtain another Leech lattice using a translation map $\mathtt{X}_u$. In fact, we can use octonion translations to describe a single orbit of 8640 octonion integer Leech lattices of the form $\Lambda(s, s')$. In order to describe this orbit, we can use certain helpful properties of translation maps due to properties of the octonion algebra as a Moufang loop.

Lemma 6.27. *For any x, y, u in O with u a unit, the translations given above satisfy the following:*

$$\mathtt{L}_u(xy) = \mathtt{B}_u(x)\mathtt{L}_{\overline{u}}(y), \quad \mathtt{R}_u(xy) = \mathtt{R}_{\overline{u}}(x)\mathtt{B}_u(y), \quad \mathtt{B}_u(xy) = \mathtt{L}_u(x)\mathtt{R}_u(y).$$

Proof. These are another form of the Moufang identities that the octonions satisfy, as described in [**CS03**, p. 74]:

$$u(xy) = (uxu)(\overline{u}y), \quad (xy)u = (x\overline{u})(uyu), \quad u(xy)u = (ux)(yu).$$

□

These properties of translations allow us to permute $\mathtt{E}_8$ sublattices of O containing $2\mathsf{O}$, which all have the form $\mathsf{O}s$ or $s\mathsf{O}$ for some norm 2 octonion integer s.

LEMMA 6.28. *Let* u *be a unit and* s *a norm* 2 *element in* O. *Then we have,*

$$u(\mathsf{O}s) = \mathsf{O}(\overline{u}s), \qquad (\mathsf{O}s)u = \mathsf{O}(usu), \qquad u(\mathsf{O}s)u = \mathsf{O}(su),$$
$$u(s\mathsf{O}) = (usu)\mathsf{O}, \qquad (s\mathsf{O})u = (s\overline{u})\mathsf{O}, \qquad u(s\mathsf{O})u = (us)\mathsf{O}.$$

which can also be written as,

$$\mathtt{L}_u(\mathsf{O}s) = \mathsf{O}\mathtt{L}_{\overline{u}}(s), \qquad \mathtt{R}_u(\mathsf{O}s) = \mathsf{O}\mathtt{B}_u(s), \qquad \mathtt{B}_u(\mathsf{O}s) = \mathsf{O}\mathtt{R}_u(s),$$
$$\mathtt{L}_u(s\mathsf{O}) = \mathtt{B}_u(s)\mathsf{O}, \qquad \mathtt{R}_u(s\mathsf{O}) = \mathtt{R}_{\overline{u}}(s)\mathsf{O}, \qquad \mathtt{B}_u(s\mathsf{O}) = \mathtt{L}_u(s)\mathsf{O}.$$

PROOF. These follow from the Moufang identities, the fact that $\overline{u}$ is a unit in O if and only if u is also a unit in O, and the fact that any translation of O by a unit simply permutes the elements of O. □

THEOREM 6.29. *Let* u *be a norm 1 element and let* s, s' *be norm 2 elements in* O. *Let* $s + s'$ *have odd norm (i.e.,* $\mathsf{O}s \cap \mathsf{O}s' = 2\mathsf{O}$*). Then the* $2 \cdot \mathrm{O}_8^+(2)$ *isometry group, generated by translations* $\mathtt{X}_u$ *for all units* u*, permutes the lattices of Definition 6.21 as follows:*

$$\mathtt{L}_u\Lambda(s, s') = \Lambda(\mathtt{L}_{\overline{u}}(s), \mathtt{L}_{\overline{u}}(s')),$$
$$\mathtt{R}_u\Lambda(s, s') = \Lambda(\mathtt{B}_u(s), \mathtt{B}_u(s')),$$
$$\mathtt{B}_u\Lambda(s, s') = \Lambda(\mathtt{R}_u(s), \mathtt{R}_u(s')).$$

PROOF. The identities follow from the definition $\Lambda(s, s') = \Lambda(\mathsf{O}s, \mathsf{O}s')$ and the Moufang identities expressed in terms of $\mathtt{L}$, $\mathtt{R}$, $\mathtt{B}$. The $2 \cdot \mathrm{O}_8^+(2)$ group of translations $\mathtt{X}_u$ can be constructed and identified in GAP using a canonical copy of O. □

THEOREM 6.30. *Let* s, s' *in* O *be norm* 2 *octonion integers with* $s + s'$ *having an odd norm (i.e.,* $\mathsf{O}s \cap \mathsf{O}s' = 2\mathsf{O}$*). Then there exists a norm* 2 *root of* $x^2 + x + 2$ *in* O*, called* λ*, and unit* u *in* O *such that,*

$$\Lambda(s, s') = \mathtt{L}_u\Lambda(\overline{\lambda}, \lambda).$$

There are $8640 = 72 \cdot 120$ *choices of* λ *and* u*, taken modulo* $2\mathsf{O}$*, corresponding to the* 8640 *lattices of the form* $\Lambda(s, s')$*. These form a single orbit under the action of the* $2 \cdot \mathrm{O}_8^+(2)$ *generated by* $\mathtt{X}_u$ *for* u *any octonion unit and* $\mathtt{X} = \mathtt{L}, \mathtt{R}, \mathtt{B}$.

PROOF. First, we construct a graph on the 135 norm 2 octonion integers modulo $2\mathsf{O}$ where two vertices are adjacent when the sum of their representatives has odd norm. This graph is a $\mathrm{srg}(135, 64, 28, 32)$ and each *directed edge* $s \to s'$ represents a Leech lattice of the form $\Lambda(s, s')$ (in general $\Lambda(s, s') \neq \Lambda(s', s)$). Second, we can construct an edge-transitive automorphism group of this graph generated by the translation maps $\mathtt{X}_u$ acting on vertices. Indeed, this automorphism group is transitive on directed edges. Third, we can recover all 8640 directed edges by computing $(\mathtt{L}_{\overline{u}}(\overline{\lambda}), \mathtt{L}_{\overline{u}}(\lambda)) \pmod{2\mathsf{O}}$

Type	Number	Comment
$(2u, 0, 0)$	3×240	for u a unit in O
$(s, \pm s, 0)$	6×240	for s a root in $\mathsf{O}\overline{\lambda}$
$(\pm 1, \pm 1, \lambda')$	12×16	for $\lambda' \equiv \lambda \pmod{2\mathsf{O}}$

TABLE 6.1. Commutative vectors r in $\Lambda(\overline{\lambda}, \lambda)$ defining reflection symmetries $\mathtt{W}_r$ of $\Lambda(\overline{\lambda}, \lambda)$.

for the 72 choices of λ and the 120 choices of u modulo $2\mathsf{O}$. These facts can be confirmed by computation using GAP. $\square$

REMARK 6.31. The 2×8640 Leech lattices of the form $\Lambda(\mathsf{O}s, \mathsf{O}s')$ or $\Lambda(s\mathsf{O}, s'\mathsf{O})$ are identified and described in the mathematics blog posts [**Bae14b**] and [**BE14a**], by John Baez and Greg Egan. They use a combination of direct computation and analysis of $\mathtt{E}_8$ lattice properties to identify these lattices. The approach shown here instead makes use of the ring homomorphism $\mathsf{O} \mapsto \mathsf{O}/2\mathsf{O}$ and Theorem 6.9 of [**LM82**] to simplify the calculations to properties of a strongly regular graph on 135 points, which can easily be explored by computation or other graph theory techniques.

6.4. Leech Lattice Symmetries

By construction the lattice $\Lambda(\overline{\lambda}, \lambda)$ is symmetric under all coordinate permutations and coordinate sign changes, which is a $2 \times S_4$ group action. We will call these the *coordinate automorphisms* of $\Lambda(\overline{\lambda}, \lambda)$. The lattice $\Lambda(\overline{\lambda}, \lambda)$ is also fixed under the octonion automorphisms that preserve our canonical choice of arithmetic O and residue class $\lambda + 2\mathsf{O}$. We call automorphisms belonging to this $\mathrm{PSL}_2(7) \subset \mathrm{Aut}(\mathsf{O})$ the *scalar automorphisms* of $\Lambda(\overline{\lambda}, \lambda)$.

We now introduce *reflection automorphisms* of $\Lambda(\overline{\lambda}, \lambda)$. A computer search using GAP of a canonical copy of $\Lambda(\overline{\lambda}, \lambda)$ shows that the short vectors of this Leech lattice contain 2×1260 commutative octonion integer triples, namely triples where the three coefficients generate a commutative subalgebra of $\mathbb{O}$. Of these triples, 2×1176 define reflections $\mathtt{W}_r$ acting on O^3 that preserve the Leech lattice $\Lambda(\overline{\lambda}, \lambda)$. These include 720 of the form $(2u, 0, 0)$ for u a unit in O and 1440 of the form $(s, \pm s, 0)$ for s any $\mathtt{E}_8$ root in $\mathsf{O}\overline{\lambda}$. For r a commutative vector in $\Lambda(\overline{\lambda}, \lambda)$ of the form $(2u, 0, 0)$ or $(s, \pm s, 0)$, the corresponding reflection $\mathtt{W}_r$ is a coordinate symmetry in $2 \times S_4$. The remaining 2×96 commutative short vectors in $\Lambda(\overline{\lambda}, \lambda)$ that define reflection symmetries have the form $(1, 1, \lambda')$ for $\lambda' \equiv \lambda \pmod{2\mathsf{O}}$, under all coordinate permutations and sign changes.

