THEORY OF DECAYING POLYNOMIAL SPACES $P_N^\lambda(\Omega)$ AND THEIR APPLICATIONS TO APPROXIMATIONS USING ASYMPTOTIC SERIES EXPANSION, LAPLACE TRANSFORM MOMENT MATCHING, AND INTERPOLATION

THÉORIE DE ESPACES DE POLYNÔME DÉCROISSANT $P_N^\lambda(\Omega)$ ET LEURS APPLICATIONS À L'AIDE DES APPROXIMATIONS PAR DÉVELOPPEMENT EN SÉRIE ASYMPTOTIQUE, TRANSFORMÉE DE LAPLACE A MOMENT ÉGAL, ET L'INTERPOLATION

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Dedication

This thesis is dedicated to my beloved father, mother, sister, and brother for their love, support, and encouragement.

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Abstract

In this thesis, we construct a real analytic function space, called the decaying polynomial space, on the non-negative real axis. This space has multiple linear structures and favorable properties useful for approximations of continuous functions vanishing at infinity. We introduce the weak norms and metrics on the space so member functions can be weakly measured on the non-compact interval $[0, \infty)$. Then we develop three kinds of approximation methods – the asymptotic series expansion with variants, the Laplace transform moment matching, and the interpolation, all of which are based on the new space. We prove the weak uniform convergence for all the approximation methods and give an illustrative example of each method.

Résumé

Dans cette thèse, nous construisons un véritable espace de fonction analytique, appelé l'espace polynôme décroissant, sur l'axe réel non négatif. Cet espace a une structure linéaire et d'autre propriétés favorables utiles pour approxime des fonctions continu et zero à l'infini. Nous introduisons les normes et les mesures faibles sur l'espace pour ces fonctions qui peuvent être faiblement mesurées sur la intervalle non-compact $[0, \infty)$. Ensuite, nous développons trois types de méthodes d'approximation – le développement en série asymptotique avec des variantes, transformée de Laplace a moment égal, et l'interpolation, qui sont toutes basées sur le nouvel espace. Nous prouvons la convergence faible uniforme pour toutes les méthodes d'approximation et incluans des exemple illustratif pour chaque méthode.

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1. Introduction

1.1. **Overview.**

1.1.1. *Motivation.* I began my study of queueing theories by investigating the basic model, the M/M/1 queue, where the involved random variables have exponentially distributed probability densities. It was then natural to investigate more general queueing models, e.g. the GI/G/1 queue, where the involved random variables have arbitrary probability density functions. In some cases, where the involved probability density functions have rational Laplace transforms, the GI/G/1 queue can be solved exactly. Otherwise, one has to pursue approximate solutions [9] [13] [18].

The idea is that a GI/G/1 queue can be solved approximately if the involved probability density functions can be approximated by other functions with rational Laplace transforms. Thus, the problem becomes approximating continuous probability density functions on $[0, \infty)$.

Initially, we intended to use the Padé method (Henri Padé, 1863-1953) to find the approximate probability density functions with rational Laplace transforms. The method matches the moments of a probability density function to those of the approximation function [10]. However, we found in our numerical experiments the curve of the Padé approximation function does not always resemble the original probability density function. In some cases, the two curves are "roughly close" but inadequately so. In other cases, the two curves are totally different. To use the Padé method, one has to randomly try different parameters in hopes of finding an acceptable approximation function.

In order to find out why the Padé method behaves unsatisfactorily and to explore other alternative methods, we have studied the evolution of probability distribution function approximations in queueing or other stochastic models. Botta et al. made a good review of the early development of distribution approximation methods [4] [5]. For the recent approaches on approximations by phase-type distributions, see [2] [11].

In 1909, a Danish engineer Agner Erlang (1878-1929) first investigated the telephone network congestion problem. Not only did he introduce the exponential distribution in his famous M/M/1 queue analysis, but he also created the concept of successive stages (or phases) to analyze the general queueing models. According to Erlang's theory of stages, any random variable can be expressed as a sum of independent and identically exponentially distributed fundamental random variables or the stages, and the random variable is said to have an Erlang distribution which is the convolution of the exponential distributions. The concept of phases is useful both in theoretical analysis and in practical models.

The early researchers were interested in the following two problems on probability distributions. One problem is to generalize the exponential distribution using the concept of phases. Credit goes to Jensen, Smith, and Cox as the generalizations were successful and that gave rise to an exponential family of classes of distributions [4] [5]. Researchers also looked at the Laplace transforms of the probability density functions in the exponential family, since many problems involving stochastic processes can be easily solved in Laplace transforms. The other problem is to approximate any continuous distribution function on $[0, \infty)$ by some classes of distributions in the exponential family. Cox and Neuts made significant contributions on both problems theoretically and practically.

Based on the concept of stages, Cox developed a model of multiple stages of various rate parameters [3] [7]. In fact, this model was a special Markov process (Andrey Markov, 1856-1922). Loosely speaking, a Markov process is a stochastic process whose future states depend only on the present states and not on the past states. The Cox model implicitly has the Markov properties. In addition, all distribution functions of Cox models have rational Laplace transforms. Cox noticed this and considered all the distribution functions with rational Laplace transforms, possibly having complex poles, as an important class. He claimed the class was closed and dense in all continuous distribution functions on $[0, \infty)$, and suggested it be used to approximate any distribution functions.

Also based on Markov process, Neuts developed another model to represent what he called phase-type distributions, which were more general than the distributions of Cox models [14]. In his model, Neuts used a phase-type distribution to describe the random transient time until absorption in a Markov process. He introduced the matrix representation for the phase-type distributions, so he could study the properties of a phase-type distribution by its Markov transition matrix. Neuts' methods were called the matrix analytic or matrix geometric methods. He claimed the class of phase-type distributions was closed and dense in all distribution functions, and suggested it be used to approximate any distribution functions. Currently, matrix-analytic methods are the mainstream methods to approximate distribution functions in the fields of queueing and other stochastic process researches. For the fundamentals and the recent development of the methods, see [11].

Both Coxian distributions and phase-type distributions have rational Laplace transforms, and are more general than the Erlang distributions. The distribution function approximation methods using the Cox model or the phase-type model can be done in the time domain or in the transformed domain. In these approximation methods, certain properties or statistics, e.g. the function values at certain points or the moments, of the original functions are used to estimate a set of parameters of the approximation distribution in the time domain or the transformed domain. However, it should be noted an approximation method could directly use function values or moments without assuming any models.

In the theoretical analysis aspect, Schassberger showed any distribution function on $[0, \infty)$ could be approximated weakly in the Lévy metric sense by a sequence of mixed Erlang distributions, and showed the class of mixed Erlang distributions was dense in all distribution functions [18].

Botta et al. characterized the class of generalized hyperexponential distribution functions and showed its weak convergence and closure property [4] [5]. The difference between the hyperexponential distributions and the generalized hyperexponential distributions is that the former allow only positive coefficients and the latter allow both positive and negative coefficients. Botta et al. noted and claimed the added freedom had made the generalized hyperexponential distributions to approximate any distribution defined on $[0, \infty)$ as close as desired in the weak Lévy metric sense. They also investigated the set inclusion relations between many different types of distributions in the exponential family, and pointed out the generalized hyperexponential distributions were not in the phase-type class. They noted that although the phase-type class was computationally advantageous, its representation was not unique and there was no easy way to determine if a given distribution was in the phase-type class.

Since computers were introduced to solving queueing problems, many researches have focused on the numerical algorithms for computing the approximation functions, in particular the phase-type approximation distributions, rather than the theoretical analysis. Bux and Herzog developed a distribution function approximation method based on the Cox model with the uniform rate parameter or the mixed Erlang model [6]. The method used a set of measured data and the lower order moments to estimate the parameters of the Cox model such that the difference between the original and the approximation distributions was minimized. Its procedure began with a low number of phases for the parameter estimations by a linear optimization algorithm called simplex algorithm. Then the results were compared with the prescribed absolute error tolerance, and if the test failed, the number of phases was increased by one and the above procedure was repeated. The great idea behind this approximation method is a sequence of numerical statistical empirical algorithms, each of which tests a hypothesis from observations until the prescribed accuracy is reached.

Asmussen et al. demonstrated a numerical algorithm of approximating or curve-fitting any density functions by phase-type distributions [2]. The method is based on the maximum likelyhood estimation with expectation-maximization algorithm (MLE-EM), where the MLE is a method which maximizes the log-likelyhood function of a parameter from a sample of independent identically distributed observations and the EM is a two-step recursive algorithm for the MLE. Asmussen et al. stated the convergence of the method was not easily proved and the method generally needed about 1000 to 10000 iterations for reasonable fits. The MLE-EM method can also be used for approximating long-tail or heavy-tail distributions such as the Weibull distribution and the Pareto distribution, which are common for internet traffic modeling and analysis [8]. The MLE-EM method is still active within the research communities. For the recent development of the numerical phase-type approximation methods, see [11] [15] [20].

Distribution approximations can be done in the transformed domain. The main idea of approximating Laplace transforms is to match the moments of the original and the approximation distributions. There are some techniques on matching only the first three moments [12] [16]. But our main interest concerns matching the moments of any order with rational approximation Laplace transforms. Series expansions and continued fractions are two important rational Laplace transform approximation methods [1]. But the Padé method is more general. Harris and Marchal discussed the Padé method in approximating Laplace transforms and stated the Padé method might converge slowly as the number of matching moments increased [10]. However, it was likely the distributions generated by inverting the Padé approximation rational Laplace transforms might not converge at all and there was no proofs of convergences of the approximation Laplace transforms. In this respect, one could assume there existed a sub-sequence of the Padé inverted distributions converging to the original distribution. To fix this problem, Harris and Marchal suggested matching the original and the approximation Laplace transforms at some prescribed points along the negative real line on the complex plane. They also noticed the approximation density functions might have negative values and suggested using translation and scaling techniques to remedy it.

Another idea is to use the empirical approximation Laplace transform from a set of points of the original distribution in the moment matching. This is equivalent to using a step approximation distribution to form an empirical Laplace transform, and then using the Padé method to approximate the empirical Laplace transform [10] [19].

Not many researchers discuss convergence issues of the rational Laplace transform approximation methods. The above Laplace transform approximation methods do not guarantee convergences. In addition, these methods are essentially solving some systems of equations and may be numerically ill-conditioned, computationally inefficient, or simply ineffective.

The academic precedents and the existing theories and methods for distribution function approximations have made fruitful achievements and continue to provide ideas, models, and inspirations as well as lessons and failures for future improvements. Many existing approximation methods use a probability mixture model of one or many particular classes of distributions in the exponential family to approximate arbitrary distribution functions. A finite (or countably infinite) mixture distribution is a convex combination of a collection of other distributions, where all the weighting factors are non-negative and sum to one. These conditions impose such a strong restriction on the approximation function class, causing the class lacking useful mathematical structures, that the approximation problems can only be solved by optimization methods. This makes it difficult to discuss the convergence, in particular the uniform convergence, in these existing distribution approximation methods.

For example, the Cox model and the phase-type distributions are generally finite mixture distributions, and there is a major drawback to use them as the approximation function classes. Although the convex combination generalizes the distributions and guarantees the mixture is a distribution function, it imposes a strong restriction on the approximation distribution function class, limiting the ways of convergences and undermining the conditions for uniform convergences. The classical mathematical analysis provides many useful tools for function approximations, and the series expansion is one of the most known. Because many existing approximation methods are based on the mixed distribution models, they could not take advantage of the powerful classical series expansion method. As a result, the approximation problems become optimization problems and may only be solved numerically and iteratively.

One idea to improve the mixture distribution model is that the independent member functions to be mixed should be in the same class in the exponential family and have some relations to each other. Then, we could choose this particular class of distributions as the approximation class and take advantage of the useful relations between the independent member functions. Another idea is to remove the convex combination restriction. This implies a mixture function might not be a distribution function any more. However, such treatments result in a function space, a much simple mathematical structure where members are linear combinations and not convex combinations. In addition, norms and topologies on the space can be defined and there are infinitely many ways of convergences. From the view point of the classical mathematical analysis and the functional analysis, normed function spaces are ideal classes for function approximation problems. Finally, we may define a collection of "nearly-distribution functions" as some small neighborhood of a center distribution function in the space, whose members may assume negative values on some "insignificant" intervals, have a nearly-unity integral on $[0, \infty)$, or have other non-distribution properties and so on.

Fortunately, the above particular function class does exist in the exponential family of distributions. Take the mixed Erlang model for example. If we remove the restriction of the convex combination and consider all the Erlang terms of the increasing orders as a basis, we have an approximation function space. In fact, our function space theory in this thesis is developed independently of the mixed Erlang model. But in retrospect, our theory could be generalized from the mixed Erlang model. Because we have a much broader view of function approximation problems, we set our objective to approximate any continuous functions on $[0, \infty)$, where the distribution function approximation problems are only special cases. The weak convergence proofs on mixed Erlang distributions by Schassberger or by Botta et al. are clearly the view of function spaces [18] [4]; and Bux and Herzog's simplex optimization method to find the coefficients of the mixed Erlang distributions can be easily explained as uniform convergences in a function space [6].

In this thesis, we think approximating a probability density function, or more generally a continuous function, on $[0, \infty)$ is a fundamental mathematical problem, which could be resolved in a simple and elegant way. The difficulty of this problem lies in the underlying domain of the functions being non-compact sets, or unbounded intervals, on which there is, to our knowledge, no well established general theory for function approximations. Solving this fundamental problem will have a profound influence not only on queueing or other stochastic process researches but also on other areas such as signal processing, differential/integral equations, and numerical calculations.

1.1.2. *Functions and operations.* We shall first examine some basic analytic properties of continuous functions and the operations on them, since they are the main subjects of our approximation problem and are important to understanding general approximation theories.

Continuous functions are abstracted from the real world to describe how things change. In science, engineering, and applied mathematics, a continuous function may be used to describe the motion of a planet, the temperature fluctuation at a weather station, the voltage variation of an electronic signal, or simply a geometric curve. The theory behind continuity is the concept of limit, which is the foundation of calculus and mathematical analysis.

Operations on a continuous function affect its continuity: even simple operations such as addition may give unexpected results. For example, adding infinitely many continuous functions may result in a discontinuous function. Complicated operations including differentiation, integration, and convolution must be performed in some restricted and controllable way to achieve expected results.

There have already been a great many function approximation theories and methods developed over human history which correctly used the analytic properties of continuous functions and operations. Historically, function approximations began as some simple formulas, interpolation methods, or series expansions in the view of calculus or real and complex analysis. Since then, it has evolved into an abstract approximation function space theory in the modern view of functional analysis. By revisiting these existing approximation theories and methods and combining the early and the modern views of function approximations, we are set to build a new theory for our function approximation problem.

1.1.3. *Reviews of existing approximation theories and methods.* It is important to review the existing function approximation theories and methods before building a new one. This review has two purposes. The first purpose is to analyze why the existing methods fail for our approximation problem. The second is to learn the structures and ideas in those methods and apply the good ones in a new approximation method.