THEOREM 6.32. *Let λ be an octonion integer in O and zero of $x^2 + x + 2$. The vector $r = (1, 1, \lambda)$, for any $\lambda \equiv \lambda' \pmod{2\mathsf{O}}$, defines an automorphism $\mathtt{W}_r$ of the Leech lattice $\Lambda(\overline{\lambda'}, \lambda')$.*

Proof. Since r is a commutative vector, Lemma 5.5 ensures that $\mathtt{W}_r$ acting on $\mathbb{O}^3$ is an isometry with respect to the Euclidean inner product. It remains to confirm that $\mathtt{W}_r$ maps $\Lambda(\overline{\lambda'}, \lambda')$ to $\Lambda(\overline{\lambda'}, \lambda')$. To begin, we write $\Lambda(\overline{\lambda'}, \lambda') = \Lambda(\overline{\lambda}, \lambda)$ since $\lambda' \equiv \lambda \pmod{2\mathsf{O}}$. Note that $\lambda + \overline{\lambda} = -1$ and that $1 - \overline{\lambda} = -\lambda^2$. A vector $(a, b, c) \in \Lambda(\overline{\lambda}, \lambda)$ is reflected as follows:

$$\mathtt{W}_r(a, b, c) = (a, b, c) - 2(a, b, c)[r].$$

Since (a, b, c) is in the Leech lattice, we will have $\mathtt{W}_r(a, b, c)$ also in the Leech lattice when $2(a, b, c)[r]$ is in the Leech lattice. Let $(a', b', c') = 2(a, b, c)[r]$. We can compute (a', b', c') by first computing $[r]$.

$$[r] = \frac{r^\dagger r}{r r^\dagger} = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ \overline{\lambda} \end{pmatrix} (1, 1, \lambda) = \frac{1}{4}\begin{pmatrix} 1 & 1 & \lambda \\ 1 & 1 & \lambda \\ \overline{\lambda} & \overline{\lambda} & 2 \end{pmatrix}.$$

This means that we have,

$$\begin{aligned}(a', b', c') = 2(a, b, c)[r] &= \frac{1}{2}(a, b, c)\begin{pmatrix} 1 & 1 & \lambda \\ 1 & 1 & \lambda \\ \overline{\lambda} & \overline{\lambda} & 2 \end{pmatrix} \\ &= \frac{1}{2}\left(a + b + c\overline{\lambda}, a + b + c\overline{\lambda}, a\lambda + b\lambda + 2c\right).\end{aligned}$$

We first verify that $a' + b' + c' \in \mathsf{O}\lambda$.

$$a' + b' + c' = (a + b)\left(1 + \frac{\lambda}{2}\right) + c(\overline{\lambda} + 1).$$

By construction $a + b$ is in $\mathsf{O}\overline{\lambda}$ so we have $a + b = \alpha\overline{\lambda}$ for some α in O. Furthermore, $\alpha, \overline{\lambda}, 1 + \frac{\lambda}{2}$ belong to a common associative subalgebra of $\mathbb{O}$. This means that we use $\overline{\lambda}\lambda = 2$ and simplify as follows:

$$a' + b' + c' = (\alpha + c)(\overline{\lambda} + 1).$$

But $\overline{\lambda} + 1 = -\lambda$ so $a' + b' + c'$ is in $\mathsf{O}\lambda$. Next we show that $a' + b' \in \mathsf{O}\overline{\lambda}$.

$$a' + b' = a + b + c\overline{\lambda}.$$

By construction $a + b = \alpha\overline{\lambda}$ for some α in O. This ensures that $a' + b'$ is in $\mathsf{O}\overline{\lambda}$. Finally we check that $b' + c' = a' + c' \in \mathsf{O}\overline{\lambda}$.

$$a' + c' = \frac{1}{2}(a + b)\left(1 + \lambda\right) + \frac{1}{2}c\left(2 + \overline{\lambda}\right) = -\frac{1}{2}(a + b)\overline{\lambda} - \frac{1}{2}c\overline{\lambda}^2$$

We know that $(a + b) = \alpha\overline{\lambda}$:

$$a' + c' = -\frac{1}{2}(\alpha + c)\overline{\lambda}^2.$$

We also know that $a+b+c=\beta\lambda$ and can write $c=\beta\lambda-\alpha\overline{\lambda}$:

$$\begin{aligned} a'+c' &= -\frac{1}{2}(\alpha-\alpha\overline{\lambda})\overline{\lambda}^2 - \frac{1}{2}(\beta\lambda)\overline{\lambda}^2 \\ &= -\frac{1}{2}\alpha(1-\overline{\lambda})\overline{\lambda}^2 - \beta\overline{\lambda}. \end{aligned}$$

We use the properties $1-\overline{\lambda}=-\lambda^2$ and $\lambda^2\overline{\lambda}^2=4$:

$$\begin{aligned} a'+c' &= \frac{1}{2}\alpha\lambda^2\overline{\lambda}^2 - \beta\overline{\lambda} \\ &= 2\alpha - \beta\overline{\lambda}. \end{aligned}$$

Since 2α is in $2\mathsf{O} \subset \mathsf{O}\overline{\lambda}$, this ensures that $a'+c'=b'+c'$ are in $\mathsf{O}\overline{\lambda}$. □

Theorem 6.32 ensures that $\mathtt{W}_r$ for $r=(1,1,\lambda')$ and $\lambda' \equiv \lambda \pmod{2\mathsf{O}}$ is a reflection symmetry of Leech lattice $\Lambda(\overline{\lambda},\lambda)$. It turns out that that each of the 2×96 short vectors in $\Lambda(\overline{\lambda},\lambda)$ of the form $(\pm 1,\pm 1,\lambda')$ for $\lambda' \equiv \lambda \pmod{2\mathsf{O}}$ defines a reflection symmetry of $\Lambda(\overline{\lambda},\lambda)$. Computation in GAP on a canonical example verifies the following two theorems about how to generate the full automorphism group of the Leech lattice $\Lambda(\overline{\lambda},\lambda)$.

THEOREM 6.33. *The automorphism group* $2\cdot\mathrm{Co}_1$ *of Leech lattice* $\Lambda(\overline{\lambda},\lambda)$ *is generated by reflections* $\mathtt{W}_r$ *for* $r=(\pm 1,\pm 1,\lambda')$ *under all coordinate permutations and with* $\lambda' \equiv \lambda \pmod{2\mathsf{O}}$.

THEOREM 6.34. *The automorphism group* $2\cdot\mathrm{Co}_1$ *of Leech lattice* $\Lambda(\overline{\lambda},\lambda)$ *is generated by the* $2\times S_4$ *coordinate permutations and sign changes together with reflections* $\mathtt{W}_r$ *for* $r=(1,1,\lambda')$ *for* $\lambda' \equiv \lambda \pmod{2\mathsf{O}}$ *and* $\mathrm{Re}(\lambda)=\mathrm{Re}(\lambda')$.

REMARK 6.35. The condition that $\mathrm{Re}(\lambda)=\mathrm{Re}(\lambda')$ is simply introduced to find a smaller generating set of the group $2\cdot\mathrm{Co}_1$. From these two theorems, checked by computation, it follows that all automorphisms of $\Lambda(\overline{\lambda},\lambda)$ are compositions of octonion reflection automorphisms. Although the Leech lattice does not have any real reflection symmetries, the octonion Leech lattice $\Lambda(\overline{\lambda},\lambda)$ has octonion reflection symmetries that generate the full automorphism group.

REMARK 6.36. We can also describe automorphisms of any octonion Leech lattice of the form $\Lambda(s,s')$. Since by Theorem 6.30 we know that $\Lambda(s,s')=\mathtt{L}_u\Lambda(\overline{\lambda},\lambda)$ we also have $\mathtt{L}_{\overline{u}}\Lambda(s,s')=\Lambda(\overline{\lambda},\lambda)$. Let $r=(1,1,\lambda')$, under any coordinate permutation and sign change, and for any $\lambda' \equiv \lambda \pmod{2\mathsf{O}}$. Then $\mathtt{W}_r$ acting on $\mathbb{O}^3$ is an automorphism of $\Lambda(\overline{\lambda},\lambda)$ and $\mathtt{L}_u\mathtt{W}_r\mathtt{L}_{\overline{u}}$ is an automorphism of $\mathtt{L}_u\Lambda(\overline{\lambda},\lambda)$. We can generate the automorphism group $2\cdot\mathrm{Co}_1$ of Leech lattice $\mathtt{L}_u\Lambda(\overline{\lambda},\lambda)$ using involutions of the form $\mathtt{L}_u\mathtt{W}_r\mathtt{L}_{\overline{u}}$. In the special case where $u=1$, the involution $\mathtt{L}_u\mathtt{W}_r\mathtt{L}_{\overline{u}}$ becomes an octonion reflection.

6.5. The Common Construction and Octonion Integer Triples

We now use the common construction of [**Nas22**] to describe the correspondence between certain structures in the octonion integer Leech lattice and generalized hexagons in the octonion projective plane. We will also describe an approach to generating the Suzuki chain of Leech lattice automorphism subgroups using octonion reflections.

First introduced by Jacques Tits, a *finite generalized n-gon* of order (s,t) is a block design on v points such that (1) each block contains $s+1$ points, (2) each point belongs to $t+1$ blocks, and (3) the point-block incidence graph has diameter n and girth $2n$ (described also as a bipartite graph with diameter n and girth $2n$ in [**GR01**, 5.6]). We will write $\mathrm{Gh}(s,t)$ to denote the *generalized hexagon* $(n=6)$ of order (s,t). An ordinary hexagon, taken as a block design with edges for blocks and vertices for points, is a $\mathrm{Gh}(1,1)$. Cohen demonstrates in [**Coh83**] that the generalized hexagons $\mathrm{Gh}(2,1)$, $\mathrm{Gh}(2,2)$, and $\mathrm{Gh}(2,8)$ can be constructed as sets of Jordan frames (i.e., three orthogonal primitive idempotents) in the octonion projective plane. We will show that the common construction of [**Nas22**] puts these generalized hexagon structures in correspondence with certain integral octonion triples with reflections that generate Suzuki chain subgroups of Leech lattice automorphisms.

Second, the automorphism group of the Leech lattice is the Conway group $\mathrm{Co}_0 = 2 \cdot \mathrm{Co}_1$. The tight 5-design in $\mathbb{RP}^{23}$ corresponding to the 98280 lines spanned by the Leech lattice short vectors has the sporadic simple group Co_1 for an isometry group. The Suzuki chain is a series of subgroups of Co_1 constructed in the following manner [**Wil09a**, p. 219]. The group Co_1 has a maximal subgroup $A_9 \times S_3$. The symmetric group S_3 centralizes A_9 in Co_1. The chain of alternating subgroups $A_9 > A_8 > A_7 > \cdots > A_4 > A_3$ has the following corresponding chain of centralizers in Co_1, known as the *Suzuki chain*:

$$S_3 < S_4 < \mathrm{PSL}_2(7) < \mathrm{PSU}_3(3) < \mathrm{HJ} < G_2(4) < 3 \cdot \mathrm{Suz}.$$

Here HJ is the Hall-Janko sporadic simple group and Suz is the Suzuki sporadic simple group. In what follows we will see how the group Co_1 and the Suzuki chain subgroups can be constructed as octonion reflection groups, acting projectively on $\mathbb{R}^{24}/\{\pm 1\} \cong \mathbb{O}^3/\{\pm 1\}$, generated by reflections on $\mathbb{O}^3$ of the form $\mathtt{W}_r$ for suitable choices of octonion triple r.