There are roughly two views of existing approximation methods - the early view and the modern view.

The early view on function approximations is mainly a formula-type method, in which one can directly use the expression of a function to find its approximation. This includes polynomial interpolations, trigonometric interpolations, and Taylor series expansions, or even Fourier series expansions. Polynomial interpolations can be further classified as global lower/higher order interpolations and local piecewise interpolations, e.g. the spline interpolation. These approximation methods cannot be directly used for our approximation problem mainly because they are generally valid only on compact domains and not effective on non-compact ones. There are other specific reasons for the failure of these methods, some of which will be discussed in later chapters.

In the early view, approximation can also be done in transformed domains. For example, the Fourier series expansion method is equivalent to approximating a continuous spectral function by a discrete function in the frequency domain, which is the weak convergence under certain conditions. But the corresponding convergence theory has not been well established for Laplace transformations, in spite of general theories of complex function approximations in complex analysis. The question is how the convergence of Laplace transforms in the complex domain relates to the convergence of inverse functions in the real domain, and it has not been well addressed in some of the existing approximation methods. Lacking strict proofs for convergence is the main reason for the failure of some practical rational Laplace transform approximation methods, including the Padé method.

The modern view of function approximations is based on the theory of abstract function spaces. An abstract function space is a collection of functions where the "distance" between any two member functions is defined. Then the approximation to a function can be viewed as finding a sequence of functions in the space converging to it. This is geometrically similar to a sequence of points converging to a limit point in Euclidean spaces. What is different is the convergence of a function sequence takes various forms and should be treated carefully. Only the so-called uniform convergence resembles the point convergence in Euclidean spaces. In general, a convergent sequence in a function space may or may not have a limit in the space. In a complete space or a Banach space (Stefan Banach, 1892-1945), every convergent sequence has a limit. Thus, in order to use a function space for function approximations, one must prove it is complete.

The underlying domain of an abstract function space determines the definition of "distance" for the space and so has a great influence on the space based approximation methods. For example, the "distance" for a space may be defined as an integral over the underlying domain. In this case, the "distance" usually exists as a definite integral if functions in the space are continuous and the underlying domain is compact. If the underlying domain is non-compact, the "distance" is an improper integral over an unbounded interval and may not exist, or diverge.

In addition, the correct classification of functions in an abstract function space and the correct use of operations on the space are both important to a space based approximation theory.

Next, we shall review some of the existing approximation methods. We shall study the structures, properties, and function classifications in these methods and discuss their potentials to be used in a new approximation theory. Some of them can be directly used in the new approximation theory while others may need modifications.

Polynomial interpolations are successful approximation methods for continuous functions on closed intervals. There are also many practical algorithms to simplify the involved calculations, e.g. the Newton interpolation method (Issac Newton, 1643-1727) and the Lagrange interpolation method (Joseph-Louis Lagrange, 1736-1813). The weakness of polynomial interpolation methods is higher order polynomial interpolations may be oscillating and unstable – the well-known Runge's phenomenon (Carl Runge, 1856-1927). This is why lower order piecewise interpolations, such as the spline interpolation, are being used extensively. If the distribution of the interpolation nodes are carefully arranged, e.g. the Chebyshev distribution (Pafnuty Chebyshev, 1821-1894), then the uniform convergence for higher order interpolations can be achieved and the Runge's phenomenon can be avoided.

Applying abstract function space theories on polynomial interpolations on compact intervals gives a more general perspective on function approximations. By the Weierstrass approximation theorem (Karl Weierstrass, 1815-1897), any continuous function on a compact interval can be approximated by some polynomial at any prescribed precision. There are many proofs for the Weierstrass approximation theorem. A constructive proof by Bernstein (Sergei Bernstein, 1880-1968) has often been referred to. Bernstein polynomials can be perfectly described by the theories of abstract polynomial function spaces. This implies the Weierstrass approximation theorem has successfully expanded the polynomial approximation theory.

Another familiar early approximation method is the Taylor series expansion (Brook Taylor, 1685-1731). If a continuous function can be expanded into a power series about some center, then the partial sums of the series, which are polynomials, can be considered as approximation functions on some neighborhood of the expansion center. Thus, the Taylor series expansion is a local approximation method and the approximation is only valid inside a local interval where the Taylor series converges. If the convergence interval has a finite radius, the Taylor series expansion method cannot be extended on unbounded domains such as $[0, \infty)$.

The Fourier series expansion (Joseph Fourier, 1768-1830) is an important approximation method. It is successful in approximating continuous functions on closed intervals. More remarkably, it has profoundly changed the view of function space based approximation methods by introducing orthogonality to the space. Orthogonality is not defined in all spaces, but if it is in a space, the computation for the coefficients of the space's member functions may be greatly simplified.

Generally speaking, orthogonality comes from inner products. This implies if a space has orthogonality then it must be an inner product space, where "angles" between vectors are defined. In this case, a set of infinitely many orthogonal vectors can be used to form an orthogonal basis for the space. If the space is also complete, it becomes a Hilbert space (David Hilbert, 1862-1943). A finite Hilbert space is isomorphic to an Euclidean space. A function approximation in a Hilbert space is just a projection onto one of its subspaces.

Applying the Hilbert space theory to polynomial spaces enriches the theories and methods of polynomial approximations. Apart from the Fourier series spaces, orthogonal polynomial spaces are the most commonly used dense subsets of Hilbert spaces and there are various ways to construct orthogonal polynomial bases for a polynomial space. However, the orthogonal polynomial spaces suffer the same issue as other polynomial spaces on the unbounded domains. In addition, an approximation method does not necessarily depend on orthogonality.

Finally, we shall say a few words about iterative approximation methods. Generally, iterative approximation methods are numerical calculation techniques based on the Banach or the Hilbert space fixed point theory. The success of such methods relies on the conditions for uniform convergence. Similar to the series form solution of a differential equation, the solution of an iterative approximation method often gives finitely many terms of a convergent infinite series, which are often polynomials, and has no closed form expressions for the entire domain. This implies that the method is a local approximation method and the resulting series may have a finite radius of convergence. Thus, one should not expect the resulting approximation polynomial converges on the entire domain such as $[0, \infty)$.

1.1.4. *A new function space and new approximation methods.* In this thesis, we introduce a new function space for approximations, where the abstract function space theory is applied to the early approximation methods for the following two purposes. One purpose is to deal with the issue of non-compact domains. The other is to correctly build an approximation function class. This implies the member functions in the new function space are restricted with some prescribed properties so the new approximation method can take advantage of them.

Before we start to develop our new approximation theory, it is helpful to look at the basic definition of function approximations. Generally and intuitively, a function approximation problem may be described as "matching" an original function by another simple function such that the two functions are sufficiently "close". This definition does not imply a specific method to find the approximation function, nor does it specify what "close" means and leaves its definition for later researchers. By this primitive definition of approximations, we can usually tell if an approximation method is effective or not.

The above general approximation definition does raise a serious question: what is a good approximation? Sometimes an approximation satisfies the above definition but we do not think it to be a good one. For example, the piecewise linear or the spline interpolation is a simple and effective method to approximate a continuous function on a closed interval, and the resulting approximation function may be "smooth" enough for theoretical or practical uses and the approximation error converges to zero uniformly. However, by another objective criteria, we do not think such an approximation function is "good" enough for our approximation problem. In our extreme opinion, only a continuous function, which has a simple analytic expression on the entire domain and has infinitely many derivatives at any point, can be considered as a good approximation function. It is this criteria we believe accurately defines the essence of a good approximation.

To develop an approximation theory in which one can measure how "close" two functions are, we must introduce a new function space and define the distance between its members; this new space must be a normed function space.

Our approximation criteria implies the approximation functions in the new space must be analytic functions, which are infinitely differentiable. In addition, by choosing a particular class of analytic functions, we can define a "distance" for the space such that it always exists for any two member functions.

While we were developing the new function approximation theory, we found it makes more sense to construct not one fixed function space but a family of infinitely many function spaces for our approximation problem. This is not obvious at first sight, but as we progress, we will demonstrate this new idea is flexible and powerful in calculating member functions in the space, which is important in function approximations. Thus, instead of using a fixed function space, we introduce a family of function spaces controlled by a non-negative real parameter λ.

To develop an approximation method, we shall introduce some linear structures for the new function space. A linear structure means a basis of the space, and a space may have multiple linear structures. By analyzing the approximation problem into different linear structures, we are able to solve the problem from multiple perspectives. This implies we can have multiple solutions to an approximation problem by matching different linear structures. In fact, we have already successfully developed a few new space based approximation methods – the asymptotic (or Taylor) series expansion with variants, the Laplace transform moment matching, and the interpolation, in this thesis. By the theory of linear operators on function spaces, there is a linear mapping between every two linear structures. So the solutions from these approximation methods are all "equivalent" in some senses. However, the error pattern from each approximation method is unique and distinctive.

Finally, we introduce the concept of weak norms for the new function space in order to develop an approximation theory of convergence in weak norm on unbounded intervals. Suppose we wish to approximate a probability density function on $[0, \infty)$. There exists a point T such that the probability on $[0, T]$ is 0.95 and that on $[T, \infty)$ is 0.05. Clearly, the approximation to the probability density function on $[0, T]$ is more meaningful than on $[T, \infty)$. This does not mean the tail of the approximation function can be arbitrary. On the contrary, we wish to approximate the probability density function by a function with similar tail properties such as the boundedness with similar bounds and the tendency to zero at infinity.

To implement the convergence in weak norm, we only need to partition the underlying domain $[0, \infty)$ into two intervals: the main interval $[0, T]$ and the tail interval $[T, \infty)$. The weak norm of the function space on $[0, \infty)$ can be defined as the usual norm on the main interval $[0, T]$, a compact interval. Thus, we can apply existing techniques in the new approximation method which is convergent on $[0, T]$ in the sense of weak norm.

In summary, in this thesis, we shall develop a new theory and methods to approximate any bounded continuous vanishing at infinity functions on $[0, \infty)$. We shall construct an approximation function space, where every member function is continuous, bounded, integral-convergent, and vanishing at infinity on $[0, \infty)$, and the usual norm for each member function on $[0, \infty)$ is always well-defined. In order to address the problem the usual norm may diverge for some original functions on the unbounded interval $[0, \infty)$, we shall arbitrarily select a compact interval $[0, T]$, $T > 0$, where the main part of the original function is defined, and define the weak norm on it for both the original functions and the approximation functions. This can certainly be done because $[0, T]$ is compact. Thus, we can discuss the uniform convergence on $[0, T]$ instead of on $[0, \infty)$. In addition, by choosing some particular values for the parameter(s) of the approximation functions, we can make the approximation functions "nicely" bounded on the unbounded tail interval $[T, \infty)$.

1.1.5. *The future research: other applications of the new function space beyond function approximations.* Although the goal to build a new function space was initially for function approximations, we found the new space has many nice fundamental properties such that it can be useful in areas beyond function approximations. One idea is to study the operations on the new space. This will give new meanings to some old mathematical theories and methods.

Our new function space is useful where Laplace transforms are involved. In practical areas of probability and stochastic processes, signal processing and control, and numerical computations, rational Laplace transforms, which are meromorphic functions, play an important role in performing complicated operations such as convolutions. The new function space provides an algebraic way of computing convolutions or deconvolutions, since all the functions in the space have rational Laplace transforms. It is well-known the Laplace transform of a convolution of two functions is the product of their Laplace transforms. It follows that a convolution of two functions with rational Laplace transforms is also a rational Laplace transform.

In addition, since a Laplace transformation and its inverse transformation are linear operations, they can be represented by matrices in the new function space, which is commonly known in the theory of linear operators on function spaces. This idea of matrix representations for transformations may simplify the process of computing the transforms or the inverse transforms.

The new function space theory can help to approximately solve ordinary differential or integral equations. Problems in science and engineering often lead to ordinary differential equations with constant coefficients. Laplace transformation is usually used for solving such ordinary differential equations. If the operation can be represented by a rational Laplace transform, then the solution is also a rational Laplace transform. If the involved functions in the differential equation do not have rational Laplace transforms, or not even Laplace transforms, we can always find their approximations with rational Laplace transforms and solve the equation approximately. The argument is also true for integral equations as well.

The new function space and the new approximation methods can be used in numerical calculations. Evaluating a function at any point or calculating its integral over any interval can be done in the new function space approximately with an explicit analytical expression. This is of great advantage over some existing numeric methods, which only provide a numerical result at a point and do not provide the approximate expressions on the entire domain.

1.2. **Organizations.** In Chapter 2, we discuss the preliminary mathematical definitions and theorems on continuous functions, normed vector spaces, infinite series, and Laplace transforms, and in particular on asymptotic series expansions and the associated convergence issues. We include these generally accepted mathematical theories in this almost self-contained monograph not only for the purposes of definitions and notations but also as the necessary parts of a rigorous theory of a new function space and new approximation methods.

In Chapter 3, we introduce the decaying polynomial space $P_n^\lambda(\Omega)$. We discuss some of its elementary algebraic and topological properties, the isomorphism between the new space and the polynomial space or the Euclidean space, its subspace structures, and the linear operations on the new space.

In Chapter 4, we discuss a new function approximation theory based on $P_n^\lambda(\Omega)$ spaces and develop three kinds of approximation methods and their variants. Each approximation method depends on a particular linear structure of $P_n^\lambda(\Omega)$ and has a unique error pattern. Examples are given to demonstrate these new approximation methods.

Chapter 5 concludes the theory of $P_n^\lambda(\Omega)$ spaces and the new function approximation methods.

2. Preliminaries

This chapter contains the preliminary definitions, notations, propositions, and theorems used throughout this monograph. The monograph is almost self-contained. We give proofs of the propositions and theorems important to our new theory, but may exclude them for some generally accepted theorems. For references, see [17] [21].

In this section and throughout the monograph, we use $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ for real, complex, and natural numbers, respectively. Unless otherwise specified, we always denote $\Omega = [0, \infty)$ the non-negative real axis and assume all functions we will approximate are real-valued continuous functions defined or supported on the domain Ω.

2.1. **Continuous functions.**

2.1.1. *Limits of a function.* Let $A \subset \mathbb{R}$ be a subset and $c \in \mathbb{R}$ be an accumulation point of A. A real-valued function $f : A \to \mathbb{R}$ is said to have a finite *limit* L as t approaches c, denoted by

$$\lim_{t \to c} f(t) = L,$$

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $0 < |t - c| < \delta$, $t \in A$, we have

$$|f(t) - L| < \epsilon.$$

Note in the above definition, c need not be in A and $f(t)$ need not be defined at $t = c$.

We say $f(t)$ approaches positive infinity as t approaches c, denoted by

$$\lim_{t \to c} f(t) = \infty,$$

if for every $M > 0$, there exists a $\delta > 0$ such that for all $0 < |t - c| < \delta$, we have

$$f(t) > M.$$

Similarly, we can define

$$\lim_{t \to c} f(t) = -\infty.$$

It is convenient to define the one-sided limit of $f(t)$ at c. The left limit of $f(t)$ at c is denoted as

$$\lim_{t \to c^-} f(t) = L$$

if t approaches c and $t < c$. Similarly, we can write

$$\lim_{t \to c^+} f(t) = L$$

if t approaches c and $t > c$.