Remark 6.37. Consider the following related sequence of groups:

$$\mathrm{PSU}_3(3) < \mathrm{HJ} < G_2(4) < \mathrm{Suz}.$$

The groups in this sequence have permutation representations respectively of degrees 63, 100, 416, and 1782. The Suz group action on 1782 points has rank 3, and the point stabilizer is the group $G_2(4)$ with orbits of lengths $1+416+1365$. The action of $G_2(4)$ on the 416 points also has rank 3, and the point stabilizer is the group HJ with orbits of lengths $1+100+315$. The action of HJ on the 100 points also has rank 3, and the point stabilizer is the group $\mathrm{PSU}_3(3)$ with orbits of lengths $1 + 36 + 63$. The action of $\mathrm{PSU}_3(3)$ on both the 36 or 63 length orbits is rank 4, so the sequence of rank 3 permutation groups stops (contrary to [**Gri98**, p. 124]). As mentioned in [**Gri98**, p. 124], there is no rank 3 permutation group containing Suz as a point stabilizer.

Having described generalized hexagons and the Suzuki chain of Leech lattice symmetries, we now define certain octonion integer triples for use in the common construction in order to link the two concepts. Consider the following vectors with λ some zero of $x^2 + x + 2$ in O:

$$S_\lambda = \left\{(2,0,0), (\overline{\lambda}, 0, \overline{\lambda}), (1,1,\lambda)\right\}.$$

Since the coefficients belong to a common complex subalgebra $\mathbb{C}$, we have S_λ in $\mathbb{C}^3 \subset \mathbb{O}^3$ and the projectors $\{[x] \mid x \in S_\lambda\}$ in the complex projective subplane $\mathbb{CP}^2 \subset \mathbb{OP}^2$.

Example 6.38. The common construction of [**Nas22**] applied to initial vectors $\{r_1, r_2, r_3\} = S_\lambda$ yields $G = 2 \times \mathrm{PSL}_2(7)$ acting on $\mathbb{C}^3$, with a 42 point design on Ω_6. The common construction also yields $H = \mathrm{PSL}_2(7)$ acting on $\mathbb{CP}^2$ and the orbit of the projectors of S_λ define a 21 point design. The 21 points of the design in $\mathbb{CP}^2$, and the blocks of three mutually orthogonal points, form a $\mathrm{Gh}(2,1)$ structure. Here we have a link between the Suzuki chain group $H = \mathrm{PSL}_2(7) \cong G/\{\pm 1\}$ and a generalized hexagon $\mathrm{Gh}(2,1)$ of Jordan frames on $\mathbb{CP}^2$.

Remark 6.39. The 2-design in $\mathbb{CP}^2$ of Example 6.38 is equivalent to Example 12 of [**Hog82**]. However, this design is mislabeled in [**Hog82**] as a 3-design at the special bound. A calculation confirms that it is neither a 3-design nor at the special bound.

Remark 6.40. The group $G = 2 \times \mathrm{PSL}_2(7)$ acting on $\mathbb{C}^3$ of Example 6.38 contains a real reflection subgroup $2 \times S_4$. Indeed, the 42 vectors generated by S_λ under reflection $\mathtt{W}_r$ are precisely the orbits of the vectors in S_λ under all coordinate permutations and sign changes, which is the action of $2 \times S_4$. The projectors generating these reflections are the nine contained in the real projective subplane $\mathbb{RP}^2 \subset \mathbb{CP}^2$, which form a 1-design. Their reflection action on $\mathbb{O}^3$ generate the $2 \times S_4$ coordinate symmetries of $\Lambda(\overline{\lambda}, \lambda)$ discussed above.

We call the stabilizer in $\mathrm{Aut}(\mathsf{O})$ of the norm 2 representatives of residue class $\lambda + 2\mathsf{O}$ a *frame stabilizer*. This frame stabilizer is a group of type $\mathrm{PSL}_2(7)$. Under the action of this $\mathrm{PSL}_2(7)$ frame stabilizer, the vector λ has an orbit of length 8. The corresponding scalar action on the set of triples S_λ also has length 8, as does the scalar action on the $\mathrm{Gh}(2,1)$ structures generated by reflections $\mathtt{W}_r$ for r in S_λ vectors. Each $S_{\lambda'}$ in this orbit is defined by $\lambda' \equiv \lambda \pmod{2\mathsf{O}}$ and $\mathrm{Re}(\lambda') = \mathrm{Re}(\lambda)$.

The permutation action of $\mathrm{PSL}_2(7)$ on the eight $\lambda' \equiv \lambda \pmod{2\mathrm{O}}$ with $\mathrm{Re}(\lambda) = \mathrm{Re}(\lambda')$ is 2-transitive. However, this action is transitive on *unordered* subsets $\{\lambda_i, \lambda_j, \lambda_k, \ldots\}$ of size n, except when $n = 4$. That is, the frame stabilizer is also transitive on pairs $\{\lambda_i, \lambda_j\}$ and triples $\{\lambda_i, \lambda_j, \lambda_k\}$. It is therefore also transitive on subsets of cardinality $5, 6, 7, 8$. In contrast, there are three orbits of quadruples $\{\lambda_i, \lambda_j, \lambda_k, \lambda_l\}$, with lengths $14 + 42 + 14$. Both length 14 orbits define Steiner systems $S(3, 4, 8)$ on the eight λ' with $\lambda' \equiv \lambda \pmod{2\mathrm{O}}$ and $\mathrm{Re}(\lambda') = \mathrm{Re}(\lambda)$.

We can now describe the Suzuki chain groups and their correspondence to certain generalized hexagons via the common construction of Definition 6.1. In each case, the result is obtained by computation in GAP on a representative example.

EXAMPLE 6.41. If we apply the common construction to any pair $S_{\lambda_i} \cup S_{\lambda_j}$ we obtain $G/\{\pm 1\} = \mathrm{PSU}_3(3)$ and $H = \mathrm{PSU}_3(3)$. The corresponding design on $\mathbb{HP}^2 \subset \mathbb{OP}^2$ defines a $\mathrm{Gh}(2, 2)$ finite geometry.

EXAMPLE 6.42. If we instead apply the common construction to any triple $S_{\lambda_i} \cup S_{\lambda_j} \cup S_{\lambda_k}$ we obtain $G/\{\pm 1\} = \mathrm{HJ}$ and H is ${}^3D_4(2)$. The corresponding design on $\mathbb{OP}^2$ defines a $\mathrm{Gh}(2, 8)$ finite geometry.

EXAMPLE 6.43. The two Steiner systems on quadruples $\{\lambda_i, \lambda_j, \lambda_k, \lambda_l\}$ behave differently. One, but not the other, has the special property that $\{r_1, \ldots, r_n\} = S_{\lambda_i} \cup S_{\lambda_j} \cup S_{\lambda_k} \cup S_{\lambda_l}$ yields the two strictly projective tight 5-designs under the common construction of [**Nas22**]. That is, for one of the two length 14 orbits of quadruples, the common construction applied to the vectors of $S_{\lambda_i} \cup S_{\lambda_j} \cup S_{\lambda_k} \cup S_{\lambda_l}$ yields $G/\{\pm 1\} = G_2(4)$ and H is ${}^3D_4(2)$. The corresponding spherical design on Ω_{24} is the same as that given by the short vectors of Leech lattice $\Lambda(\overline{\lambda}, \lambda)$, with the corresponding tight projective 5-design in $\mathbb{RP}^{23}$. The corresponding design on $\mathbb{OP}^2$ defines a $\mathrm{Gh}(2, 8)$ finite geometry. This is the same finite geometry and ${}^3D_4(2)$ group obtained by the common construction using any three $S_{\lambda_i}, S_{\lambda_j}, S_{\lambda_k}$ contained in the quadruple.

EXAMPLE 6.44. The quadruples $\{\lambda_i, \lambda_j, \lambda_k, \lambda_l\}$ of the other length 14 orbit and of the length 42 orbit no longer yield the $\mathrm{Gh}(2, 8)$ finite geometry on $\mathbb{OP}^2$ under the common construction, although they still yield $G/\{\pm 1\} = G_2(4)$ and the Leech lattice short vectors on $\mathbb{O}^3$. It is unclear whether the projectors of the generating set have a finite orbit under the action of corresponding group H, or whether the group H is finite.

EXAMPLE 6.45. Any quintuple $S_{\lambda_i} \cup \ldots \cup S_{\lambda_j}$ yields $G/\{\pm 1\} = 3 \cdot \mathrm{Suz}$ and the same spherical design on Ω_{24} as its $G_2(4)$ subgroups do.

$\{r_1, \ldots, r_n\}$	$G/\{\pm 1\} \subset \mathrm{Co}_1$	$H \subset F_4$	Example
S_{λ_i}	$\mathrm{PSL}_2(7)$	$\mathrm{PSL}_2(7)$	6.38
$S_{\lambda_i} \cup S_{\lambda_j}$	$\mathrm{PSU}_3(3)$	$\mathrm{PSU}_3(3)$	6.41
$S_{\lambda_i} \cup S_{\lambda_j} \cup S_{\lambda_k}$	HJ	${}^3D_4(2)$	6.42
$S_{\lambda_i} \cup S_{\lambda_j} \cup S_{\lambda_k} \cup S_{\lambda_l}$	$G_2(4)$	${}^3D_4(2)$	6.43
$S_{\lambda_i} \cup S_{\lambda_j} \cup S_{\lambda_k} \cup S_{\lambda_l}$	$G_2(4)$		6.44
$S_{\lambda_i} \cup S_{\lambda_j} \cup S_{\lambda_k} \cup S_{\lambda_l} \cup S_{\lambda_m}$	$3 \cdot \mathrm{Suz}$		6.45
$S_{\lambda_i} \cup S_{\lambda_j} \cup S_{\lambda_k} \cup S_{\lambda_l} \cup S_{\lambda_m} \cup S_{\lambda_n}$	Co_1		6.46

TABLE 6.2. The common construction of [**Nas22**] applied to orbits of combinations of S_λ under $\mathrm{PSL}_2(7) \subset \mathrm{Aut}(\mathsf{O})$.

EXAMPLE 6.46. Any sextuple $S_{\lambda_i} \cup \ldots \cup S_{\lambda_j}$ yields $G/\{\pm 1\} = \mathrm{Co}_1$ and the same spherical design on Ω_{24} as its $3 \cdot \mathrm{Suz}$ subgroups do. The same is true of a septuple or the union of the full set of eight S_λ.

The results described above are summarized in Table 6.2. The Suzuki chain subgroups satisfy the following theorem.