Consider a function $f(t)$ on $\mathbb{R}$. $f(t)$ has a limit L at infinity, denoted by

$$\lim_{t\to\infty} f(t) = L,$$

if for every $\epsilon > 0$, there is a $T > 0$ such that whenever $t > T$, we have

$$|f(t) - L| < \epsilon.$$

Similarly, we can define

$$\lim_{t\to-\infty} f(t) = L.$$

2.1.2. *Continuous functions.* Let $f(t)$ be a real-valued function defined on an interval $I \subset \mathbb{R}$. Then f is *continuous at* an interior point $c \in I$, if for every ϵ-neighborhood V of $f(c)$, there exists a δ-neighborhood U of c, contained in I, such that $f(U) \subseteq V$.

If f is continuous at every point in I, f is said to be *continuous on* I, which implies

$$f(c) = \lim_{t\to c} f(t) = \lim_{t\to c^+} f(t) = \lim_{t\to c^-} f(t)$$

for every interior point $c \in I$. If c is a boundary point of I, then the corresponding one-sided limit of f at c must equal to $f(c)$.

If f is not continuous at a point $c \in \mathbb{R}$, f is said to be *discontinuous* at c or has a *singularity* at c, which may be classified as one of the following cases:

(1) $\lim_{t\to c} f(t) = \pm\infty$;
(2) if f has a limit at c but it does not equal to the function value $f(c)$, then f is said to have a *removable discontinuous* point at c;
(3) if f has both left limit and right limit at c and they are different, f is said to have a *jump* at c; or
(4) if f has neither left limit nor right limit at c, then f is said to have an *essential discontinuous* point at c.

A continuous function f on an interval $I \subseteq \mathbb{R}$ is said to be *monotonically increasing* if $f(t_2) \geq f(t_1)$ whenever $t_2 \geq t_1,\ t_1, t_2 \in I$. f is said to be *monotonically decreasing* if $f(t_2) \leq f(t_1)$ whenever $t_2 \geq t_1,\ t_1, t_2 \in I$.

2.1.3. *Boundedness.* A function f is *bounded* on an interval $I \subseteq \mathbb{R}$, if there is a number $M > 0$ such that $|f(t)| \leq M$ for every $t \in I$. M is then called a *bound* for f on I.

Proposition 2.1. *Assume f and g are bounded on I. Then the following properties are true:*

(1) $f + g$ is bounded; and
(2) αf, $\alpha \in \mathbb{R}$, is bounded.

Proof. By hypothesis, f and g are bounded on I. Then there exist $M_1, M_2 > 0$ such that

$$|f(t)| \leq M_1 \quad \text{and} \quad |g(t)| \leq M_2$$

for all $t \in I$. It follows that

$$|f(t) + g(t)| \le |f(t)| + |g(t)| \le M_1 + M_2.$$

Thus, $f + g$ is bounded and the property (1) is proved. It also follows that

$$|\alpha f(t)| = |\alpha||f(t)| \le |\alpha| M_1.$$

Thus, αf is bounded and the property (2) is proved. □

2.1.4. *Compact sets.* A set K is *compact* if every sequence in K has a subsequence converging to a limit in K. An equivalent definition of compact set is that a set K is *compact* if every open cover of K has a finite subcover. For example, in $\mathbb{R}$, the closed interval $[0, 1]$ is compact but an open interval $(0, 1)$ is not.

Proposition 2.2. *A set $K \subset \mathbb{R}$ is compact if and only if it is closed and bounded.*

Proof. Let K be compact. Then every sequence in K has a subsequence converging to a limit in K. This implies K is closed. Assume K is not bounded. Then there exists a divergent sequence $\{x_i\}_{i\in\mathbb{N}}$ in K. For any $M > 0$, there is at least one x_j for some $j \in \mathbb{N}$ such that $|x_j| > M$. Consider an open cover $\{(-i, i)\}_{i\in\mathbb{N}}$. Clearly,

$$K \subset \cup_{i\in\mathbb{N}}(-i, i).$$

For any finite subcover

$$\cup_{i=1}^{N}(-n_i, n_i),\ n_1, n_2, \cdots, n_N \in \mathbb{R},$$

set $M = \max(n_1, n_2, \cdots, n_N)$. The fact there exists an x_j such that $|x_j| > M$ implies K is not covered by the subcover. This contradicts the compactness of K. Thus K is bounded.

Conversely, let K be closed and bounded. Clearly, every sequence in K is bounded since K is bounded. By the Bolzano-Weierstrass theorem, each bounded sequence has a convergent subsequence. Since K is closed, the limit of the subsequence is in K. Thus K is compact. □

Proposition 2.3. *Let K be a compact set and $f : K \to \mathbb{R}$ be a continuous function. Then $f(K)$ is compact. In other words, f is bounded and attains its maximum/minimum on K.*

Proof. (First proof). Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence in $f(K)$, not necessarily convergent. By continuity of f, there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \in K$ and $y_n = f(x_n)$ for all n. Since K is compact, $\{x_n\}$ has a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ converging to a limit $x \in K$. Set $y = f(x)$. By continuity of f, the subsequence of $\{y_n\}$ converges to y, i.e.

$$\lim_{i\to\infty} y_{n_i} = y.$$

Since $y \in f(K)$, $f(K)$ is compact. This also implies $f(K)$ is bounded and f attains its maximum/minimum for some $x \in K$. □

Proof. (Second proof). Let $\cup_{n\in\mathbb{N}} V_n$ be an open cover for $f(K)$. Since f is continuous, for each open set V_n for $f(K)$, there is an open set U_n of K such that $f(U_n) \subseteq V_n$. Then there is an open cover $\cup_{n\in\mathbb{N}} U_n \supset K$. Since K is compact, there is a finite subcover $U_{n_1} \cup U_{n_2} \cup \cdots \cup U_{n_N} \supset K$. For every point $x \in K$, $x \in U_{n_i}$ for some i, $1 \le i \le N$. Set $y = f(x)$. Then $y \in f(U_{n_i})$ implies $y \in V_{n_i}$. Thus $V_{n_1} \cup V_{n_2} \cup \cdots \cup V_{n_N} \supset f(K)$ and $f(K)$ is compact. □

2.1.5. *Vanishing at infinity.* Let f be a continuous function on Ω. Then f is said to *vanish at infinity*, denoted by

$$\lim_{t\to\infty} f(t) = 0,$$

if for every $\epsilon > 0$, there exists a $T > 0$ such that whenever $t > T$, we have

$$|f(t) - 0| < \epsilon.$$

Proposition 2.4. *(Tail boundedness). If a continuous function f on Ω vanishes at infinity, then for any $\epsilon > 0$ there is a compact set $K \subset \Omega$ such that $|f| < \epsilon$ on $\Omega \backslash K$, the complement of K in Ω.*

Proof. By our definition of vanishing at infinity, for every $\epsilon > 0$, there is a $T > 0$ such that whenever $t > T$, we have

$$|f(t) - 0| < \epsilon.$$

Let $K = [0, T]$. Then $|f(t)| < \epsilon$ on $\Omega \backslash K$. □

Proposition 2.5. *(Global boundedness). A continuous function f which vanishes at infinity is bounded on Ω.*

Proof. This follows from Propositions 2.3 and 2.4. We can also prove it by contradiction. Assume f is unbounded on Ω. Since f vanishes at infinity, then for any $\epsilon > 0$, there exists a number $b \in \Omega$ such that $|f(t)| < \epsilon$ for $t > b$. Let $a = 0$. Then f is unbounded on $[a, b]$. Let t be such that $a < t < b$. Then $f(t)$ is unbounded on either $[a, t]$ or $[t, b]$. Let the closed interval where f is unbounded be $[a_1, b_1]$. Then f is unbounded on $[a_1, b_1]$. Repeat this process and we obtain a sequence of intervals $[a_n, b_n], n = 1, 2, \cdots$, with $[a_n, b_n] \subset [a_{n-1}, b_{n-1}]$, on which f is unbounded. Thus there is a unique number $\xi \in [a_n, b_n]$ for all $n = 1, 2, \cdots$, such that $|f(\xi)|$ is greater than any given number. This implies that

$$\lim_{t\to\xi} |f(t)| = \infty$$

and $f(t)$ has a singularity at $t = \xi$. This is a contradiction, since $f(t)$ is continuous at ξ. Thus, f is bounded on Ω. □

Proposition 2.6. *Let f and g vanish at infinity on Ω. Then the following properties are true:*

(1) $f + g$ vanishes at infinity;
(2) αf, $\alpha \in \mathbb{R}$, vanishes at infinity; and
(3) fg vanishes at infinity.

Proof. We shall repeatedly use Proposition 2.4, the tail boundedness, in the proof.

Let $\epsilon > 0$. By hypothesis, f and g on Ω vanish at infinity. Then there is a compact set $K_1 \subset \Omega$ such that $|f| < \frac{\epsilon}{2}$ on $\Omega \backslash K_1$ and a compact set $K_2 \subset \Omega$ such that $|g| < \frac{\epsilon}{2}$ on $\Omega \backslash K_2$. Then

$$|f + g| \leq |f| + |g| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

on $\Omega \backslash K_1 \cup K_2$. Thus, $f + g$ vanishes at infinity and the property (1) is proved.

Let $\alpha \in \mathbb{R}$. By hypothesis, f on Ω vanishes at infinity. Then there is a compact set $K \subset \Omega$ such that $|f| < \frac{\epsilon}{|\alpha|}$ on $\Omega \backslash K$. Then

$$|\alpha f| \leq |\alpha||f| < |\alpha| \frac{\epsilon}{|\alpha|} = \epsilon$$

on $\Omega \backslash K$. Thus, αf vanishes at infinity and the property (2) is proved.

Let $M > 0$. By hypothesis, f and g on Ω vanish at infinity, Then there is a compact set $K_1 \subset \Omega$ such that $|f| < M$ on $\Omega \backslash K_1$ and a compact set $K_2 \subset \Omega$ such that $|g| < \frac{\epsilon}{M}$ on $\Omega \backslash K_2$. Then

$$|fg| \leq |f||g| < \frac{\epsilon}{M} M = \epsilon$$

on $\Omega \backslash K_1 \cup K_2$. Thus, fg vanishes at infinity and the property (3) is proved. $\square$

2.1.6. *Integration.* Let $A \subset \mathbb{R}$. A function $f : A \to \mathbb{R}$ is *Lebesgue measurable*, or *measurable*, if for any c real, the set

$$\{t \in A \mid f(t) > c\}$$

is measurable. Since we only concerns continuous function on intervals in this thesis and intervals are measurable, we shall not discuss more general cases for measurable sets. A continuous function is measurable.

Let f be a measurable function on an interval $I \subset \mathbb{R}$. Then f is said to be *integrable* or *Riemann integrable* on I, if the *Riemann integral*

$$\int_I f(t) dt < \infty.$$

Riemann integrable implies *Lebesgue integrable.* If the *integrand* f is continuous, then both its Riemann integral and Lebesgue integral exist and are equal.

The integral of a continuous function on a closed interval is called a *definite integral.*

For a continuous integrand, an *improper integral of the first kind* is the limit of a definite integral over an interval whose upper or lower end-point tends to infinity. If this limit is finite, the improper integral *exists* or *converges*. Otherwise, it *diverges*. In particular, let f be a continuous function on Ω. Then f is said to be *integral-convergent* on Ω, denoted by

$$\int_0^\infty f(\tau)d\tau = \lim_{T\to\infty}\int_0^T f(\tau)d\tau = L$$

for some finite L, if for any $\epsilon > 0$, there exists a $T > 0$ such that whenever $t > T$, we have

$$\left|\int_0^t f(\tau)d\tau - L\right| = \left|\int_t^\infty f(\tau)d\tau\right| < \epsilon.$$

f is said to be *integral-divergent* on Ω if it is not integral-convergent.

Integral-convergent continuous functions on Ω are very common in the fields of applied mathematics, science, and engineering. For example, a continuous probability density function on Ω is integral-convergent.

If the integrand of an integral on a closed interval is unbounded and tends to positive or negative infinity at a point in the interval, then the integral is said to be an *improper integral of the second kind*. This kind of improper integral is not our main subject for this monograph.

Proposition 2.7. *Let f and g be integral-convergent on Ω. Then the following properties are true:*

(1) $f + g$ is integral-convergent on Ω; and
(2) αf, $\alpha \in \mathbb{R}$, is integral-convergent on Ω.

Proof. By hypothesis, f and g on Ω are integral-convergent on Ω. Then

$$\int_0^\infty f(\tau)d\tau = L_1 \quad \text{and} \quad \int_0^\infty g(\tau)d\tau = L_2$$

for some finite L_1 and L_2. It follows that

$$\int_0^\infty f(\tau) + g(\tau)d\tau = \int_0^\infty f(\tau)d\tau + \int_0^\infty g(\tau)d\tau = L_1 + L_2.$$

Thus, $f+g$ is integral-convergent and the property (1) is proved. It also follows that

$$\int_0^\infty \alpha f(\tau)d\tau = \alpha\int_0^\infty f(\tau)d\tau = \alpha L_1.$$

Thus, αf is integral-convergent and the property (2) is proved. □

Proposition 2.8. *(Cauchy convergence criterion for improper integrals). A continuous function f is integral-convergent on Ω if and only if for any $\epsilon > 0$, there exists a $T > 0$ such that whenever $t_1, t_2 > T$, we have*

$$\left|\int_{t_1}^{t_2} f(\tau)d\tau\right| < \epsilon.$$

Proof. Let f be integral-convergent on Ω and

$$\int_0^\infty f(\tau)d\tau = L$$

for some finite L. Then for any $\epsilon > 0$, there exists a $T > 0$ such that whenever $t > T$, we have

$$\left| \int_0^t f(\tau)d\tau - L \right| < \frac{\epsilon}{2}.$$

Thus,

$$\begin{aligned} \left| \int_{t_1}^{t_2} f(\tau)d\tau \right| &= \left| \int_0^{t_2} f(\tau)d\tau - \int_0^{t_1} f(\tau)d\tau \right| \\ &= \left| \left(\int_0^{t_2} f(\tau)d\tau - L \right) - \left(\int_0^{t_1} f(\tau)d\tau - L \right) \right| \\ &\leq \left| \left(\int_0^{t_2} f(\tau)d\tau - L \right) \right| + \left| \left(\int_0^{t_1} f(\tau)d\tau - L \right) \right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

The proof for the converse statement is omitted.