THEOREM 6.47. *The Suzuki chain subgroups of* Co_1*,*

$$\mathrm{PSL}_2(7) < \mathrm{PSU}_3(3) < \mathrm{HJ} < G_2(4) < 3 \cdot \mathrm{Suz},$$

are the quotients modulo $\{\pm 1\}$ *of octonion reflection groups generated by reflections* $\mathtt{W}_r$ *acting on* $\mathbb{O}^3$ *with* r *in the union of any subset of* $\{S_{\lambda_i}, \cdots, S_{\lambda_j}\}$ *of cardinality respectively* $1, 2, 3, 4, 5$.

PROOF. The proof involves checking representative examples in GAP. $\square$

The following remarks describe two remaining open questions.

REMARK 6.48. Using the notation given above, the element $\mathtt{W}_r$ for $r = (1, 1, \lambda')$ is an involution in $2 \cdot \mathrm{Co}_1$. What is the conjugacy class of this involution? Likely the full conjugacy class contains involutions that are not octonion reflections. Of note, an octonion reflection may have determinant -1 as an octonion matrix even though the 24×24 real matrix for the endomorphism acting on O^3 as a real 24-vector has determinant 1.

REMARK 6.49. An open question is whether there is some algebraic property of $\{\lambda_i, \lambda_j, \lambda_k, \lambda_l\}$ in the $S(3, 4, 8)$ of Example 6.43 that yields the two tight projective 5-designs that distinguishes it from the other length 14 orbit (which is also an $S(3, 4, 8)$) in Example 6.44.

6.6. Albert Isotope Integer Rings and Tight 5-Designs

In this section we describe the work of [**EG96**] and [**EG01**] in terms of Jordan algebra isotopes and connect those papers to our approach in this chapter. Let $V = \mathrm{Herm}(3, \mathbb{O})$ be the Albert algebra with identity e. The product and quadratic map are given by,

$$x \circ y = \frac{1}{2}(xy + yx) = L(x)y, \quad P(x) = 2L(x)^2 - L(x^2).$$

Let $\mathsf{A} = \mathrm{Herm}(3, \mathsf{O}) \subset V$ be the subset of the Albert algebra restricted to octonion integer matrix entries. Each element x in V satisfies the cubic equation with real-valued $\mathrm{tr}(x)$, $\sigma(x)$, $\det(x)$ [**FK94**, chap. 2]:

$$x^3 - \mathrm{tr}(x)x^2 + \sigma(x)x - \det(x)e = 0.$$

For x in A, we have $\mathrm{tr}(x)$, $\sigma(x)$, $\det(x)$ in $\mathbb{Z}$. The positive-definite elements of V (in terms of eigenvalues, not discussed here) form the *symmetric cone* Ω. For any element in Ω, the cubic equation coefficients $\mathrm{tr}(x)$, $\sigma(x)$, $\det(x)$ are greater than zero. All elements in Ω are invertible and we compute the inverse as $x^{-1} = P(x)^{-1}x$. Also, every element q in Ω has the form $q = a^2 = P(a)e$ for some a with $\det(a) \neq 0$. Not all squares are invertible and not all invertible elements are squares. But the symmetric cone Ω is precisely the invertible squares of V [**FK94**, chap. 3].

As described in [**McC04**, p. 86], given any $q = P(a)e$ in Ω we can construct a *q-isotope algebra* $V^{(q)}$ with identity q^{-1} using product and quadratic map,

$$x \circ_q y = x \circ (q \circ y) + (x \circ q) \circ y - q \circ (x \circ y), \quad P^{(q)}(x) = P(x)P(q).$$

The algebra V is isomorphic to isotope $V^{(q)}$ with $P(a)^{-1} = P(a^{-1})$ the isomorphism map. Isotopy is an equivalence relation. It is reflexive, since $V^{(q)}$ is its own q^{-1}-isotope. It is symmetric, since $V^{(q)}$ is the q-isotope of V and V is the $P(q)^{-1}e$-isotope of $V^{(q)}$. It is transitive, since $(V^{(q)})^{(r)} = V^{(P(q)r)}$.

We are interested in using isotopy to describe the results of [**EG96**] (as also done recently in [**GPR22**]). For q in $\Omega \cap \mathsf{A}$ with $\det(q) = 1$, we define the *q-isotope Albert integer ring* $\mathsf{A}^{(q)}$ as the set $\mathsf{A} = \mathrm{Herm}(3, \mathsf{O})$ with the product $2x \circ_q y$ and quadratic map $P^{(q)}(x)$.

Let $G(\Omega \cap \mathsf{A})$ be the group of general linear transformations on V that preserve Ω and also preserve A. This group is studied in [**Gro96**, sec. 4] and [**GPR22**, sec. 18]. Group $G(\Omega \cap \mathsf{A})$ preserves the determinant $\det(x)$ and the corresponding symmetric trilinear form,

$$\begin{aligned} 6\langle x, y, z\rangle &= \det(x + y + z) - \det(x + y) - \det(y + z) - \det(x + z) \\ &\quad + \det(x) + \det(y) + \det(z). \end{aligned}$$

The crucial observation of [**EG96**] and [**Gro96**] is that there are precisely two orbits of determinant 1 elements in Ω under the action of $G(\Omega \cap \mathsf{A})$, with the following representatives, for λ any root of $x^2 + x + 2$ in O:

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 & \lambda & \overline{\lambda} \\ \overline{\lambda} & 2 & \lambda \\ \lambda & \overline{\lambda} & 2 \end{pmatrix}.$$

The group $G(\Omega \cap \mathsf{A})$ is infinite, but the stabilizers of e and b^{-1} are respectively the finite groups $2^2 \cdot \mathrm{O}_8^+(2) : S_3$ and ${}^3D_4(2) : 3$ [**Gro96**]. These are respectively the isotope ring automorphism groups $\mathrm{Aut}(\mathsf{A}^{(e)})$ and $\mathrm{Aut}(\mathsf{A}^{(b)})$.

The Jordan Euclidean inner product in the q-isotope is defined as,

$$\langle x, y \rangle^{(q)} = 9\langle x, q^{-1}, q^{-1} \rangle \langle y, q^{-1}, q^{-1} \rangle - 6\langle x, y, q^{-1} \rangle.$$

Using q as either $e = (1, 1, 1 \mid 0, 0, 0)$ and $b = (2, 2, 2 \mid \lambda, \lambda, \lambda)$ for the two orbits of $G(\Omega \cap \mathsf{A})$ described above, we will characterise the geometry of the isotope rings $\mathsf{A} = \mathsf{A}^{(e)}$ and $\mathsf{A}^{(b)}$. Although $\mathsf{A}^{(e)}$ and $\mathsf{A}^{(b)}$ are the same set of elements in $\mathrm{Herm}(3, \mathbb{O})$, they have different ring products, quadratic maps, traces, and inner products. As described in [**EG96**] and [**EG01**], with respect to their respective inner products $\langle x, y \rangle^{(e)}$ and $\langle x, y \rangle^{(b)}$, both $\mathsf{A}^{(e)}$ and $\mathsf{A}^{(b)}$ are odd unimodular lattices. The b-isotope ring $\mathsf{A}^{(b)}$ has a different lattice geometry, described in [**Bor84**, 5.7], [**EG96**], [**EG01**], and [**GPR22**].

First consider $\mathsf{A}^{(e)}$. There are 723 elements in $\mathsf{A}^{(e)}$ with $\langle x, x \rangle^{(e)} = 4$ and that satisfy $x \circ_e x = 2x$. Each of these elements is twice a primitive idempotent relative to product $\circ_e$. They are the elements in $\mathrm{Herm}(3, \mathsf{O})$ of the following form, for any octonion integer unit u in O:

$$\begin{array}{ccc} (0, 1, 1 \mid u, 0, 0), & (1, 0, 1 \mid 0, u, 0), & (1, 1, 0 \mid 0, 0, u), \\ (2, 0, 0 \mid 0, 0, 0), & (0, 2, 0 \mid 0, 0, 0), & (0, 0, 2 \mid 0, 0, 0). \end{array}$$

The maps $P(e - x) = P(x - e)$ for x any of these 723 vectors generate the ring automorphism group $\mathrm{Aut}(\mathsf{A}^{(e)}) = 2^2 \cdot \mathrm{O}_8^+(2) : S_3$ [**Gro96**].

The lattice defined by the ring $\mathsf{A}^{(b)}$ is described well in [**EG01**]. There are 819 elements in $\mathsf{A}^{(b)}$ with $\langle x, x \rangle^{(b)} = 4$ and that satisfy $x \circ_b x = 2x$. We can verify by a brief computation that these define the tight projective 5-design on the octonion projective plane in the $\mathrm{Herm}(3, \mathbb{O})^{(b)}$ isotope algebra. The shortest vectors of $\mathsf{A}^{(b)}$ are those with $\langle x, x \rangle^{(b)} = 3$, which consist of $\pm b$ and $\pm(b - x)$ for x one of the 819 vectors described above. The maps $P^{(b)}(b - x)$ and $P^{(b)}(x - b)$ generate a simple group of type ${}^3D_4(2)$. The full automorphism group $\mathrm{Aut}(\mathsf{A}^{(b)})$ is ${}^3D_4(2) : 3$, which includes these quadratic maps as well as an outer automorphism of order 3 [**Gro96**] [**EG96**].

Remark 6.50. A GAP computation confirms a conjecture of [**EG96**, p. 688] that the group ${}^3D_4(2)$ is transitive on the 69888 elements proportional to primitive idempotents with trace 3 in $\mathsf{A}^{(b)}$.

The trace in the q-isotope is defined as $\mathrm{tr}^{(q)}(r) = 3\langle r, q^{-1}, q^{-1}\rangle$. The authors of [**EG96**] define the following inner product on octonion integer triples O^3:

$$\{x,y\}^{(q)} = \mathrm{tr}^{(q)}(x^\dagger y + y^\dagger x) = 3\langle x^\dagger y + y^\dagger x, q^{-1}, q^{-1}\rangle = 2\mathrm{Re}(xqy^\dagger).$$

Since $x^\dagger y + y^\dagger x$ and $x^\dagger x$ are in A, we know that $\{x,y\}^{(q)}$ is $\mathbb{Z}$-valued and that $\{x,x\}^{(q)}$ is $2\mathbb{Z}$-valued. This inner product therefore defines an even integral lattice geometry on O^3. We can verify on any basis of O^3, by checking the Gram matrix determinant and computing the shortest vector lengths, that $\{x,y\}^{(e)}$ defines an $\mathtt{E}_8^3$ lattice and that $\{x,y\}^{(b)}$ defines a Leech lattice.