□

Proposition 2.9. *An integral-convergent continuous function f on Ω is bounded.*

Proof. By contradiction. Let f be continuous and integral-convergent on Ω. By Proposition 2.8, for any $\epsilon > 0$, there exists a $T > 0$ such that whenever $t_1, t_2 > T$, we have

$$\left| \int_{t_1}^{t_2} f(\tau)d\tau \right| < \epsilon.$$

Clearly, f is bounded on $[0, T]$. Without loss of generality, assume f is positively unbounded at infinity on $[T, \infty)$. Then there exists an S such that whenever $t > S$, we have $f(t) > 1$. Let $t_1 > \max(T, S)$ and $t_2 = t_1 + \epsilon$. Then

$$\left| \int_{t_1}^{t_2} f(\tau)d\tau \right| > (t_2 - t_1) = \epsilon.$$

This is a contradiction. Thus, f is bounded on $[T, \infty)$ and consequently on Ω. □

Proposition 2.10. *Let f be continuous and integral-convergent on Ω. If f has a limit at infinity, then f vanishes at infinity.*

Proof. By contradiction. Without loss of generality, we assume

$$\lim_{t \to \infty} f(t) = l > 0.$$

Then for any sufficiently small $\epsilon > 0$, there exists a $T > 0$ such that whenever $t > T$, we have

$$0 < l - \epsilon < f(t) < l + \epsilon.$$

Since f is integral-convergent, by Proposition 2.8, there exists an S such that whenever $t_1, t_2 > S$, we have

$$\left| \int_{t_1}^{t_2} f(t)dt \right| < l - \epsilon.$$

Let $t_1 > \max(T, S)$ and $t_2 = t_1 + 1$. Then

$$\left| \int_{t_1}^{t_2} f(t)dt \right| > \left| \int_{t_1}^{t_2} l - \epsilon dt \right| = l - \epsilon.$$

This is a contradiction. Therefore, l cannot be positive. Similarly, l cannot be negative. It follows that $l = 0$ and f vanishes at infinity. □

Proposition 2.11. *If $|f|$ is integral-convergent on Ω, then f is also integral-convergent on Ω.*

Proof. By hypothesis, $|f|$ is integral-convergent on Ω. Then for any $\epsilon > 0$, there exists a $T > 0$ such that whenever $t > T$, we have

$$\left| \int_{t}^{\infty} |f(\tau)|d\tau \right| < \epsilon.$$

It follows that

$$\left| \int_{t}^{\infty} f(\tau)d\tau \right| \leq \left| \int_{t}^{\infty} |f(\tau)|d\tau \right| < \epsilon$$

and f is integral-convergent on Ω. □

If $|f|$ is integral-convergent on Ω, then f is said to be *absolutely integral-convergent* on Ω. The converse of Proposition 2.11 is not always true.

Proposition 2.12. *Let f be absolutely integral-convergent and g continuous and bounded on Ω. Then fg is also absolutely integral-convergent on Ω.*

Proof. The hypothesis g is continuous and bounded implies $|g| \leq M$ for some $M > 0$ on Ω. By hypothesis f is absolutely integral-convergent on Ω, so for any $\epsilon > 0$, there exists a $T > 0$ such that whenever $t > T$, we have

$$\int_{t}^{\infty} |f(\tau)|d\tau < \frac{\epsilon}{M}.$$

It follows that

$$\begin{aligned}\left|\int_t^\infty |f(\tau)g(\tau)|d\tau\right| &\leq \int_t^\infty |f(\tau)||g(\tau)|d\tau \\ &\leq \int_t^\infty |f(\tau)|M d\tau \\ &< \frac{\epsilon}{M}M = \epsilon\end{aligned}$$

Thus, fg is absolutely integral-convergent. □

Let f be continous on Ω. Then f is said to be n-th *moment integral-convergent*, $n \in \mathbb{N}$, if

$$\left|\int_0^\infty \tau^n f(\tau)d\tau\right| = L$$

for some finite L.

Let f and g be two measurable functions on $\mathbb{R}$. Then f is said to be *Riemann-Stieltjes integrable* with respect to g on an interval $I \subseteq \mathbb{R}$ if

$$\int_I f(t)dg(t) = L$$

for some finite L. This is equivalent to

$$\int_I f(t)g'(t)dt = L,$$

if g is differentiable on any open subsets of $\mathbb{R}$ containing I. The Riemann-Stieltjes integration is a generalization of the usual integration and is useful in function transformations.

2.1.7. *Analytic and transcendental functions.* Let f be a real-valued function on an interval $I \subseteq \mathbb{R}$. Then f is *analytic* at a point $c \in I$ if f can be represented as a power series

$$f(t) = \sum_{n=0}^{\infty} a_n(t-c)^n, \quad a_0, a_1, \cdots \in \mathbb{R},$$

in some ϵ-neighborhood of c contained in I. If f is analytic at every point in I, then f is said to be analytic on I. If f is analytic at all points in I except b, then b is called a *singular* point of f in I. Real analytic functions are special cases of complex analytic functions. Analytic functions are strongly related to infinite function series, which will be discussed in detail in section 2.5.

An *algebraic function* is a function which can be expressed in the form of a finite sequence of algebraic operations. For example, polynomial and rational functions are algebraic. A *transcendental function* is an analytic function that does not satisfy a polynomial equation. Examples of transcendental functions are: e^t, $\sin(x)$, and Bessel functions. Transcendental functions are infinitely differentiable, and may be generated from indefinite integrals of algebraic functions.

A *bounded transcendental function* is a transcendental function which is bounded. Based on their limit at infinity, bounded transcendental functions can be classified into three categories:

(1) no limit at infinity, e.g. $\sin(x)$;
(2) a non-zero finite limit at infinity, e.g. $1 - e^{-t}$; or
(3) vanishing at infinity, e.g. e^{-t}.

2.2. **Normed vector spaces.**

2.2.1. *Linear vector spaces.* An *additive group*, or an *Abelian group*, is a set X with an operation $+$ satisfying the following axioms:

(1) $x + y \in X,$ if $x, y \in X$;
(2) $x + y = y + x,\ x, y \in X$;
(3) $x + (y + z) = (x + y) + z,\ x, y, z \in X$;
(4) $-x \in X$; and
(5) $0 \in X$.

A *linear vector space* over a field $\mathbb{F}$ (the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers) is an additive group X with an operation $\times : \mathbb{F} \times X \to X$ satisfying the following axioms:

(1) $\alpha(x + y) = \alpha x + \alpha y,\ \alpha \in \mathbb{F},\ x, y \in X$;
(2) $(\alpha + \beta)x = \alpha x + \beta x,\ \alpha, \beta \in \mathbb{F},\ x \in X$;
(3) $\alpha(\beta x) = (\alpha\beta)x,\ \alpha, \beta \in \mathbb{F},\ x \in X$; and
(4) $1x = x,\ x \in X$.

A linear vector space is denoted by $(X, \mathbb{F}, (+, \times))$, or simply by X.

A linear vector space is often simply called a *vector space* or a *linear space*. The elements of a vector space are called *vectors* or *points*. The elements of the associated field are called *scalars*. A linear vector space is a set of points equipped with a linear structure and is closed under addition and scalar multiplication.

Let X be a vector space over a field $\mathbb{F}$. If $x, y, \cdots, z \in X$ and $\alpha, \beta, \cdots, \gamma \in \mathbb{F}$, then the sum $\alpha x + \beta y + \cdots + \gamma z$ is called a *linear combination* of $x, y, \cdots, z$, which is also an element in X. This follows from the axioms of a vector space.

Let X be a vector space. Let $B = \{x, y, \cdots, z\}$ of distinct vectors and $B \subseteq X$. Then B is said to be *linearly dependent* if there exists a set of scalars $\alpha, \beta, \cdots, \gamma$, not all zeros, such that $\alpha x + \beta y + \cdots + \gamma z = 0$. B is said to be *linearly independent* if B is not linearly dependent.

Let B be linearly independent. If every vector in X can be uniquely represented by a linear combination of vectors in B, then X is said to be *spanned* by B, or a *span* of B, denoted by $\operatorname{span} B$, and B is called a *basis* of X.

Let B be a basis of a vector space X. The cardinality of B is called the *dimension* of X, denoted by $\dim X$. Thus, a vector space is finite-dimensional if $\dim X$ is finite, countably infinite-dimensional if $\dim X = \aleph_0$, the cardinality of the natural number, or uncountably infinite-dimensional if $\dim X = \mathfrak{c}$, the cardinality of the continuum.

Let X be a vector space. A subset $X_1 \subseteq X$ is called a *linear subspace* or a *subspace* in X if and only if X_1 itself is a vector space and $\dim X_1 \leq \dim X$.

Proposition 2.13. *Let B be a basis of X. Let B_1 be any non-trivial subset of B of finite or countably infinite size. Let $X_1 = \operatorname{span} B_1$. Then B_1 contains linearly independent vectors, and X_1 is a subspace of X.*

Proof. Trivial. □

X is called a *linear superspace*, or a *superspace* to X_1, if X_1 is a subspace of X. Clearly, $\dim X \geq \dim X_1$. A superspace X_n, $n \in \mathbb{N}$, may be constructed in the following way. Let $B_1 = \{x_1\}$ be a set of one vector and be the basis of X_1. Let $x_2 \notin B_1$ be a new vector such that $B_2 = \{x_2\} \cup B_1$ is linearly independent. Let $X_2 = \operatorname{span} B_2$. Then X_2 is a superspace of X_1. Repeat the above process in finitely many steps. Then there is a vector space X_n such that X_n is a superspace to every X_k, $k \leq n, k \in \mathbb{N}$. Clearly, $\dim X_n = n$. In addition, it is obvious that X_m is a superspace of X_n if and only if $m \geq n$, $m, n \in \mathbb{N}$.

Let X and Y be two vector spaces. A mapping $T : X \to Y$ is said to be a *linear operator* on X into Y if it satisfies the following conditions:

(1) $T(x_1 + x_2) = Tx_1 + Tx_2,\ x_1, x_2 \in X$; and
(2) $T(\alpha x) = \alpha Tx,\ x \in X, \alpha \in \mathbb{F}$.

The domain of T, denoted by $\mathscr{D}(T)$, is the set of $x \in X$, on which T is defined. The range of T, denoted by $\mathscr{R}(T)$, is the set of $y \in Y$ such that $y = Tx$ for every $x \in X$. A linear operator is also called a *linear transformation*, a *linear mapping*, or a *linear function*. In particular, if Y is a field $\mathbb{F}$, a linear operator is also called a *linear functional*.

Let T and S be two linear operators on X into Y. Define the sum of T and S and the scalar multiplication of T as

(1) $(T + S)x = Tx + Sx$, $x \in X$; and
(2) $(\alpha T)x = \alpha(Tx)$, $x \in X$, $\alpha \in \mathbb{F}$,

respectively. All linear operators from X to Y form a vector space, called a *linear operator space*.

Let X, Y, and Z be vector spaces and $T : X \to Y$ and $S : Y \to Z$ be two linear operators. Then the *composite operator* ST is a linear operator from X to Z formed by the product of operators S and T such that $(ST)x = S(Tx)$.

Let T be a linear operator on X to X. Let the symbol T^2 represents the operator TT and inductively T^n for $T^{n-1}T$ or TT^{n-1}. An operator $I : X \to X$

is called an *identity operator* if $Ix = x$, for all $x \in X$. An operator O is called a *zero operator* if $Ox = 0$, the *zero vector* in X, for all $x \in X$. Thus, the following is a valid expression of a linear operator:

$$P(T) = \alpha_0 I + \alpha_1 T + \cdots + \alpha_n T^n,$$

where $\alpha_i \in \mathbb{F}$, $i = 0, 1, \cdots, n$.

Two vector spaces X and Y over the same field $\mathbb{F}$ are said to be *isomorphic* if there exists a one-to-one operator T that maps every vector of X onto a vector of Y. This definition also implies there exists an *inverse operator* T^{-1} mapping Y onto X.

Let X and Y be two subspaces of a vector space Z and $X \cap Y = \{0\}$. If each $z \in Z$ can be written uniquely as $z = x + y$, where $x \in X$ and $y \in Y$, then Z is said to be a *direct sum* of X and Y, denoted by $Z = X \oplus Y$. The idea of a direct sum is to expand a vector space to a higher dimensional vector space.

2.2.2. *Metric spaces.* Let X be a non-empty set. A *metric* or *distance* on X is a non-negative function $\rho : X \times X \to [0, \infty)$ such that for every x, y, and z in X and α real,

(1) $\rho(x, y) \geq 0$, and $\rho(x, y) = 0$ iff $x = y$;
(2) $\rho(x, y) = \rho(y, x)$; and
(3) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

The set X together with the metric ρ, denoted by a couple (X, ρ), or simply by X, is called a *metric space*.

In a metric space, the metric measures how close one element is to another. The smaller the metric value, the closer the two elements. If the metric value is zero, the two elements are said to be *equal* in the metric sense.

In the above metric definition, if $\rho(x, y) = 0$ does not imply $x = y$, then ρ is called a *pseudometric* on X and (X, ρ) is called a *pseudometric space*.

In a metric space (X, ρ), the *diameter* of a subset U of X is defined as

$$\operatorname{diam} U = \sup \{\rho(x, y) \mid x, y \in U\}.$$

Thus, if $U \subseteq V$, then $\operatorname{diam} U \leq \operatorname{diam} V$. If $\operatorname{diam} U < M$, $M > 0$, then U is said to be *bounded*.

Let (X, ρ) be a metric space and $r > 0$. For a point $x \in X$, the set of points of X, denoted by

$$N_r(x) = N(x, r) = \{y \in X \mid \rho(x, y) < r\},$$

is called an *open ball*, or an *open neighborhood*, centered at x with a radius r. In $\mathbb{R}$ with Euclidean distance, an open ball $N_r(x)$ is an *open interval* $(x - r, x + r)$.

In a metric space (X, ρ), a subset U of X is said to be *open* if, for each point $x \in U$, there exists an $r > 0$ such that $N_r(x) \subset U$. A subset U of X is said to be *closed* if the complement of U in X is open.

Open set is a concept of topological space, generalizing the idea of an open interval in $\mathbb{R}$ and is fundamental to such concepts as limit, convergence, and continuity.

A family τ of subsets of a non-empty set X is called a *topology* on X if

(1) $X, \phi \in \tau$;
(2) if $A, B \in \tau$, then $A \cap B \in \tau$; and
(3) if $A_\alpha \in \tau$, $\alpha \in I$, an index set, then $\cup_{\alpha \in I} A_\alpha \in \tau$.

The pair (X, τ), or simply τ, denotes a *topological space*, whose members are called *open sets* .

Proposition 2.14. *Let (X, ρ) be a metric space. Then the collection of usual open sets defined by the metric ρ forms a topology τ on X. This topology is also called a topology* induced by the metric ρ, *or a* metric topology*.*

Proof. We shall prove the members of τ satisfy the axioms of topology.

(1) Both X and ϕ are open sets by definition. Thus, $X, \phi \in \tau$.
(2) Let $A, B \in \tau$. Then A and B are open. For every $x \in A \cap B$, we have $x \in A$ and $x \in B$. By definition of open sets, there exist an $N_{r_1}(x) \subset A$ and an $N_{r_2}(x) \subset B$. Let $r = \min(r_1, r_2)$. Then $N_r(x) \subset A$ and $N_r(x) \subset B$ implies $N_r(x) \subset A \cap B$. Thus $A \cap B$ is open, i.e. $A \cap B \in \tau$.
(3) Let $A_\alpha \in \tau$, $\alpha \in I$, an index set. Then A_α are open. For every $x \in \cup_{\alpha \in I} A_\alpha$, $x \in A_\alpha$ for some $\alpha \in I$. Then there exists an $N_r(x) \subset A_\alpha \subset \cup_{\alpha \in I} A_\alpha$. It implies that $\cup_{\alpha \in I} A_\alpha$ is open, i.e. $\cup_{\alpha \in I} A_\alpha \in \tau$.

Therefore, τ is indeed a topology (induced by ρ) on X and (X, τ) is a topological space. Clearly, every metric induces a topology on a metric space. □

Let X be a topological space and x, y distinct elements of X. If there always exist neighborhoods $N_{\epsilon_1}(x)$ and $N_{\epsilon_2}(y)$ for some $\epsilon_1, \epsilon_2 > 0$ such that $N_{\epsilon_1}(x) \cap N_{\epsilon_2}(y) = \phi$, then X is called a *Hausdorff* space. This definition implies that any two distinct elements can be separated in a Hausdorff space. Since any two distinct elements in a metric space can be separated by open sets, a metric space is a Hausdorff space. We have already implicitly used this separation property in the above proofs.

Let (X, τ_1) and (X, τ_2) be two topological spaces. If $\tau_1 \subseteq \tau_2$, then τ_2 is *stronger* than τ_1, and τ_1 is *weaker* than τ_2.

Let (X, ρ) be a metric space. A sequence $\{x_n\}$ in X is said to *converge* to a *limit* point x if

$$\lim_{n \to \infty} \rho(x_n, x) = 0.$$

Note x may not be in X. A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if

$$\lim_{m,n \to \infty} \rho(x_m, x_n) = 0.$$

Proposition 2.15. *Every convergent sequence in a metric space is Cauchy.*

Proof. Let (X, ρ) be a metric space. Consider a sequence $\{x_n\}$ in X converging to an $x \in X$. Then for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\rho(x_m, x) < \frac{\epsilon}{2} \quad \text{and} \quad \rho(x_n, x) < \frac{\epsilon}{2},$$

whenever $m, n > N$. Thus,

$$\rho(x_m, x_n) \leq \rho(x_n, x) + \rho(x_m, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

whenever $m, n > N$. Therefore,

$$\lim_{m,n\to\infty} \rho(x_m, x_n) = 0,$$

i.e. $\{x_n\}$ is Cauchy. $\square$

A Cauchy sequence in a metric space may converge to a point not in the space. Consider a metric space $(X, \rho(x, y) = |x - y|)$ with $X = (0, 1]$. A convergent sequence $\{\frac{1}{n}\}$ in X, which is Cauchy, converges to $0 \notin X$. Thus, not every Cauchy sequence in a metric space is convergent to a point in the space.

A metric space X is *complete* if every Cauchy sequence in X converges to a limit point that is also in X. For example, the set of rational numbers $\mathbb{Q}$ is not complete but the set of real numbers $\mathbb{R}$ is, if both sets are equipped with the metric $\rho(x, y) = |x - y|$. Furthermore, the set of complex numbers $\mathbb{C}$ equipped with the modulus metric $|\cdot|$ is a complete metric space. The proofs of the above statements are trivial.

If a metric space is not complete, it can be extended into a larger metric space by adding all the limit points of its Cauchy sequences. Then the new metric space is complete. This process is called the completion of a metric space.

Let (X_1, ρ_1) and (X_2, ρ_2) be two metric spaces. Then a function $f : X_1 \to X_2$ is said to be *uniformly continuous* if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $x, y \in X_1$ with $\rho_1(x, y) < \delta$, we have $\rho_2(f(x), f(y)) < \epsilon$.

In particular, a function $f : I \to \mathbb{R}$, where I is an interval, is uniformly continuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $t_1, t_2 \in I$, whenever $|t_1 - t_2| < \delta$, we have $|f(t_1) - f(t_2)| < \epsilon$.

2.2.3. *Normed vector spaces.* A *norm* on a vector space (or a metric space) X is a functional $\|\cdot\| : X \to [0, \infty)$ satisfying, for all $x, y \in X$ and α real, the following properties:

(1) $\|x\| \geq 0$, $\|x\| = 0$ iff $x = 0$; (Positivity)
(2) $\|\alpha x\| = |\alpha|\|x\|$; (Homogeneity) and
(3) $\|x + y\| \leq \|x\| + \|y\|$. (The triangle inequality)

A vector space X equipped with a norm $\|\cdot\|$ is called a *normed vector space*, or a *normed space*, denoted by $(X, \|\cdot\|)$, or simply by X, if the norm is implied. A vector space X may be equipped with different norms, resulting in different normed spaces.

A normed space is also a metric space and a topological space.

Proposition 2.16. *Every norm on X induces a metric and therefore induces a topology on X.*

Proof. Let $(X, \|\cdot\|)$ be a normed space and $x, y \in X$. Define $\rho(x, y) = \|x - y\|$. Clearly, ρ satisfies all the axioms of a metric. Thus ρ is a metric induced by the norm $\|\cdot\|$ and (X, ρ) is a metric space. By Proposition 2.14, ρ induces a metric topology on X. $\square$

Proposition 2.17. *A norm is uniformly continuous.*

Proof. Let X be a vector space and consider a norm $\|\cdot\| : X \to [0, \infty)$. By Proposition 2.16, the norm $\|\cdot\|$ induces a metric $\rho(x, y) = \|x - y\|$ on X. For every $x, y \in X$, by the triangle inequality in the definition of norm,

$$\|x\| = \|(x - y) + y\| \le \|y\| + \|x - y\|$$

and

$$\|y\| = \|(y - x) + x\| \le \|x\| + \|y - x\|.$$

This implies

$$\|x\| - \|y\| \le \|x - y\|$$

and

$$\|y\| - \|x\| \le \|y - x\|.$$

That is

$$|\|x\| - \|y\|| \le \|x - y\|.$$

Thus, for every $\epsilon > 0$, there is a $\delta = \epsilon$ such that whenever $\rho(x, y) = \|x - y\| < \delta$, $|\|x\| - \|y\|| < \epsilon$. It follows from the definition of uniform continuity that $\|\cdot\|$ is uniformly continuous. $\square$

Let X be a vector space equipped with two norms $\|\cdot\|_a$ and $\|\cdot\|_b$. Then $\|\cdot\|_a$ and $\|\cdot\|_b$ on X are *equivalent* if for each $x \in X$ there exist some real α and β such that

$$\alpha\|x\|_b \le \|x\|_a \le \beta\|x\|_b.$$

In this monograph, we use a generic norm symbol $\|\cdot\|$ for a normed space unless a specific norm needs to be emphasized.

Furthermore, a complete normed space is called a *Banach space*.

2.2.4. *Convergence in weak norm.* In a normed space, a norm and its induced metric are called the *strong norm* and the *strong metric*, respectively. There are many other ways to define "weak" norms and metrics for the space.

Let X be a normed space. A functional $\|\cdot\|^w : X \to [0,\infty)$, different from the usual norm inducing the topology and satisfying the norm axioms, is called a *weak norm*, if for every convergent sequence $\{x_n\}$ in X converging to a point $x \in X$ in the usual norm, we have

$$\lim_{n\to\infty} \|x_n\|^w = \|x\|^w.$$

The functional $\rho^w(x,y) = \|x-y\|^w$ is called a *weak metric* induced by this weak norm $\|\cdot\|^w$, if

$$\lim_{n\to\infty} \|x_n - x\|^w = 0.$$

In this case, $\{x_n\}$ is said to *converge in weak norm* to x.

2.3. **Normed continuous function spaces on Ω.** This section discusses various continuous function spaces.

2.3.1. *The continuous function space $C(a,b)$.* Let $C(a,b)$ be the set of all real-valued continuous functions on $I = [a,b]$. Define the *essential norm* $\|\cdot\|_\infty : C(a,b) \to [0,\infty)$ as

$$\|x\|_\infty = \sup_{t\in I} |x(t)| = \max_{t\in I} |x(t)|.$$

It follows from Proposition 2.3 that the definition of $\|\cdot\|_\infty$ is valid and can be used for other normed continuous function spaces as well.

Proposition 2.18. *$(C(a,b), \|\cdot\|_\infty)$ is a normed vector space.*

Proof. Firstly, it is obvious the set of continuous functions are closed under addition and scalar multiplication. Thus, $C(a,b)$ is a vector space. It is left to prove $\|\cdot\|_\infty$ satisfies the axioms of a norm.

(1) For any $x \in C(a,b)$, if $\|x\|_\infty = 0$, then $\sup_{t\in[a,b]} |x(t)| = 0$. This implies $x(t) = 0$ for all $t \in [a,b]$.
(2) For any $x \in C(a,b)$,

$$\begin{aligned}\|\alpha x\|_\infty &= \sup_{t\in[a,b]} |\alpha x(t)| \\ &= |\alpha| \sup_{t\in[a,b]} |x(t)| = |\alpha| \|x(t)\|_\infty.\end{aligned}$$

(3) For any $x, y \in C(a,b)$,

$$\begin{aligned}\|x+y\|_\infty &= \sup_{t\in[a,b]} |x(t)+y(t)| \\ &\le \sup_{t\in[a,b]} |x(t)| + \sup_{t\in[a,b]} |y(t)| \\ &= \|x(t)\|_\infty + \|y(t)\|_\infty.\end{aligned}$$

Thus, $\|\cdot\|_\infty$ is a norm. It follows that $(C(a,b), \|\cdot\|_\infty)$ is a normed vector space. $\square$

Moreover, define a functional $\rho : C(a,b) \times C(a,b) \to [0,\infty)$ as

$$\rho(x,y) = \|x-y\|_\infty = \sup_{t\in[a,b]} |x(t)-y(t)| = \max_{t\in[a,b]} |x(t)-y(t)|.$$

Clearly, ρ is a metric function induced by the norm $\|\cdot\|_\infty$, and thus $(C(a,b), \rho)$ is a metric space.

Unlike $C(a,b)$, $C(\Omega)$ is only a vector space. It is neither a normed space nor a metric space with respect to the essential norm defined above. It is technically possible to define a norm so that $C(\Omega)$ is a normed space. But under the usual norm, $C(\Omega)$ is only a vector space and not a normed space.

2.3.2. *The bounded continuous function space $B(\Omega)$.* Let $B(\Omega)$ be the class of all bounded continuous functions on Ω equipped with the essential norm $\|\cdot\|_\infty$. The definition of the essential norm $\|\cdot\|_\infty$ is valid since every function in $B(\Omega)$ is bounded and has a maximum.

Proposition 2.19. *$(B(\Omega), \|\cdot\|_\infty)$ is a normed vector space and $B(\Omega)$ is a proper subspace of $C(\Omega)$.*

Proof. It follows directly from Proposition 2.1 that the properties (1) and (2) of bounded continuous functions on Ω satisfy the vector space axioms. In addition, the zero function is in $B(\Omega)$. Therefore, $B(\Omega)$ is a vector space.

Moreover, both $B(\Omega)$ and $C(\Omega)$ contain bounded continuous functions but $C(\Omega)$ also contains unbounded ones. It follows that $B(\Omega)$ is a proper subspace of $C(\Omega)$.

The statement $(B(\Omega), \|\cdot\|_\infty)$ is a normed space follows a similar proof to the one for $(C(a,b), \|\cdot\|_\infty)$. $\square$

2.3.3. *The continuous vanishing at infinity function space $V(\Omega)$.* Let $V(\Omega)$ be the class of all continuous vanishing at infinity functions on Ω equipped with the essential norm $\|\cdot\|_\infty$. It follows from Proposition 2.5 the definition of $\|\cdot\|_\infty$ is valid.

Proposition 2.20. *$(V(\Omega), \|\cdot\|_\infty)$ is a normed vector space and $V(\Omega)$ is a proper subspace of $B(\Omega)$.*

Proof. It follows directly from Proposition 2.6 that the properties (1) and (2) of continuous vanishing at infinity functions on Ω satisfy the vector space axioms. In addition, the zero function is in $V(\Omega)$. Therefore, $V(\Omega)$ is a vector space.

By Proposition 2.5, all continuous vanishing at infinity functions on Ω are bounded. But there are other bounded functions which do not vanish at infinity. Thus, $V(\Omega)$ is a proper subspace of $B(\Omega)$ and consequently of $C(\Omega)$.

The statement $(V(\Omega), \|\cdot\|_\infty)$ is a normed space follows a similar proof to the one for $(C(a,b), \|\cdot\|_\infty)$. $\square$

2.3.4. *The continuous integral-convergent function space $D(\Omega)$.* Let $D(\Omega)$ be the class of all continuous integral-convergent functions on Ω equipped with the essential norm $\|\cdot\|_\infty$. It follows from Proposition 2.9 the definition of $\|\cdot\|_\infty$ is valid.

Proposition 2.21. *$(D(\Omega), \|\cdot\|_\infty)$ is a normed vector space and $D(\Omega)$ is a proper subspace of $V(\Omega)$.*

Proof. It follows directly from Proposition 2.7 that the properties (1) and (2) of continuous integral-convergent functions on Ω satisfy the vector space axioms. In addition, the zero function is in $D(\Omega)$. Therefore, $D(\Omega)$ is a vector space.

By Proposition 2.10, all continuous integral-convergent functions on Ω vanish at infinity. There are other vanishing at infinity functions which are integral-divergent. Thus, $D(\Omega)$ is a proper subspace of $V(\Omega)$. Clearly, $D(\Omega)$ is also a proper subspace of $B(\Omega)$ and $C(\Omega)$.

The statement $(D(\Omega), \|\cdot\|_\infty)$ is a normed space follows a similar proof to the one for $(C(a,b), \|\cdot\|_\infty)$. □

2.3.5. *The restricted function space $V(I)$ of $V(\Omega)$.* Let $I = [0, T] \subseteq \Omega$, $T > 0$, be the restricted underlying domain. Then the restricted bounded continuous vanishing at infinity function space $V(I)$ of $V(\Omega)$ is defined as

$$V(I) = \{g(t), t \in I \mid g(t) = f(t), \text{ for } t \in I, f(t) \in V(\Omega)\}.$$

Proposition 2.22. *$(V(I), \|\cdot\|_\infty)$ is a normed space.*

Proof. Trivial due to isomorphism. □

2.3.6. *Other norms.* Norms other than the essential norm (or the sup norm) $\|\cdot\|_\infty$ may be defined for a subspace of $D(\Omega)$. For example, the L_1 norm is

$$\|f\|_1 = \int_0^\infty |f(t)|\, dt,$$

or the L_2 norm is

$$\|f\|_2 = \left[\int_0^\infty (f(t))^2\, dt\right]^{\frac{1}{2}},$$

if the corresponding improper integral converges. Clearly, the L_1 or the L_2 norm is not well-defined for $D(\Omega)$, $V(\Omega)$, or $B(\Omega)$.