We see then that, as described in [**EG96**], the b-isotope yields the Leech lattice via inner product $\{x,y\}^{(b)} = \mathrm{tr}^{(b)}(x^\dagger y + y^\dagger x)$ on O^3, which provides us with the tight 5-design in $\mathbb{RP}^{23}$, and that the b-isotope $\mathsf{A}^{(b)}$ contains 819 elements defining the tight projective 5-design in $\mathbb{OP}^2$. To connect the examples of [**EG96**] to our examples in this chapter, note that we can decompose b as $2b = 2(2,2,2 \mid \lambda,\lambda,\lambda) = M_\lambda M_\lambda^\dagger$ with M_λ the following matrix for λ in O a root of x^2+x+2:

$$M_\lambda = \begin{pmatrix} 2 & 0 & 0 \\ \overline{\lambda} & \overline{\lambda} & 0 \\ \lambda & -1 & 1 \end{pmatrix}.$$

It turns out that the Leech lattice $\Lambda(\overline{\lambda},\lambda)$ defined earlier in this chapter is an O-linear combination of the rows of M_λ:

$$\Lambda(\overline{\lambda},\lambda) = \left\{ xM_\lambda \mid x \in \mathsf{O}^3 \right\}.$$

We can verify on a basis for a canonical example that the following inner product on O^3 is a Leech lattice,

$$\{x,y\}^{(b)} = 2\mathrm{Re}(xby^\dagger) = \mathrm{Re}(x(MM^\dagger)y) = \mathrm{Re}((xM)(yM)^\dagger).$$

We have used M_λ because it has the property $2b = MM_\lambda$, which corresponds to the use of b in [**EG96**]. To correspond more closely to the notation of this chapter, we can write,

$$S_\lambda = \begin{pmatrix} 2 & 0 & 0 \\ \overline{\lambda} & 0 & \overline{\lambda} \\ 1 & 1 & \lambda \end{pmatrix}, \quad c = \frac{1}{2}S_\lambda S_\lambda^\dagger.$$

We previously used S_λ to denote a set of triples, but now denote a matrix with those triples for rows. We can confirm by computations on a suitable basis that the isotope ring $\mathsf{A}^{(c)}$ is isomorphic to $\mathsf{A}^{(b)}$. The 819 elements of $\mathsf{A}^{(c)}$ that satisfy $x \circ_c x = 2x$ define a tight projective 5-design on the octonion projective plane and the 98280 lines spanned by elements of O^3 of norm $\{x, x\}^{(c)} = 2\mathrm{Re}(xcx^\dagger) = 4$ define the tight projective 5-design due to the Leech lattice on $\Lambda(\overline{\lambda}, \lambda)$.

6.7. Isotope Rings, Orbits, and Squares

Based on the results of [**Gro96**] and [**GPR22**], we know that for q in $\Omega \cap \mathsf{A}$ with $\det(q) = 1$, the isotope ring $\mathsf{A}^{(q)}$ is isomorphic to either $\mathsf{A}^{(e)}$ or $\mathsf{A}^{(b)}$, which are described above. We are interested in the relation between orbits of $G(\Omega \cap \mathsf{A})$ acting on determinant 1 elements in $\Omega \cap \mathsf{A}$ and these two isomorphism classes of isotope ring $\mathsf{A}^{(q)}$.

Lemma 6.51. *Let q, r be in $\Omega \cap \mathsf{A}$ with* $\det(q) = \det(r) = 1$*. Then* $\mathsf{A}^{(q)} \cong \mathsf{A}^{(r)}$ *if and only if q and r share an orbit of $G(\Omega \cap \mathsf{A})$.*

Proof. Let $G(\Omega)$ be the group of general linear transformations of Albert algebra V that preserve the positive definite cone Ω. Let $G(\Omega)_x$ be the stabilizer of some x in Ω. For a, b in Ω with $b = ga$ for $g \in G(\Omega)$, the following stabilizer groups are conjugate: $G(\Omega)_b = gG(\Omega)_a g^{-1}$ [**FK94**, p. 5]. Likewise, when q and r share an orbit of $G(\Omega \cap \mathsf{A})$, there is an element g in $G(\Omega \cap \mathsf{A})$ such that $q = gr$ and $G(\Omega \cap \mathsf{A})_q = gG(\Omega \cap \mathsf{A})_r g^{-1}$. This ensures that the stabilizers of q and r are isomorphic when they share an orbit. Equivalently, if q and r have non-isomorphic stabilizers they cannot be in the same orbit of $G(\Omega \cap \mathsf{A})$. We know that there are precisely two stabilizers up to isomorphism [**Gro96**] corresponding to two Albert isotope rings [**GPR22**], which are described above. Since there are only two orbits of $G(\Omega \cap \mathsf{A})$ on determinant 1 elements in $\Omega \cap \mathsf{A}$, there cannot be any cases of isomorphic stabilizers (or isotope rings) in distinct orbits. $\square$

Lemma 6.52. *For q in $\Omega \cap \mathsf{A}$ with* $\det(q) = 1$*, the set* $\mathsf{A} = \mathrm{Herm}(3, \mathsf{O})$ *is closed under $(x, y) \mapsto 2x \circ_q y$ and $(x, y) \mapsto P^{(q)}(x)y$.*

Proof. We can write the isotope product in terms of a Jordan triple system as follows: $2x \circ_q y = 2\{x, q, y\} = P(x + y)q - P(x)q - P(y)q$. We can check on some basis for A that $2\{x, q, y\}$ is always in A, which proves closure of A under the product $2x \circ_q y$. To prove that A is closed under the action of $(x, y) \mapsto P(x)y$, we first select some basis for A and check that all pairs of basis elements x, y satisfy that $P(x)y$ in A. Next, for any x, y, z in A we have

$$P(x + z)y = 2x \circ_y z + P(x)y + P(z)y.$$

This ensure that $P(x + z)y$ is in A when $P(x)y$ and $P(z)y$ are. Any element in A is the sum of basis vector elements. By induction, since the sum of any pair $x + y$ has $P(x + z)y$ in A when $P(x)y$ and $P(z)y$ are in A, we must have $P(a)b$ in A for any a, b in A. By composition A is also closed under

$(x, y) \mapsto P^{(q)}(x)y = P(x)P(q)y$ for q in $\Omega \cap \mathsf{A}$ with $\det(q) = 1$ (we are only interested in q-isotopes for $\det(q) = 1$ since we require that q^{-1} is also in A). $\square$

LEMMA 6.53. *For any a in A with $\det(a) = \pm 1$, the map $P(a)$ belongs to $G(\Omega \cap \mathsf{A})$.*

PROOF. By Lemma 6.52 we also have $P(a)\mathsf{A} = \mathsf{A}$. Since $\det(a) = \pm 1$, a is invertible and the map $P(a)$ is in $G(\Omega)$. Suppose that $q = P(a)r$. Then we have $\det(q) = \det(a)^2 \det(r) = \det(r)$. So the map $P(a)$ preserves determinant and we know that $P(a)^{-1}$ also preserves A. Therefore $P(a)$ is in $G(\Omega \cap \mathsf{A})$. $\square$

REMARK 6.54. Consider all elements q, r in the set $\Omega \cap \mathsf{A}$ with $\det(q) = \det(r) = 1$. There exists an a in Ω with $\det(a) = \pm 1$ such that $q = P(a)r$. When a is also in A we call q a *r-square*. Otherwise, q is a *r-nonsquare*. By Lemma 6.53, the r-squares are all contained in $G(\Omega \cap \mathsf{A})r$. Therefore, by Theorem 6.51, $\mathsf{A}^{(q)} \cong \mathsf{A}^{(r)}$ when q is an r-square. An open question is whether $\mathsf{A}^{(q)} \cong \mathsf{A}^{(r)}$ if and only if q is an r-square. An equivalent open question is whether q and r share an orbit of $G(\Omega \cap \mathsf{A})$ if and only if q is an r-square (and r is a q-square). A related question is whether $P(a)\mathsf{A} = \mathsf{A}$ if and only if a is in A, which is a stronger claim than Lemma 6.52.

REMARK 6.55. It also remains to be proven that $\{x, y\}^{(q)}$ is an $\mathtt{E}_8^3$ geometry on O^3 for any q in the orbit $G(\Omega \cap \mathsf{A})e$ and a Leech lattice geometry for any q in the orbit $G(\Omega \cap \mathsf{A})b$, although this seems very likely to be true.

6.8. Conclusion

We have seen that the Leech lattice can be constructed as a sublattice of octonion integer triples, determined up to unit u and root λ of x^2+x+2 taken modulo $2\mathsf{O}$, as described in Theorem 6.30. We have also seen that the automorphism group of the Leech lattices with $u = 1$ are generated by octonion reflections $\mathtt{W}_r$, as described in Theorems 6.33 and 6.34. For $u \neq 1$ these generating reflections are replaced by generating involutions $\mathtt{L}_u \mathtt{W}_r \mathtt{L}_{\overline{u}}$. We have also seen that this Leech lattice construction (for $u = 1$) provides a simple means of generating the two strictly projective tight 5-designs, given in Example 6.43. Specifically, the generating reflections used in the common construction satisfy a $S(3, 4, 8)$ Steiner system structure determined via $\mathrm{PSL}_2(7)$ octonion integer automorphisms stabilizing $\lambda + 2\mathsf{O}$. We also see that the Suzuki chain subgroups of Co_1 have a simple octonion reflection construction given by Theorem 6.47. Finally, we have described the prior results of [**EG96**] in terms of Jordan isotopes and made a connection between the examples used there and those of this chapter. Some open questions remain about the role of squares in the orbits of $G(\Omega \cap \mathsf{A})$.

CHAPTER 7

Conclusion

This thesis explored the four tight projective 5-designs and their connections to various exceptional structures. In what follows we review the previous chapters and describe some areas for further exploration.

7.1. The Regular Hexagon

We began by reviewing how the vertices of a regular hexagon, also called a star, define an important root lattice. Through the process of one-line extension, we can construct all irreducible root lattices from the star. This process involves identifying the glue vectors of a root lattice, which in turn define the one-line extensions of that lattice. The glue vectors of a lattice also define the three-gradings of an irreducible root lattice. Given an irreducible root lattice with non-trivial glue vectors we can both construct larger root systems via one-line extension and also construct smaller root systems via three-gradings. In the introduction we also reviewed how the irreducible root lattices and their root systems govern the structure of Lie, Jordan, and composition algebras. In particular, irreducible root systems are in one-to-one correspondence with simple Lie algebras over $\mathbb{C}$ and Jordan structures correspond to three-gradings on root systems.