Proposition 2.23. *Assume norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ are well-defined for some subspace of $D(\Omega)$. Then they are topologically equivalent.*

Proof. Omitted. □

2.3.7. *Weak norms and weak metrics for $V(\Omega)$.* Consider a closed interval $I = [0, T] \subseteq \Omega$, $T > 0$. For every $f \in V(\Omega)$, define the weak essential norm as

$$\|f(t)\|_\infty^w = \max_{t\in[0,T]} |f(t)| .$$

Then $(V(I), \|\cdot\|_\infty^w)$ is a weak normed vector space. The corresponding weak metric induced by $\|\cdot\|_\infty^w$ is defined as

$$\rho^w(f,g) = \|f(t) - g(t)\|_\infty^w = \max_{t\in[0,T]} |f(t) - g(t)| ,$$

for every $f, g \in V(\Omega)$. The definitions of weak norm and weak metric can be viewed as restricting the usual norm and metric to the compact interval $[0, T]$. In addition, the above weak norms and weak metrics may be defined for $B(\Omega)$ or $C(\Omega)$.

Proposition 2.24. *In the above definitions for $V(\Omega)$, the weak metric ρ^w is weaker than the usual metric ρ.*

Proof. Firstly, we will show that ρ implies ρ^w. Consider a sequence in $V(\Omega)$ converging to the zero function in the usual metric. Then we can find a subsequence $\{f_n\}$ in $V(\Omega)$ converging to the zero function in the usual metric. This implies

$$\lim_{n\to\infty} \|f_n(t) - 0\| = 0,$$

or

$$\lim_{n\to\infty} \max_{t\in[0,\infty)} |f_n(t)| = 0,$$

which is

$$\lim_{n\to\infty} \max\left(\max_{t\in[0,T]} |f_n(t)|, \max_{t\in[T,\infty)} |f_n(t)| \right) = 0.$$

For every $n \in \mathbb{N}$, we have

$$0 \le \max_{t\in[0,T]} |f_n(t)| \le \max_{t\in\Omega} |f_n(t)|.$$

By Proposition 2.27 sequence comparison test, we have

$$\lim_{n\to\infty} \max_{t\in[0,T]} |f_n(t)| = 0,$$

which is

$$\lim_{n\to\infty} \|f_n(t)\|^w = 0.$$

Conversely, consider the sequence

$$f_n(t) = \sum_{k=0}^{n} (-t)^k e^{-t}, \quad n = 0, 1, \cdots$$

converging to $\frac{e^{-t}}{1+t}$ on $[0, r]$, $r < 1$. But $f_n(t)$ does not converge on Ω. This means the convergence in $\|\cdot\|^w$ or ρ^w does not imply the convergence in $\|\cdot\|$ or ρ.

Therefore, $\|\cdot\|^w$ and ρ^w are weaker than $\|\cdot\|$ and ρ, respectively. $\square$

Similarly, we can define the weak L_1 norm and metric ($\|\cdot\|_1^w$ and ρ_1^w) or the weak L_2 norm and metric ($\|\cdot\|_2^w$ and ρ_2^w) for $V(\Omega)$, $B(\Omega)$, or $C(\Omega)$, since continuous functions on a closed interval are integrable.

2.4. **Bounded linear operators on normed spaces.** The theory of linear operators on normed spaces is important for studying and developing function approximation methods, in particular iterative ones. For example, a linear operator satisfying certain conditions becomes a contraction operator and, by the Banach's fixed point theorem, one can approximate the exact solution to a differential equation iteratively by a convergent function sequence. However, the above iterative approximation is only a local approximation method and fails if the contraction condition is not satisfied. Although the theory of operators on $P_n^\lambda(\Omega)$ is less important to our new approximation theory, it is one of the main topics in future research. In this section, we only introduce some basic concepts of linear operators and will not discuss the general theory of operators.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces with the induced metrics ρ_X and ρ_Y, respectively. An operator $A : X \to Y$ is *linear* if

(1) $A(\alpha x) = \alpha A(x)$; and
(2) $A(x_1 + x_2) = A(x_1) + A(x_2)$,

for all $x, x_1, x_2 \in X$ and $\alpha \in \mathbb{R}$. We shall use the term operators and linear operators synonymously from here on out.

An operator $A : X \to Y$ is *continuous* at a point $x_0 \in X$ if for every ϵ-neighborhood V of $Ax_0 \in Y$, there exists a δ-neighborhood U of $x_0 \in X$ such that for every point $x \in U$, $Ax \in V$, i.e.

$$\rho_Y(A(x), A(x_0)) < \epsilon,$$

whenever

$$\rho_X(x, x_0) < \delta.$$

If δ is independent of x_0, A is said to be *uniformly continuous.*

A continuous operator $A : X \to Y$ is said to be *Lipschitz continuous* (Rudolf Lipschitz, 1832-1903) if there exists a constant $C > 0$ such that

$$\rho_Y(A(x), A(x_0)) \leq C\rho_X(x, x_0)$$

for all $x, x_0 \in X$.

Proposition 2.25. *Every Lipschitz continuous operator is uniformly continuous.*

Proof. Let $A : X \to Y$ be continuous at $x_0 \in X$. Then for any $\epsilon > 0$, there exists a $\delta > 0$ such that whenever

$$\rho_X(x, x_0) < \delta,$$

we have
$$\rho_Y(A(x), A(x_0)) < \epsilon.$$
Since A is Lipschitz continuous, there exists a constant $C > 0$ such that
$$\rho_Y(A(x), A(x_0)) \le C\rho_X(x, x_0),$$
for all $x, x_0 \in X$. Let $\delta = \frac{\epsilon}{C}$. Then whenever
$$\rho_X(x, x_0) < \delta = \frac{\epsilon}{C},$$
we have
$$\rho_Y(A(x), A(x_0)) < C\delta = C\frac{\epsilon}{C} = \epsilon.$$
This implies the choice of δ is independent of $x, x_0 \in X$. Thus, A is uniformly continuous. □

An operator $A : X \to Y$ is *bounded* if there exists an $M > 0$ such that for every $x \in X$
$$\|Ax\|_Y \le M\|x\|_X, \quad \text{for all } x \in X.$$
Let $A : X \to Y$ be a bounded operator. The *norm* of A is defined as
$$\|A\| = \inf\{M \mid \|Ax\|_Y \le M\|x\|_X, \text{ for all } x \in X\},$$
or equivalently as
$$\|A\| = \sup_{x\in X} \frac{\|Ax\|_Y}{\|x\|_X}, \quad x \neq 0.$$
Let all the bounded linear operators from X to Y form a set denoted by $L(X, Y)$. Let the norm of any operator in $L(X, Y)$ be defined as above. Then $(L(X, Y), \|\cdot\|)$ is a vector space in its own right. The proof is omitted.

Let $A \in L(X, Y)$ and $B \in L(Y, Z)$. Then there is an operator $BA \in L(X, Z)$ such that $BA(x) = B(A(x))$ for all $x \in X$. BA is called the *composite operator* of A and B. Let $A \in L(X, X)$. Denote $A^2 = AA$, $A^3 = AAA$, etc. Then $a_0 + a_1A + \cdots + a_nA^n$, for $a_i \in \mathbb{R}$, $i = 0, 1, \cdots, n$, is a valid operator in $L(X, X)$. In some cases, there may exist operator power series.

2.5. **Infinite function series.**

2.5.1. *Infinite number series.* A set of numbers $\{u_0, u_1, \cdots\}$, of countably many elements, is called an *infinite number sequence* or a *sequence* in a metric space $\mathbb{R}$ or $\mathbb{C}$. A sequence is also denoted by $\{u_n, n = 0, 1, \cdots\}$, $\{u_n\}_{n=0}^{\infty}$, $\{u_n\}$, or simply by u_n.

Let $\{u_n\}$ be a sequence and L a number. If for any $\epsilon > 0$, there is an index N such that whenever $n > N$, we have
$$|u_n - L| < \epsilon,$$
then $\{u_n\}$ is said to *converge* to the *limit* L, denoted by
$$\lim_{n\to\infty} u_n = L \qquad \text{or} \qquad u_n \to L, \text{ as } n \to \infty.$$

A sequence is said to be *divergent* if it is not convergent.

Let $\{u_n\}$ be a sequence. An *infinite number series*, or a *series*, associated with $\{u_n\}$, is called a *formal sum*

$$\sum_{n=0}^{\infty} u_n = u_0 + u_1 + \cdots$$

or simply $\sum u_n$. Define the *partial sums* of the series $\sum u_n$ as

$$s_n = \sum_{k=0}^{n} u_k, \quad n = 0, 1, \cdots .$$

Then $\{s_n\}$ is a sequence of partial sums. If $\{s_n\}$ is convergent to a limit L, then the series $\sum u_n$ is said to be *convergent* to the *limit* L. If $\{s_n\}$ is divergent, then $\sum u_n$ is said to be *divergent*.

For example, a *geometric series* is a series $\sum_{n=0}^{\infty} u_n$ such that $\frac{u_{n+1}}{u_n} = r < \infty$ for all $n \in \mathbb{N}$. If $|r| < 1$, then

$$\lim_{n\to\infty} \sum_{k=0}^{n} u_k = \lim_{n\to\infty} \frac{u_0(1-r^n)}{1-r} = \frac{u_0}{1-r}$$

and the series is convergent. If $|r| \geq 1$, the series is divergent.

Proposition 2.26. *If $\sum u_n$ converges, then*

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} |u_n| = 0.$$

Proof. Assume $\sum u_n$ converges to L. Since $u_n = s_n - s_{n-1}$. It follows that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0.$$

This implies for any $\epsilon > 0$, there is an index N such that whenever $n > N$, we have

$$||u_n| - 0| = |u_n - 0| < \epsilon.$$

Thus,

$$\lim_{n\to\infty} |u_n| = 0.$$

□

Note $\lim_{n\to\infty} u_n = 0$ is only a necessary and not a sufficient condition for $\sum u_n$ to be convergent. The converse statement is not always true: the harmonic series $\sum \frac{1}{n}$ diverges.

A series $\sum u_n$ is said to *converge absolutely* if $\sum |u_n|$ converges. The series $\sum u_n$ is said to *converge conditionally* if it is convergent but not absolutely convergent. Informally speaking, if a series converges absolutely, then its terms are commutative and associative.

The following propositions are useful in series convergence tests.

Proposition 2.27. *(Comparison test). Consider two positive series* $\sum u_n$ *and* $\sum v_n$ *such that*

$$0 \leq u_n \leq v_n.$$

If $\sum v_n$ *converges, then* $\sum u_n$ *converges absolutely. If* $\sum u_n$ *diverges, then* $\sum v_n$ *diverges.*

Proof. Omitted. A bounded monotonically increasing sequence converges to a limit. $\square$

Proposition 2.28. *(Ratio test). Consider a series* $\sum u_n$*. Assume*

$$r = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

exists. Then $\sum u_n$ *converges if* $0 \leq r < 1$*, and diverges if* $r > 1$*.*

Proof. The hypothesis implies there exists a t, $r < t < 1$, such that for $N \in \mathbb{N}$ sufficiently large

$$\left| \frac{u_{n+1}}{u_n} \right| < t,$$

or

$$|u_n| < |u_N| t^{n-N},$$

for all $n = N+1, N+2, \cdots$. Since $t < 1$, we have

$$\sum_{n=N+1}^{\infty} |u_n| < |u_N| \sum_{n=N+1}^{\infty} t^{n-N} = \frac{|u_N| t}{1-t}.$$

Thus, $\sum_{n=0}^{\infty} |u_n|$ converges, and $\sum_{n=0}^{\infty} u_n$ converges absolutely for $r < 1$.

Let $1 < t < r$. The statement about divergence can be proved similarly. $\square$

Proposition 2.29. *(Root test). Consider a series* $\sum u_n$*. Assume*

$$r = \lim_{n\to\infty} \sup |u_n|^{\frac{1}{n}}$$

exists. Then $\sum u_n$ *converges if* $0 \leq r < 1$*, and diverges if* $r > 1$*.*

Proof. The hypothesis implies there exists a t, $r < t < 1$, such that for $N \in \mathbb{N}$ sufficiently large

$$|u_n|^{\frac{1}{n}} < t,$$

or

$$|u_n| < t^n,$$

for all $n = N+1, N+2, \cdots$. Clearly, $\sum_{n=N+1}^{\infty} t^n$, $t < 1$, is convergent, and so is $\sum_{n=N+1}^{\infty} |u_n|$. Thus, $\sum_{n=0}^{\infty} |u_n|$ converges, and $\sum_{n=0}^{\infty} u_n$ converges absolutely.

Let $1 < t < r$. The statement about divergence can be proved similarly. $\square$

Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two convergent series and α real. Then the *sum* of the two series is defined as

$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} (a_n + b_n)$$

and the *scalar multiplication* as

$$\alpha \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \alpha a_n.$$

Clearly, $\sum (a_n + b_n)$ and $\sum \alpha a_n$ are convergent. Thus, convergent series have a linear structure.

Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two convergent series. The *Cauchy product* of the two series is defined as

$$\left(\sum_{n=0}^{\infty} a_n \right) \cdot \left(\sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} c_n,$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad n = 0, 1, \cdots .$$

Clearly, $\{c_n\}$ is the discrete convolution of $\{a_n\}$ and $\{b_n\}$. The following proposition discusses when a Cauchy product converges.

Theorem 2.30. *If $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$ absolutely, then their Cauchy product converges to AB absolutely.*

Proof. By hypothesis both series converge absolutely. Assume

$$\sum_{n=0}^{\infty} |a_n| = A^* \quad \text{and} \quad \sum_{n=0}^{\infty} |b_n| = B^*.$$

Define

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n \right) \cdot \left(\sum_{n=0}^{\infty} b_n \right),$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad n = 0, 1, \cdots .$$

The table below shows all the product terms $a_i b_j$, $i, j = 0, 1, \cdots$.

By the triangle inequality,

$$\sum_{n=0}^{\infty} |c_n| \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_k b_{n-k}|.$$

$$\begin{array}{ccccc} a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \cdots \\ a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 & \cdots \\ a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 & \cdots \\ a_3b_0 & a_3b_1 & a_3b_2 & a_3b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Let

$$u_n = \sum_{i=0}^{n}\sum_{j=0}^{n} |a_i b_j| = \left(\sum_{i=0}^{n} |a_i|\right) \cdot \left(\sum_{j=0}^{n} |b_j|\right).$$

Clearly,

$$u_n \le \sum_{k=0}^{2n} |c_k| \le u_{2n}.$$

Then

$$\lim_{n\to\infty} u_n \le \lim_{n\to\infty} \sum_{k=0}^{2n} |c_k| \le \lim_{n\to\infty} u_{2n}.$$

Thus,

$$\lim_{n\to\infty} \sum_{k=0}^{n} |c_k| = \lim_{n\to\infty} \sum_{k=0}^{2n} |c_k| = \lim_{n\to\infty} u_n = A^* B^*.$$

This implies $\sum_{k=0}^{n} c_k$ converges absolutely.