In Chapter 3 we asked whether any combinatorial properties of root lattices might provide a partial explanation for certain seemingly arbitrary properties of the standard model of particle physics. Specifically, we sought to answer whether the Lie algebra of the standard model and its particular representation, as three generations of fermions, could be attributed to any exceptional structures or properties of root systems. We found that by considering all possible sequences of three-gradings, the following sequence emerges as exceptional:

$$\mathtt{E}_7 \to \mathtt{E}_6 \to \mathtt{D}_5 \to \mathtt{A}_4 \to \mathtt{A}_1 \times \mathtt{A}_2.$$

First, the $\mathtt{E}_7$ system is unique in that it is the only irreducible root lattice that admits a three-grading without itself being obtainable by three-grading. This means that any sequence of three-gradings can be extended further left unless it begins with $\mathtt{E}_7$. Second, this sequence is local in the following sense. Each three-grading defines a graph with the roots of the 1-component for vertices and two roots adjacent when they are not orthogonal. A sequence of three-gradings is *local* when the graph of each three-grading is the local subgraph of the previous one. The set of local sequences of three-gradings are well-defined and this is the only local sequence that begins with $\mathtt{E}_7$. Finally, this sequence is maximal in the following sense. Given a root system that admits multiple three-gradings, the cardinality of the 1-component of each three-grading will generally differ. A three-grading is *maximal* when it yields the largest 1-component. The exceptional sequence given above only contains maximal three-gradings.

Having identified this sequence as exceptional for these reasons, we find that the Lie algebra of the standard model corresponds to the last Lie algebra in this exceptional sequence, of type $\mathtt{A}_1 \times \mathtt{A}_2$. The branching rules for $\mathtt{A}_1 \times \mathtt{A}_2$ within the adjoint representation of Lie algebra $\mathtt{E}_7$, the first Lie algebra in the exceptional sequence, yields the three generations of standard model fermions. This provides some combinatorial justification for the seemingly accidental properties of the standard model of particle physics—namely, the physical choice of g_{SM} and representation $\rho_{SM} \oplus \rho_{SM} \oplus \rho_{SM}$ among all possiblities—in terms of exceptional structures in the landscape of irreducible root systems and their corresponding Lie and Jordan structures.

There are some additional properties of this exceptional sequence that are not yet fully developed. For instance, the three-grading graph of $\mathtt{E}_7 \to \mathtt{E}_6$ is the Schläfli graph, the unique $\mathrm{srg}(27, 16, 10, 8)$. This graph also appears in the context of certain interesting t-designs. For example, there is a 2-design in $\mathbb{HP}^3$ with $|X| = 64$ and angle set $A = \{\frac{1}{9}, \frac{1}{3}\}$ [**Hog82**, Example 22]. This design can be obtained as the derived design of the tight 3-design in $\mathbb{HP}^4$ (and can be converted to the Hoggar lines in $\mathbb{CP}^7$). Of note, the neighbours of any point with angle $\frac{1}{9}$ form the 27 vertices of a graph, with edges when the angle between the pair is $\frac{1}{3}$. This graph is also a Schläfli graph. The local sequence of subgraphs may yield an analogous set of structures.

For a second related example, consider the tight 3-design in $\mathbb{CP}^5$ with angle set $A = \{0, \frac{1}{4}\}$ (Example A.15). The graph on orthogonal elements is a $\mathrm{srg}(126, 45, 12, 18)$, and the induced subgraph on the neighbours of any element is a $\mathrm{srg}(45, 12, 3, 3)$. This graph has 27 maximal cliques, each of size 5, and the graph on non-intersecting cliques is the Schläfli graph. These two examples suggest that the exceptional sequence, which begins with the Schläfli graph, may appear in the context of other t-designs. Our exceptional sequence and these two examples suggest that the Schläfli graph and the sequence of its local subgraphs provides a tool for identifying interesting substructures of certain exceptional objects. Of note, the Schläfli graph is also the Jordan grid graph of the exceptional Jordan algebra.

Another approach worth exploring involves nested sequences of rank 3 permutation representations (related to towers of permutations groups [**Car72**, pp. 306-307]). Strongly regular graphs, such as the Schläfli graph, correspond to rank 3 permutation groups. The stabilizer of a point in a rank 3 permutation group has two non-trivial orbits. When the action of the stabilizer on one of these orbits is also a rank 3 permutation group, then we can begin to form a chain of rank 3 permutation groups. The exceptional sequence above is one such chain, acting on the 1-component of each three-grading, corresponding to groups $\mathrm{U}_4(2) : 2 \to 2^4 : S_5 \to S_5 \to D_{12}$. Other interesting nested sequences of rank 3 permutation groups exist, often including sporadic simple groups. For instance, the Suzuki group acting on 1782 points has the following rank 3 chain: $\mathrm{Suz} \to G_2(4) \to \mathrm{HJ} \to \mathrm{PSU}_3(3)$. The McLaughlin group acting on 275 points has the chain $\mathrm{McL} \to \mathrm{PSU}_4(3) \to \mathrm{PSL}_3(4) \to A_6$. The Fischer sporadic simple groups also form a rank 3 chain, given by $\mathrm{Fi}_{24}' \to \mathrm{Fi}_{23} \to \mathrm{Fi}_{22} \to \mathrm{PSU}_6(2)$ acting respectively on 306936, 31671, 3510, and 693 points. Although this chain terminates in $\mathrm{PSU}_6(2)$ acting on 693 points, the group $\mathrm{PSU}_6(2)$ acting instead on 672 points yields another long rank 3 chain: $\mathrm{PSU}_6(2) \to \mathrm{PSU}_5(2) \to \mathrm{U}_4(2) \to (3^2 : Q_8) : 3 \to 3 \times S_3$. As these examples suggest, rank 3 sequences may be an interesting aspect of exceptional symmetries worth exploring further.

7.2. The Regular Icosahedron

The vertices of a regular icosahedron in $\mathbb{CP}^1 \cong \Omega_3$ form our second tight projective 5-design. In Chapter 4 we verified that it is exceptional among tight t-designs outside of the unit circle because it has an irrational angle set. We reviewed how prior proofs of this fact either included errors or excluded the full range of spherical and projective spaces. The proof given in Chapter 4 addressed the previous errors and also included all possible cases, since we used the manifolds of Jordan primitive idempotents to treat spherical and projective cases together in a unified way. In this way we verified that the regular icosahedron vertices are the unique example of a non-polygon tight t-design with an irrational angle set.

The icosahedron is an extremely well-studied object that continues to reward careful attention. The edges of an icosahedron define the $\mathtt{H}_3$ vectors, which extend to $\mathtt{H}_4$ and which can be assigned a quaternion product to produce the icosian ring. Triples of icosian ring elements can be used to construct the Leech lattice, using a special inner product [**Wil09a**, pp. 220-223]. This suggests that many exceptional structures, including the sporadic Co_1 symmetries of the Leech lattice, can be traced back to the icosahedron.

Another fascinating aspect of the icosahedron is the fact that it has six axes. The exceptional properties of the symmetric group S_6 can be used to construct a number of sporadic permutation groups, including M_{22}, M_{23}, M_{24}, HS, and McL. Finally, although the angle set of the icosahedron is not rational, it does belong to the golden field $\mathbb{Q}(\sqrt{5})$. Many exceptional structures are can be constructed from this starting point.

7.3. The Leech Lattice and the Octonion Projective Plane

In Chapters 5 and 6 we explored properties of the Leech lattice related to the octonion algebra, especially properties related to the octonion projective plane. The Leech lattice is a well-studied object but its relation to octonions has received less attention. Twelve of the twenty-six sporadic simple groups are symmetries of the Leech lattice, namely subgroups of Co_1.

In Chapter 5 we explored Robert Wilson's octonion construction of the Leech lattice and described how certain short vectors can define reflections, despite the non-associativity of the octonion algebra. Wilson describes the Leech lattice symmetries using right multiplication by octonion matrices. In contrast, we described how to use reflections to generate the full symmetry group of the Leech lattice, $2 \cdot \mathrm{Co}_1$. We also describe how to choose reflections that generate the Suzuki chain subgroups of Co_1. It turns out that the generators of the subgroup $G_2(4)$ both yield the tight 5-design on $\mathbb{RP}^{23}$ and also have a ${}^3D_4(2)$ action on $\mathbb{OP}^2$ that also yields a tight 5-design. We explore the orbits of other subgroups as well.

The Leech lattice reflections described above correspond to primitive idempotents in the octonion projective plane. We use this correspondence to define a common construction of designs in Ω_{24}, $\mathbb{RP}^{23}$, and $\mathbb{OP}^2$. This common construction allows us to link the two tight strictly projective 5-designs by producing them together using well-chosen vectors to generate corresponding involutions on $\mathbb{O}^3 \cong \mathbb{R}^{24}$ and $\mathbb{OP}^2$. Despite the non-associativity of the octonion algebra, the common construction remains well-defined because we use suitable vectors to define the needed involutions rather than requiring all vectors of a certain length to define involutions. One interesting property of the common construction is that it yields groups G and H of relatively prime order in the non-associative case responsible for the two strictly projective tight 5-designs.

In Chapter 6 we move beyond Wilson's definition of an octonion Leech lattice and explore Leech lattices constructed from octonion integer ring triples. In contrast to other authors exploring octonion integer Leech lattices, we focus on the properties of the octonion integers modulo 2. This permits us to understand a family of Leech lattices using the properties of a small strongly regular graph defined on the 135 non-units in $\mathsf{O}/2\mathsf{O}$. We then generalize the results from Chapter 5 to a larger family of octonion integer Leech lattices, defining generators of the automorphism group $2 \cdot \mathrm{Co}_1$ and the Suzuki chain subgroups.

Finally, in Chapter 6 we return to a result due to Elkies and Gross [**EG96**] that provides a different link between the Leech lattice and the tight 5-design in the octonion projective plane. We interpret these results using the concept of Jordan algebra isotopes and describe our integer octonion Leech lattice $\Lambda(\overline{\lambda}, \lambda)$ as the O-span of three octonion vectors used to define a corresponding isotope.

This Jordan isotope approach suggests some new techniques for identifying exceptional structures in the context of t-designs. Specifically, suppose that we select an integer ring $\mathsf{F} \subset \mathbb{F}$ within composition algebra $\mathbb{F}$ and also select a determinant 1 element u in $\mathrm{Herm}(\rho, \mathsf{F})$. In certain cases, the u-isotope ring $\mathrm{Herm}(\rho, \mathsf{F})^{(u)}$ will contain vectors that define an interesting t-design on $\mathbb{FP}^{\rho-1}$. In addition to the octonion projective plane example examined in Chapter 6, a brief computation in GAP confirms that we can use this technique to obtain the tight 3-designs in $\mathbb{HP}^4$ and $\mathbb{CP}^5$, Examples A.15 and A.16.

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APPENDIX A

Tight T-Designs

This Appendix describes the known tight t-designs. We begin with examples of *infinite families* of tight t-designs. These include a pair of antipodal points generalized to any rank and degree as a Jordan frame, the regular polygons on the unit circle Ω_2, and orbits of the Weyl groups $W(\mathtt{A}_{d+1})$ and $W(\mathtt{D}_{d+1})$ corresponding to the simplex and cross polytope in Ω_{d+1}. The glue vectors $[n]$ described below are special vectors of the dual lattice to a root lattice, labelled according to the convention given in [**CS13**, chap. 4] and described in Chapter 1. These examples correspond to the entries of Table 1.1. More details are available in [**DGS77**], [**Hog84**], [**CD07**, chap. 54], and [**BB09**].

Example A.1. A *Jordan frame* is a set of ρ orthogonal primitive idempotents in a simple Jordan algebra of rank ρ. A Jordan frame is a tight 1-design.

Example A.2. The $t+1$ vertices of a regular polygon in the circle Ω_2 constitute a tight t-design. The symmetries of the polygon are given by the dihedral group $D_{2(t+1)}$, which is the Coxeter group of type $\mathtt{I}_2(t+1)$. Since the unit circle is isomorphic to the real projective line, $\Omega_2 \cong \mathbb{RP}^1$, we also have a tight t-design in $\mathbb{RP}^1$ for each value of t. We treat both cases as equivalent, since they have the same rank $\rho = 2$ and degree $d = 1$. The $t = 5$ case is the first of our four tight projective 5-designs.

Example A.3. Begin with a root system in $\mathbb{R}^{d+1}$ of type $\mathtt{A}_{d+1}$ and glue vector [1] (or $[d+1]$). The orbit of glue vector [1] under the action of Weyl group $W(\mathtt{A}_{d+1})$ is a tight spherical 2-design in Ω_{d+1}, also called a *simplex*.

Example A.4. Begin with a root system in $\mathbb{R}^{d+1}$ of type $\mathtt{D}_{d+1}$ with glue vector [2]. The orbit of glue vector [2] under the action of Weyl group $W(\mathtt{D}_{d+1})$ is a tight spherical 3-design in Ω_{d+1}, also called a *cross polytope*. This example consists of the unit vectors spanning a set of $d+1$ mutually orthogonal lines.

The next three examples are tight t-designs corresponding to systems of *equiangular lines* in a real vector space. Equiangular lines define a strength $s = 3$ set of antipodal (so $\varepsilon = 1$) points on the sphere. For a tight spherical design corresponding to equiangular lines, we have $t = 2s - 1 = 5$. In addition to the vertices of a regular hexagon, which is the $t = 5$ case of Example A.2, there are three known additional systems of tight equiangular lines, described below. In each case, the equiangular lines define a tight 2-design on the corresponding real projective space, and a tight 4-design taken by selecting one point on the sphere and projecting its nearest neighbours into the portion of the sphere perpendicular to it (also called the *derived design*). Every tight spherical 5-design corresponds to a tight spherical 4-design [**BB09**, p. 1401] and tight real projective 2-design. The search for additional examples can be reduced to searching for additional tight spherical 5-designs.

The classification of tight spherical 5-designs remains open, but [**Gil18**] proves that the known examples for $d = 1, 2, 6, 22$ are the only cases that contain subsets of $d + 1$ vectors with positive pairwise inner product. Any new tight spherical 5-designs will lack this property. Another special property of the known tight 5-designs is that they have a 2-transitive automorphism group action on the lines. However, the 2-transitive automorphism groups on equiangular lines (equivalent to two-graphs) have been classified in [**Tay92**]. This suggests that any future spherical tight 5-designs will lack a 2-transitive automorphism group, although this requires some more work to establish.

Example A.5. The vertices of a regular icosahedron defines a tight 5-design in $\Omega_3 \cong \mathbb{CP}^1$, forming a single orbit under the action of Coxeter group $W(\mathtt{H}_3) = 2 \times A_5$. The corresponding icosahedron axes define a tight 2-design in $\mathbb{RP}^2$. The corresponding tight 4-design in Ω_2 is the regular pentagon of Example A.2 for $t = 4$.

Example A.6. Begin with the $\mathtt{E}_7$ root systems and glue vector [1]. The orbit of [1] under the action of Weyl group $W(\mathtt{E}_7)$ is a tight spherical 5-design in Ω_7, also known as the axes of the Hess polytope. These antipodal points define a corresponding tight 2-design in $\mathbb{RP}^6$. The corresponding tight 4-design in Ω_6 can also be obtained by beginning with the $\mathtt{E}_6$ root system and glue vector [1] or [2]. The orbit of either glue vector under the action of $W(\mathtt{E}_6)$ is a tight 4-design.

Example A.7. Begin with the Leech lattice Λ_{24} and any vector r of norm 6. The projection of the 552 norm 4 vectors with smallest angle relative to r onto the subspace perpendicular to r is a tight 5-design in Ω_{23}. These points form an orbit of $2{\times}\mathrm{Co}_3$. The corresponding tight 4-design is an orbit of McL : 2 in Ω_{22} (McL is the McLaughlin sporadic simple group). The corresponding projective tight 2-design is an orbit of Co_3 in $\mathbb{RP}^{22}$.

There are only three known tight 7-designs: the $t = 7$ instance of Example A.2 and the following two spherical examples. Any additional tight 7-designs, if found, will be spherical [**BH89**].

Example A.8. The $\mathtt{E}_8$ lattice is the lattice with most dense sphere-packing in $\mathbb{R}^8$ [**Via17**]. The $\mathtt{E}_8$ root system defines a tight 7-design in Ω_8, forming a single orbit under the action of Weyl group $W(\mathtt{E}_8)$. The corresponding tight projective 3-design is an orbit of $\mathrm{O}_8^+(2):2$ in $\mathbb{RP}^7$.

EXAMPLE A.9. Begin with the Leech lattice Λ_{24} and select a norm 4 vector r. The 4600 norm 4 vectors with smallest angle relative to r, projected onto the subspace orthogonal to r, form a tight 7-design in Ω_{23} and is an orbit of Co_2. The corresponding tight 3-design is an orbit of Co_2 in $\mathbb{RP}^{22}$. Both designs are orbits of the sporadic simple group Co_2 acting on the respective spaces.

The Leech lattice short vectors are the only tight 11-design [**BS81**], aside from the regular dodecagon in Ω_2. We describe this rare structure in the following example.

EXAMPLE A.10. The Leech lattice is the lattice with the most dense sphere-packing in $\mathbb{R}^{24}$ [**CKM$^+$17**]. It is spanned by its 196560 shortest vectors, which span a system of 98280 lines. There are numerous constructions of the Leech lattice, many of which are given in [**CS13**]. The lines spanned by these short vectors define the tight projective 5-design in $\mathbb{RP}^{23}$, the third of our four examples. Both designs are orbits of the group $2\cdot\mathrm{Co}_1$ acting on the vectors and the lines respectively.

We now describe an important family of tight 2-designs in complex projective spaces. A tight 2-design in $\mathbb{CP}^{\rho-1}$ ($\rho>1$, $d=2$) is also known in the literature as a SIC-POVM (*Symmetric Informationally Complete Positive Operator Valued Measure*) [**Sta20**]. An important open question is whether a SIC-POVM exists for each positive integer $\rho>1$.

EXAMPLE A.11. According to [**Sta19**], exact solutions for tight 2-designs with $d=2$ are known for the following ranks:

$$\begin{aligned}\rho = {} & 2\text{–}28, 30, 31, 35, 37\text{–}39, 42, 43, 48, 49, 52, 53, 57, 61\text{–}63,\\ & 67, 73, 74, 78, 79, 84, 91, 93, 95, 97\text{–}99, 103, 109, 111, 120,\\ & 124, 127, 129, 134, 143, 146, 147, 168, 172, 195, 199, 228,\\ & 259, 292, 323, 327, 399, 489, 844, 1299.\end{aligned}$$

The $d=2$ case is already described in Example A.3. All SIC-POVMs are defined as an orbit of a group acting on $\mathbb{CP}^{\rho-1}$. Indeed, except in the case of $\rho=8$, the group is known as the *Weyl-Heisenberg Group* [**Sta20**, 5]. We can construct this group using the following two operators acting on orthonormal basis vectors $e_0, e_1, e_2, \ldots, e_{\rho-1}$:

$$X : e_n \mapsto e_{(n+1) \pmod{\rho}}, \quad Z : e_n \mapsto \exp\left(\frac{2\pi n i}{d}\right) e_n.$$

The Weyl-Heisenberg group is the group generated by X and Z acting on $\mathbb{C}^\rho$, and the corresponding action on $\mathbb{CP}^{\rho-1}$. Apart from $\rho=8$, then, the task of describing a SIC-POVM in a known dimension is just the task of identifying an orbit representative, known in the SIC-POVM literature as the *fiducial vector*, for the orbit of $X \subset \mathbb{CP}^{\rho-1}$ under the Weyl-Heisenberg group action. The angle set is always given by $A = \{\frac{1}{\rho+1}\}$ and the cardinality is given by $|X| = \rho^2$.

In the case of $\rho = 2, 3, 8$ there also exist *sporadic SICs*, which have sporadic symmetries either beyond or instead of the Weyl-Heisenberg group symmetries [**Sta21**]. Specifically, the sporadic SICs of $\rho = 2, 3, 8$ are the only complex tight 2-designs (i.e. degree $d = 2$) with a group action that is 2-transitive on the points [**Zhu15**]. The $\rho = 2$ case is given by Examples A.3 and A.11, namely the vertices of a tetrahedron in $\mathbb{CP}^1 = \Omega_3$. The Weyl reflections defined by the $\mathtt{A}_3$ roots in $\mathbb{CP}^1 = \Omega_3$ permute the tetrahedron vertices with a S_4 group action. This group action is 4-transitive. The A_4 alternating subgroup is 2-transitive. The remaining two sporadic examples are given separately below. A recent detailed study of these three sporadic SICs is available in [**Sta21**].

Example A.12. [**Hog82**, Example 5] There exists a sporadic SIC-POVM in $\rho = 3$, called the *Hesse SIC*, that is a tight 2-design. The automorphism group is $\mathrm{SU}_3(2)$ and has order 216. This group is 2-transitive on the points. Each pair of points defines a $\mathbb{RP}^1$ subspace containing three points. These triples form the blocks of the unique Steiner triple system STS(9).