Therefore, by commutative and associative properties of absolute convergence,

$$\lim_{n\to\infty} \sum_{k=0}^{n} c_k = \lim_{n\to\infty} \sum_{i=0}^{n}\sum_{j=0}^{n} a_i b_j = \left(\sum_{i=0}^{n} a_i\right) \cdot \left(\sum_{j=0}^{n} b_j\right) = AB.$$

□

There is a further generalization of this theorem by Mertens.

Theorem 2.31. *(Mertens' theorem). If $\sum_{n=0}^{\infty} a_n = A$ absolutely and $\sum_{n=0}^{\infty} b_n = B$ conditionally, then their Cauchy product converges to AB.*

Proof. Omitted. □

2.5.2. *Infinite function series.* The concept of a function sequence is similar to that of a number sequence. If each term of a sequence is a function in a real variable t on a set $A \subseteq \mathbb{R}$, i.e.

$$\{u_0(t), u_1(t), \cdots\}, \quad t \in A \subseteq \mathbb{R},$$

then the sequence is called an *infinite function sequence*, a *function sequence*, or simply a *sequence* on A, denoted by $\{u_n(t), t \in A\}_{n=0}^{\infty}$, $\{u_n(t)\}_{n=0}^{\infty}$, $\{u_n(t)\}$, or simply $u_n(t)$, if the underlying domain is implied.

Correspondingly, an *infinite function series*, also called a *function series* or simply a *series*, associated with the function sequence $\{u_n(t)\}$ is a formal expression

$$\sum_{k=0}^{\infty} u_k(t), \quad t \in A.$$

For each $t \in A$, the function series $\sum_{k=0}^{\infty} u_k(t)$ is a number series. If this number series converges to a finite limit, then $\sum_{k=0}^{\infty} u_k(t)$ is said to be *convergent* at t. Otherwise, the function series is *divergent* at t. The above convergence is called *pointwise convergence*, because it only concerns the convergence of a function series at a single point. Generally, a function series is convergent at some points and divergent at others.

Let $I \subseteq A$ be the set of all points at which function series $\sum_{k=0}^{\infty} u_k(t)$ is convergent. Then I is called the *region of convergence* (or the *interval of convergence*) of the function series. Assume I is an interval. Then there exists a function $s : I \to \mathbb{R}$ such that for every $t \in I$

$$s(t) = \sum_{k=0}^{\infty} u_k(t).$$

The function $s(t)$ is called the *limit function* of the function series. It is clear $s(t)$ has domain $I \subseteq A$ and has no definition on $A \backslash I$. Define the partial sums of the function series restricted to I as

$$s_n(t) = \sum_{k=0}^{n} u_k(t), \quad n = 0, 1, \cdots, \ t \in I.$$

Then

$$s(t) = \lim_{n \to \infty} s_n(t), \quad t \in I.$$

Although pointwise convergence is a simple notion of convergence for a function sequence or series, it does not consider the overall convergence of a function series on its domain, nor does it describe the intrinsic properties of the limit function.

Another notion of convergence for a function sequence or series is uniform convergence. Uniform convergence considers the overall convergence of a function series on an interval. It is an important concept in functional analysis, in that treating a uniformly convergent sequence of functions in a function space is very similar to treating a convergent sequence of points in an Euclidean space.

A function series $\sum_{k=0}^{\infty} u_k(t)$ is said to *uniformly converge* to a limit function $s(t)$ on an interval I if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $t \in I$ and all $n > N$, we have

$$\left| \sum_{k=0}^{n} u_k(t) - s(t) \right| < \epsilon.$$

This implies

$$\lim_{n\to\infty} \sup_{t\in I} \left| \sum_{k=0}^{n} u_k(t) - s(t) \right| = 0.$$

Uniform convergence implies pointwise convergence. The converse is not always true. For example, the function sequence $\{t^n\}$ converges on $[0,1]$ pointwise but not uniformly.

In addition, linear operations preserve uniform convergence.

2.5.3. *Power series.* A *power series* in a real variable t on $\mathbb{R}$ is a particular function series of the form

$$\sum_{k=0}^{\infty} a_k (t - t_0)^k,$$

where the *coefficients* a_k are real and t_0 is called the *center* of the series. A convergent power series is also called a *Taylor series*. A Taylor series centered at the origin is called a *MacLaurin series* and is denoted by

$$\sum_{k=0}^{\infty} a_k t^k.$$

Any power series can be made into a MacLaurin series by translating the independent variable. In the following sections, we describe and prove propositions and theorems about power series in the form of MacLaurin series for the sake of simplicity.

Let $\sum_{k=0}^{\infty} a_k t^k$ and $\sum_{k=0}^{\infty} b_k t^k$ be two power series and α real. Then the *sum* of the two power series is defined as

$$\sum_{k=0}^{\infty} a_k t^k + \sum_{k=0}^{\infty} b_k t^k = \sum_{k=0}^{\infty} (a_k + b_k) t^k,$$

and the *scalar multiplication* as

$$\alpha \sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} \alpha a_k t^k.$$

Thus, power series have a linear structure.

Let the partial sums of the power series $\sum_{k=0}^{\infty} a_k t^k$ be

$$s_n(t) = \sum_{k=0}^{n} a_k t^k, \quad n = 0, 1, \cdots .$$

Then $s_n(t)$ is a polynomial of degree n on $\mathbb{R}$. Define formally the limit function of the power series as

$$s(t) = \lim_{n\to\infty} s_n(t) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k t^k.$$

Since $\sum_{k=0}^{\infty} a_k t^k$ may converge at some real values of t and diverge at others, there exists a convergence region for the power series. Clearly, $s(t)$ is only defined on the convergence region.

It can be shown by the following Abel's theorems (Niels Abel, 1802-1829) the convergence region for a power series is an interval (or a disk for a complex power series).

Theorem 2.32. *(Abel's convergence theorem). If the power series $\sum_{k=0}^{\infty} a_k t^k$ converges at $t = r$, then it converges at any point $-|r| < t < |r|$ absolutely.*

Proof. We show this by the *Weierstrauss M-test*. For any fixed number t such that $|t| < |r|$, the power series $\sum_{k=0}^{\infty} a_k t^k$ is a number series and

$$\left|a_k t^k\right| < \left|a_k r^k\right|$$

for all $k \in \mathbb{N}$.

By hypothesis, $\sum_{k=0}^{\infty} a_k t^k$ converges at $t = |r|$. Then we have

$$\lim_{k\to\infty} \left|a_k r^k\right| = 0.$$

This implies for any given $M > 0$, there exists an $n \in \mathbb{N}$ such that

$$\left|a_k r^k\right| < M$$

for all $k > n$.

Thus,

$$\begin{aligned}\left|\sum_{k=n+1}^{\infty} a_k t^k\right| &\leq \sum_{k=n+1}^{\infty} \left|a_k t^k\right| \\ &= \sum_{k=n+1}^{\infty} \left|a_k r^k\right| \left|\frac{t^k}{r^k}\right| \\ &< \sum_{k=n+1}^{\infty} M \left|\frac{t}{r}\right|^k \\ &= \frac{M|r|}{|r|-|t|} \left|\frac{t}{r}\right|^{n+1}.\end{aligned}$$

As n tends to infinity, $\left|\sum_{k=n+1}^{\infty} a_k t^k\right|$ tends to zero. Thus, $\sum_{k=0}^{\infty} a_k t^k$ converges at t absolutely for all t such that $|t| < |r|$. □

Theorem 2.33. *(Abel's divergent theorem). If the power series $\sum_{k=0}^{\infty} a_k t^k$ diverges at $t = r$, then it diverges at any point t such that $|t| > |r|$.*

Proof. By contradiction. Assume the power series $\sum_{k=0}^{\infty} a_k t^k$ converges at a particular t, $|t| > |r|$. By Theorem (2.32), it converges at $t = r$ absolutely. This contradicts the hypothesis the power series diverges at $t = r$. Therefore, $\sum_{k=0}^{\infty} a_k t^k$ diverges at any t such that $|t| > |r|$. □

Theorem 2.34. *(Abel's theorem on radius of convergence). Suppose the power series $\sum_{k=0}^{\infty} a_k t^k$ does not converge for some real values of t. Then there is an $R > 0$ such that $\sum_{k=0}^{\infty} a_k t^k$ converges on $(-R, R)$ and diverges on $(-\infty, -R) \cup (R, +\infty)$. R is called the* radius of convergence*.*

Proof. By hypothesis, we can find two points a_1 and b_1 such that the power series $\sum_{k=0}^{\infty} a_k t^k$ converges at a_1 and diverges at b_1. Then by Theorem (2.32), $|a_1| < |b_1|$. Let $a = |a_1|$, $b = |b_1|$, and $n = 1$. Then repeatedly do the following steps:

(1) choose a number p such that $a < p < b$ and test its convergence.
(2) if the power series converges at p, set $a_{n+1} = a = p$ and $b_{n+1} = b$; if it diverges at p, set $a_{n+1} = a$ and $b_{n+1} = b = p$.
(3) increase n by one and return to (1).

The interval sequence formed by two sequences $\{a_n\}$ and $\{b_n\}$ has the property

$$[|a_1|, |b_1|] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots$$

for all $n \in \mathbb{N}$. By the completeness of $\mathbb{R}$, there is a unique positive number

$$R \in \cap_{n \in \mathbb{N}} [a_n, b_n]$$

such that the power series converges on $(-R, R)$ and diverges on $(-\infty, -R) \cup (R, +\infty)$. □

For example, the power series $\sum_{k=0}^{\infty} t^k$ converges to the limit function $\frac{1}{1-t}$ on $(-1, 1)$ with a radius of convergence $R = 1$.

If $\sum_{k=0}^{\infty} a_k t^k$ has a radius of convergence $R = 0$, the series converges only at the center $t = 0$ and diverges elsewhere. If $R = \infty$, the series converges everywhere. For example, $\sum_{k=0}^{\infty} k! t^k$ only converges at $t = 0$ and diverges at any $t \neq 0$, while $\sum_{k=0}^{\infty} \frac{1}{k!} t^k$ converges to e^t at any $t \in \mathbb{R}$.

Theorem 2.35. *(Cauchy-Hadamard's formula). Consider a power series $\sum a_n t^n$. Let*

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} .$$

Then R is the radius of convergence of the power series.

Proof. Let t be fixed and consider the number series $\sum |a_n t^n|$. Using root test and by hypothesis, we have

$$\limsup_{n\to\infty} |a_n t^n|^{\frac{1}{n}} = |t| \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{|t|}{R}.$$

For any $0 \leq |t| < R$, $\frac{|t|}{R} < 1$ and by root test $\sum |a_n t^n|$ converges and $\sum a_n t^n$ converges absolutely. For any $|t| > R$, $\frac{|t|}{R} > 1$ and by root test $\sum |a_n t^n|$ diverges and $\sum a_n t^n$ also diverges or else there is a contradiction. Thus, R is the radius of convergence of the power series $\sum a_n t^n$. □

Proposition 2.36. *A power series converges pointwise and absolutely on its convergence interval.*

Proof. Trivial. The proof is similar to the one for Theorem 2.32. □

Proposition 2.37. *A power series converges uniformly on its convergence interval (or on any closed interval inside its convergence interval).*

Proof. Assume the power series $\sum_{k=0}^{\infty} a_k t^k$ converges to $f(t)$ on the convergence interval $(-R, R)$. Consider any closed interval $[A, T]$ such that $-R < A < T < R$. Let

$$r = \max(|A|, |T|).$$

Since $r \in (-R, R)$, the number series $\sum_{k=0}^{\infty} a_k r^k$ converges absolutely, i.e. $\sum_{k=0}^{\infty} |a_k r^k|$ converges. This implies for any given $\epsilon > 0$,

$$\sum_{k=n+1}^{\infty} \left| a_k r^k \right| < \epsilon$$

for n sufficiently large.

Furthermore, for every $t \in [A, T]$,

$$\left| a_k t^k \right| \leq \left| a_k r^k \right|$$

for all $k > n$, $k \in \mathbb{N}$. It follows that

$$\left|\sum_{k=n+1}^{\infty} a_k t^k\right| \leq \sum_{k=n+1}^{\infty} \left|a_k t^k\right| \leq \sum_{k=n+1}^{\infty} \left|a_k r^k\right|.$$

This implies

$$\max_{t\in[A,T]} \left|\sum_{k=n+1}^{\infty} a_k t^k\right| \leq \sum_{k=n+1}^{\infty} \left|a_k r^k\right|,$$

which tends to zero independently of t as n tends to infinity. Thus, the power series $\sum_{k=0}^{\infty} a_k t^k$ converges uniformly on $[A, T]$ or on $(-R, R)$. □

Proposition 2.38. *The limit function of a power series is continuous on its convergence interval.*

Proof. Consider

$$s(t) = \lim_{n\to\infty} s_n(t) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k t^k$$

on the convergence interval I. Then for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > N$, we have

$$|s(t) - s_n(t)| < \frac{\epsilon}{3}$$

for all $t \in I$ due to uniform convergence.

Let $t_0, t \in I$. By continuity, there exists a $\delta > 0$ such that when

$$|t - t_0| < \delta,$$

we have

$$|s_n(t) - s_n(t_0)| < \frac{\epsilon}{3}.$$

Thus, when $n > N$ and $|t - t_0| < \delta$, we have

$$\begin{aligned} |s(t) - s(t_0)| &= |s(t) - s_n(t) + s_n(t) - s_n(t_0) + s_n(t_0) - s(t_0)| \\ &\leq |s(t) - s_n(t)| + |s_n(t) - s_n(t_0)| + |s_n(t_0) - s(t_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus, $s(t)$ is continuous at t_0. Since t_0 is arbitrary, $s(t)$ is continuous on I. □

In general, the above proposition is true only if each term of the series is continuous and the convergence is uniform. Under uniform convergence, operations such as differentiation or integration can be performed term by term for a power series on the convergence interval.

2.5.4. *The Taylor series expansion.* The converse of a power series converging problem is to expand a function into a power series, the so-called Taylor series expansion problem, which may be summarized into the following questions: Given a continuous function on some domain, is there a power series converging to it on the domain? If such power series exists, is it unique? These questions are partially answered by the following Taylor expansion theorem.

Theorem 2.39. *(Taylor series expansion and uniform convergence theorem.) Let $f(t)$ be a real-valued continuous function on $\mathbb{R}$ and infinitely differentiable at t_0. Then f can be uniquely* expanded *into a power series about t_0 as*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(t_0)}{n!}(t-t_0)^n, \tag{2.1}$$

which converges to $f(t)$ uniformly on the convergence interval $(t_0 - R, t_0 + R)$ for some $R > 0$.

Proof. Omitted. □

In the above theorem, the power series (2.1) is called a *Taylor series*, where t_0 is the *expansion center*, R is the radius of convergence, and f is said to be *analytic* at t_0.