Example A.13. [**Hog82**, Example 8] A sporadic SIC-POVM also exists in $\rho = 8$, known as the *Hoggar SIC* or *Hoggar lines*. It is a tight 2-design. This design is an orbit of a 2-transitive group of order 64×6048 and has point stabilizer $G_2(2) = \mathrm{PSU}_3(3) : 2$.

The following three examples of complex or quaternion projective tight 3-designs are collected from [**Hog82**]. The classification of tight complex and quaternion projective 2- and 3-designs is open.

Example A.14. [**Hog82**, Example 6] This tight 3-design in $\mathbb{CP}^3$ corresponds to a complex $\mathtt{E}_8$ lattice, where the vectors of six roots map to the same primitive idempotent. The strongly regular graph of orthogonal idempotents has parameters $\mathrm{srg}(40, 12, 2, 4)$ and the 40 maximal cliques, which are Jordan frames, form a generalized quadrangle $\mathrm{Gq}(3, 3)$. This 3-design is an orbit of $\mathrm{PSU}_4(2) : 2$ acting on $\mathbb{C}^4$.

Example A.15. [**Hog82**, Example 7] The complex K_{12} Coxeter-Todd lattice short vectors define a tight 3-design in $\mathbb{CP}^5$ ($\rho = 6$, $d = 2$) with cardinality $|X| = 126$ and angle set $A = \left\{0, \frac{1}{4}\right\}$. This design is an orbit of a rank 3 action of $\mathrm{PSU}_4(3)$ on $\mathbb{CP}^5$. The strongly regular graph constructed from orthogonal idempotents has parameters $\mathrm{srg}(126, 45, 12, 18)$. There are 567 Jordan frames corresponding to maximal cliques in this graph. Unlike Examples A.14 and A.16, the Jordan frames do not define a generalized quadrangle geometry.

EXAMPLE A.16. [**Hog82**, Example 9] The quaternion reflection group $W(U) = 2 \times \mathrm{PSU}_5(2)$ has an orbit in $\mathbb{HP}^4$ ($\rho = 5$, $d = 4$) that forms a tight 3-design with angle set $\{0, \frac{1}{4}\}$ and cardinality $|X| = 165$. There are 297 Jordan frames in this set and each point belongs to 9 Jordan frames. The Jordan frames form the lines of a $\mathrm{Gq}(4, 8)$ geometry. We can recover the Hoggar lines, Example A.13, from this design by taking a derived design and then converting to the corresponding complex design.

An existence proof in [**CKM16**] establishes that the following two examples exist, although explicit vectors are not yet available.

EXAMPLE A.17. As described in [**CKM16**], a tight 2-design exists in $\mathbb{HP}^2$, with $A = \{\frac{2}{7}\}$ and $|X| = 15$.

EXAMPLE A.18. As described in [**CKM16**], a tight 2-design exists in $\mathbb{OP}^2$, with $A = \{\frac{4}{13}\}$ and $|X| = 27$.

The final example is the fourth and final instance of tight projective 5-designs. We discuss it in detail in Chapters 5 and 6.

EXAMPLE A.19. The generalized hexagon $\mathrm{Gh}(2, 8)$ can be described as a tight octonion projective 5-design in $\mathbb{OP}^2$ of cardinality $|X| = 819$ and angle set $A = \left\{0, \frac{1}{4}, \frac{1}{2}\right\}$. One construction, corresponding to [**Wil09a**, 162], involves taking the $819 = 3 \times (1 + 16 + 16^2)$ row vectors $(1, 0, 0)$, $(1, j, 0)$, and $(sj, j, 1)$ under all cyclic coordinate permutations, where $j = \pm i_t$ for $t \in \mathrm{PL}(7)$ and where $s = \frac{1}{2}(\pm i_1 \pm i_2 \pm i_3 \pm i_4 \pm i_5 \pm i_6 \pm i_7 \pm i_\infty)$ with the positions of the plus signs forming the extended Hamming code generated by the quaternion subalgebra bases $\{i_t, i_{t+1}, i_{t+3}, i_\infty\}$. Another construction of the same idempotents comes from setting $s = \frac{1}{2}(i_1 + i_2 + i_3 + i_4 + i_5 + i_6 + i_7 - i_\infty)$ and constructing the idempotents corresponding to the row vectors $(\pm \overline{s} j, \pm j, 1)$ for j in the octonion double basis. Under Weyl reflection, these elements close to the $\mathrm{Gh}(2, 8)$ structure given above. If we instead begin with $(\pm sj, \pm j, 1)$ for j in some *quaternion* double basis, $\{\pm i_t, \pm i_{t+1}, \pm i_{t+3}, \pm i_\infty\}$ with t in $\mathbb{F}_7$, then the corresponding idempotents close to seven distinct $\mathrm{Gh}(2, 8)$ tight 5-designs.

APPENDIX B

Jacobi Polynomials

The Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$ are defined in [**AS72**, 22.2.1]. Let V be a simple Euclidean Jordan algebra of rank ρ and degree d. In order to study a design modeled as a finite subset X of Jordan primitive idempotents $\mathcal{J}(V)$, we employ the following *renormalized Jacobi polynomials* $Q_k^\varepsilon(x)$:

$$Q_k^\varepsilon(x) = \frac{(\alpha+\beta+1-\varepsilon)_{k+\varepsilon}}{(\beta+1-\varepsilon)_{k+\varepsilon}}\left(\frac{\alpha+\beta+2k}{\alpha+\beta+k}\right)P_k^{(\alpha-1,\beta)}(2x-1).$$

Here we use $\alpha = \frac{1}{2}(\rho-1)d$ and $\beta = \frac{1}{2}d - 1 + \varepsilon$. We also use Pochhammer symbol $(a)_k = a(a+1)\cdots(a+k-1)$. For the projective cases ($d = 1, 2, 4, 8$) this definition of $Q_k^\varepsilon(x)$ corresponds to the definition given in [**Hog82**]. This appendix verifies that this definition also applies to the spherical cases ($\rho = 2$) and confirms an important Jacobi polynomial identity used in the construction of the annihilator polynomial $\operatorname{ann}(x)$.

B.1. Application to the Spherical Cases

In the spherical cases we have $\rho = 2$. This means that $\alpha = \frac{1}{2}d$, $\beta = \alpha - 1 + \varepsilon$, and $Q_k^\varepsilon(x)$ has the form,

$$Q_k^\varepsilon(x) = \frac{(2\alpha)_{k+\varepsilon}}{(\alpha)_{k+\varepsilon}}\left(\frac{2\alpha+2k-1+\varepsilon}{2\alpha+k-1+\varepsilon}\right)P_k^{(\alpha-1,\alpha-1+\varepsilon)}(2x-1).$$

Using [**AS72**, 22.5.20] and $(x)_n = \Gamma(x+n)/\Gamma(x)$, we can write,

$$P_k^{(\alpha-1,\alpha-1)}(2x-1) = \frac{(\alpha)_k}{(2\alpha-1)_k}C_k^{(\alpha-\frac{1}{2})}(2x-1).$$

Therefore, in the $\varepsilon = 0$ case we have,

$$Q_k^0(x) = \frac{(2\alpha)_k}{(\alpha)_k}\left(\frac{2\alpha+2k-1}{2\alpha+k-1}\right)\left(\frac{(\alpha)_k}{(2\alpha-1)_k}C_k^{(\alpha-\frac{1}{2})}(2x-1)\right).$$

Since $(2\alpha)_k/(2\alpha-1)_k = (2\alpha+k-1)/(2\alpha-1)$, the expression simplifies as follows:

$$Q_k^0(x) = \left(\frac{2\alpha+2k-1}{2\alpha-1}\right)C_k^{(\alpha-\frac{1}{2})}(2x-1).$$

This expression matches the one given in [**DGS77**]. (N.B. the expression in [**DGS77**] uses d to denote the sphere's dimension. Here we use d to denote Jordan algebra degree, so that $\mathcal{J}(V) \cong \Omega_{d+1}$).

B.2. An Annihilator Polynomial Identity

To simplify computations with $\operatorname{ann}(x) = x^\varepsilon R^\varepsilon_{s-\varepsilon}(x)$, the annihilator polynomial, with $R^\varepsilon_{s-\varepsilon}(x)$ defined as,

$$R^\varepsilon_{s-\varepsilon}(x) = Q^\varepsilon_0(x) + Q^\varepsilon_1(x) + \cdots + Q^\varepsilon_{s-\varepsilon}(x),$$

this section confirms that we can also write,

$$R^\varepsilon_{s-\varepsilon}(x) = \frac{(\alpha+\beta+1-\varepsilon)_s}{(\beta+1-\varepsilon)_s} P^{(\alpha,\beta)}_{s-\varepsilon}(2x-1).$$

We begin with a recurrence relation for the Jacobi polynomials $P^{(\alpha,\beta)}_n(x)$, given in [**AS72**, 22.7.18]:

$$P^{(\alpha,\beta)}_n(2x-1) = \frac{(\alpha+\beta+2n)}{(\alpha+\beta+n)} P^{(\alpha-1,\beta)}_n(2x-1) + \frac{(\beta+n)}{(\alpha+\beta+n)} P^{(\alpha,\beta)}_{n-1}(2x-1).$$

Applying this recurrence relation repeatedly, we obtain $P^{(\alpha,\beta)}_n(2x-1)$ as a linear combination of $P^{(\alpha-1,\beta)}_k(2x-1)$ polynomials:

$$P^{(\alpha,\beta)}_n(2x-1) = \sum_{k=0}^{n} \frac{(\beta+k+1)_{n-k}}{(\alpha+\beta+k+1)_{n-k}} \frac{(\alpha+\beta+2k)}{(\alpha+\beta+k)} P^{(\alpha-1,\beta)}_k(2x-1).$$

We now substitute this expression into our second expression for $R^\varepsilon_{s-\varepsilon}(x)$ given above:

$$R^\varepsilon_{s-\varepsilon}(x) = \sum_{k=0}^{s-\varepsilon} \frac{(\beta+k+1)_{s-\varepsilon-k}(\alpha+\beta+1-\varepsilon)_s}{(\beta+1-\varepsilon)_s(\alpha+\beta+k+1)_{s-\varepsilon-k}} \frac{(\alpha+\beta+2k)}{(\alpha+\beta+k)} P^{(\alpha-1,\beta)}_k(2x-1).$$

Using the expression $(x)_n/(x)_m = (x+m)_{n-m}$ for $n \geq m$ [**Bor04**, p. 17] we recover our first expression for $R^\varepsilon_{s-\varepsilon}(x)$:

$$R^\varepsilon_{s-\varepsilon}(x) = \sum_{k=0}^{s-\varepsilon} Q^\varepsilon_k(x).$$