Let $t_0 = 0$ and the domain of $f(t)$ be Ω. Then (2.1) is a MacLaurin series

$$f(t) = \sum_{k=0}^{\infty} a_k t^k, \quad t \in [0, R),$$

where the coefficients are

$$a_k = \frac{f^{(k)}(0)}{k!}, \quad k = 0, 1, \cdots .$$

The above MacLaurin series can also be written as the sum of a polynomial of degree n and a remainder series as

$$\begin{aligned} f(t) &= \sum_{k=0}^{n} a_k t^k + \sum_{k=n+1}^{\infty} a_k t^k \\ &= p_n(t) + r_n(t), \quad t \in [0, R), \end{aligned}$$

where the partial sum

$$p_n(t) = \sum_{k=0}^{n} a_k t^k$$

is called the *Taylor polynomial* and the power series

$$r_n(t) = \sum_{k=n+1}^{\infty} a_k t^k$$

is the *remainder series*. For every $t \in [0, R)$, the remainder series $r_n(t)$ has a representation of the Peno form

$$r_n(t) = \mathcal{O}(t^{n+1}),$$

or of the Lagrange form,

$$r_n(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!} t^{n+1}, \quad \xi \in [0, t),$$

or of the Cauchy integral form,

$$r_n(t) = \int_0^t \frac{f^{(n+1)}(\tau)}{n!} (t - \tau)^n d\tau,$$

respectively. These representations of $r_n(t)$ can be used in various situations to estimate the approximation error of $p_n(t)$ to $f(t)$ at any $t \in [0, R)$.

An analytic function $f(t)$ on domain Ω can be expanded into a MacLaurin series in one of the following three cases:

(1) at $t = 0$ only, when $R = 0$;
(2) on $[0, R)$, when $0 < R < \infty$; or
(3) on Ω, when $R = \infty$.

If $R < \infty$, $f(t)$ cannot be expanded as a MacLaurin series on (R, ∞).

2.5.5. *Divergent series.* Divergent number series are series that do not converge. Long before Newton (1643-1727) and Leibniz (1646-1716), and as early as Archimedes (287-212 BC), mathematicians began to use series without discussing their convergence. It was Cauchy (1789-1857) and Abel (1802-1829) who introduced the concept of convergence of series, and people started to use only the convergent series and forbid the divergent ones. In the late 1900s, Poincaré (1854-1912) re-discovered that although a divergent series does not have an infinite sum, it may still be operated on as a convergent series.

A *formal* number series is a number series regardless of its convergence. We may want to associate a formal number series with a number called the *infinite sum* of the series. This number is equal to the limit of its partial sums if the series is convergent. Otherwise, we assume its existence without actually calculating its value. In some situations, the infinite sum of a divergent series may be calculated by a technique called "telescoping". This implies the common operations are valid for divergent series as well as for convergent ones.

A divergent function series is a function series which does not converge at some or all of its points. Consider a real continuous function on Ω expanded as a power series about the origin with a finite radius of convergence $0 < R < \infty$. Then, by Theorems 2.34, the domain Ω is divided into two regions: the convergence region $[0, R)$ and the divergence region (R, ∞). If we use the expression of the power series on (R, ∞) as a formal power series regardless of its convergence, then we have a divergent power series on (R, ∞).

The divergent power series generated from the Taylor series expansions is interesting. Such a divergent series always has a convergent sibling. Together, they form a formal power series, which can be analyzed as a whole on its domain regardless of its convergence.

It is confusing that a power series in one expression on Ω can have two contradictory properties: convergent inside some interval (disk) and divergent outside. This phenomenon may be explained by the theory of analytic continuity.

Let $\sum_{k=0}^{\infty} a_k t^k$ and $\sum_{k=0}^{\infty} b_k t^k$ be two formal power series on Ω and $\alpha \in \mathbb{R}$. Define addition, scalar multiplication, and series multiplication as

$$\sum_{k=0}^{\infty} a_k t^k + \sum_{k=0}^{\infty} b_k t^k = \sum_{k=0}^{\infty} (a_k + b_k) t^k,$$

$$\alpha \sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} (\alpha a_k) t^k,$$

and

$$\left(\sum_{k=0}^{\infty} a_k t^k \right) \left(\sum_{k=0}^{\infty} b_k t^k \right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) t^k,$$

respectively. Then the set of all formal power series on Ω is a commutative ring with addition, scalar multiplication, and series multiplication operations (The proof is trivial). Obviously, the real numbers 0 and 1 are the additive and multiplicative identity of the ring. Convergence is only a property of an element of the ring and should not affect the operations within the ring.

A linear transformation on a formal power series does not affect the convergence of the series after the transformation. It transforms the convergent part of a formal power series into the convergent part of a new series and the divergent part into the divergent part of the new one. However, a linear transformation will change the rate of convergence or divergence of the series before and after the transformation. This implies, by using a transformation, we can control the convergence or divergence behavior of a series.

2.5.6. *Asymptotic series expansions.* An asymptotic series expansion is to expand a function into a formal function series. For example, if we expand a function into a formal power series regardless of its convergence, then we have an asymptotic power series expansion.

One of the most important concepts in the theory of asymptotic series expansions is the order of a function in the neighborhood of an expansion point. Order can be used to describe the relative rates of two functions approaching their common limit. Below is a set of rigorous notations about order.

Let $\gamma(t)$ and $\phi(t)$ be two real-valued continuous functions on Ω. In the neighborhood of $t = t_0$, $t_0 \in \Omega$, possibly going to ∞, the order notation $\mathcal{O}$ (big $\mathcal{O}$) is defined as

$$\gamma(t) = \mathcal{O}(\phi(t)), \quad t \to t_0,$$

if

$$\lim_{t \to t_0} \left| \frac{\gamma(t)}{\phi(t)} \right| < \infty.$$

This reads $\gamma(t)$ is in the order of or at most the order of $\phi(t)$ as t tends to t_0.

The order notation o (little o) is defined as

$$\gamma(t) = o(\phi(t)), \quad t \to t_0$$

if

$$\lim_{t \to t_0} \left| \frac{\gamma(t)}{\phi(t)} \right| = 0.$$

This reads $\gamma(t)$ has the order of infinitesimal to $\phi(t)$ as t tends to t_0.

The equivalent order notation $\sim$ is defined as

$$\gamma(t) \sim \phi(t), \quad t \to t_0,$$

if

$$\lim_{t \to t_0} \left| \frac{\gamma(t)}{\phi(t)} \right| = 1.$$

This reads $\gamma(t)$ has the same order of $\phi(t)$ as t tends to t_0.

The order notations $\mathcal{O}$, o, and $\sim$ simplify the description of relative rates of functions approaching a limit.

Proposition 2.40. *For each $n \in \mathbb{N}$,*

$$t^{n+1} = o(t^n), \quad t \to 0.$$

Proof.

$$\lim_{t \to 0} \left| \frac{t^{n+1}}{t^n} \right| = \lim_{t \to 0} t = 0,$$

which is

$$t^{n+1} = o(t^n), \quad t \to 0.$$

□

Proposition 2.41. *For each $n \in \mathbb{N}$ and every $\lambda > 0$,*

$$t^{n+1} e^{-\lambda t} = o(t^n e^{-\lambda t}), \quad t \to 0.$$

Proof.

$$\lim_{t \to 0} \left| \frac{t^{n+1} e^{-\lambda t}}{t^n e^{-\lambda t}} \right| = \lim_{t \to 0} t = 0,$$

which is

$$t^{n+1} e^{-\lambda t} = o(t^n e^{-\lambda t}), \quad t \to 0.$$

□

A function series
$$\phi_0(t) + \phi_1(t) + \phi_2(t) + \cdots$$
is said to be an *asymptotic series expansion* for a real-valued function $f(t)$ about a point $t = t_0$ if
$$f(t) - (\phi_0(t) + \phi_1(t) + \cdots + \phi_n(t)) = \mathcal{O}(\phi_{n+1}(t)), \quad t \to t_0,$$
for each $n \in \mathbb{N}$ and all t in some neighborhood of t_0, where
$$\phi_m(t) = \mathcal{o}(\phi_n(t)), \quad t \to t_0,$$
for all $m > n$, $m, n \in \mathbb{N}$. The asymptotic series expansion can also be written as
$$f(t) \sim \phi_0(t) + \phi_1(t) + \phi_2(t) + \cdots, \quad t \to t_0.$$
It follows from Proposition 2.40 that a power series expansion about $t = t_0$ is a particular asymptotic series expansion. Thus, we may write
$$f(t) \sim a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + \cdots, \quad t \to t_0,$$
$$f(t) \sim a_0 + a_1(t - t_0) + \cdots + a_n(t - t_0)^n + \mathcal{o}((t - t_0)^n), \quad t \to t_0,$$
or
$$f(t) \sim a_0 + a_1(t - t_0) + \cdots + a_n(t - t_0)^n + \mathcal{O}((t - t_0)^{n+1}), \quad t \to t_0.$$
By Proposition 2.41,
$$f(t) \sim a_0 e^{-\lambda t} + a_1(t - t_0)e^{-\lambda t} + a_2(t - t_0)^2 e^{-\lambda t} + \cdots, \quad t \to t_0,$$
is a valid asymptotic series expansion for every $\lambda > 0$. In fact, it is a family of asymptotic series expansions with a parameter λ.

In summary, a function can be asymptotically expanded into a formal function series in a half-convergent half-divergent form on its domain. Asymptotic series expansions may be used as a method to partially approximate a function on some interval.

2.6. **Laplace transforms.** The Laplace transformation is an integral transformation which maps a real-valued function in the *time domain* into a complex function in the *frequency domain*. It is extensively used in the fields of applied mathematics, science, and engineering where convolutions are involved. In general, convolutions can be very difficult to calculate. However, according to the convolution theorem, it may be easy to convolve two time domain functions in the complex frequency domain as a multiplication and then to transform the result back to the time domain. In particular, if both time domain functions in a convolution have rational Laplace transforms, then their convolution in the frequency domain is also rational, which can be easily inverted back to the time domain. This property makes Laplace transformation a powerful tool for solving linear ordinary differential or integral equations. Laplace transformations can also be used to describe the superposition relation between the input and the output of a linear time-invariant system, because superpositions imply convolutions.

Let $f(t)$ be a real-valued continuous function on Ω. If the improper integral

$$\int_0^\infty f(t)e^{-st}dt$$

exists for some real s, then $f(t)$ is said to have a *Laplace transform*, denoted by

$$\bar{f}(s) = \mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt,$$

and $\bar{f}(s)$ is said to *converge* at s. The set of all s at which $\bar{f}(s)$ converges is called its *region of convergence* (ROC).

Theorem 2.42. *(Laplace transform existence theorem). Let $f(t)$ be a continuous function on Ω. If $f(t)$ satisfies*

$$|f(t)| \leq Ke^{Dt},$$

for some $K > 0$ and $D > 0$, then $\bar{f}(s)$ exists and converges on $\mathrm{Re}(s) > D$. *This condition is equivalent to*

$$f(t) = \mathcal{O}(e^{Dt}), \quad t \to \infty.$$

Proof. Let s be real. Consider the improper integral

$$\begin{aligned}\int_0^\infty f(t)e^{-st}dt &\leq \int_0^\infty |f(t)|e^{-st}dt \\ &\leq K\int_0^\infty e^{-(s-D)t}dt.\end{aligned}$$

Clearly, the last integral exists if $s > D$. Extending s into the complex domain, the ROCs of $\bar{f}(s)$ is $\mathrm{Re}(s) > D$. □

Let α be real. Consider $f(t)$ and $g(t)$ such that both $\mathscr{L}\{f(t)\}$ and $\mathscr{L}\{g(t)\}$ converge on ROC E_f and E_g, respectively. Then the following properties are true:

(1) $\mathscr{L}\{f(t) + g(t)\} = \mathscr{L}\{f(t)\} + \mathscr{L}\{g(t)\}$ on ROC $E_f \cap E_g$;
(2) $\mathscr{L}\{\alpha f(t)\} = \alpha\mathscr{L}\{f(t)\}$ on ROC E_f.

This implies the Laplace transformation is linear.

Proposition 2.43. *The Laplace transformation is one-to-one for continuous functions on Ω.*

Proof. Omitted. □

The *inverse Laplace transform* of $\bar{f}(s)$ is a real-valued function $f(t)$ on Ω such that $\mathscr{L}\{f(t)\} = \bar{f}(s)$. It is a one-to-one mapping from a complex function space to a real function space defined by a complex contour integral

$$f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \frac{1}{2\pi i}\oint_{\Gamma} \bar{f}(s)e^{st}ds,$$

where Γ is a contour path large enough to include all the poles of $\bar{f}(s)$ in the complex plane. This contour integral is called the *Bromwich integral* (Thomas Bromwich, 1875-1929). In the case $\bar{f}(s)$ has singularities only on the left half complex plane, the Bromwich integral can be computed by the residue theorem. In particular, if $\bar{f}(s)$ is a meromorphic function (or a rational function) of s, the method of *partial fractions* can be used to find $f(t)$. In reality, only a handful of closed-form Laplace transforms have simple inverse functions, which are tabulated.

Let $\bar{f}(s)$ be a rational Laplace transform that does not have a pole at the origin. Then $\bar{f}(s)$ is analytic at the origin. In addition, $\bar{f}(s)$ has the following properties:

(1) $\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \dfrac{\bar{f}(s)}{s}$.
(2) $\mathscr{L}\{f'(t)\} = s\bar{f}(s) - f(0^+)$.
(3) $\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} s\bar{f}(s)$ (initial value theorem).
(4) $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\bar{f}(s)$ (final value theorem).

The proofs are trivial.

Proposition 2.44. *Let $f(t)$ be continuous, bounded, and vanishing at infinity on Ω, and let $\bar{f}(s) = \mathscr{L}\{f(t)\}$ converges. Then*

$$\lim_{s\to 0} s\bar{f}(s) = 0.$$

Proof. By the property (4) above, we have

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t) = 0.$$

□

Let X_{LT} be the set of all the time domain functions on Ω whose Laplace transforms exist. Then X_{LT} is a function space. In X_{LT}, those functions whose Laplace transforms are rational form a subspace X_{RLT}. Many useful time domain functions, such as polynomials, exponential functions, sine and cosine functions, are contained in X_{RLT}. The proofs of above statements are obvious. In the next chapter, we shall construct a new function space $P_n^\lambda(\Omega)$, whose Laplace transform is a subspace of X_{RLT} and X_{LT}, and build a new function approximation theory based on it.

3. Theory of $P_n^\lambda(\Omega)$ Spaces

In this chapter, we shall introduce a new function space $P_n^\lambda(\Omega)$ in order to develop a new method for approximating functions on unbounded intervals. We begin by investigating the polynomial space.

3.1. **Polynomial spaces $P_n(\Omega)$.**

Let $P_n(\Omega)$, $n \in \mathbb{N}$, be the set of all polynomials of degree up to n on Ω.

Proposition 3.1. *$P_n(\Omega)$ is a vector space.*

Proof. Let $f, g \in P_n(\Omega)$. Then f and g can be written as

$$f(t) = \sum_{k=0}^{n} a_k t^k \qquad \text{and} \qquad g(t) = \sum_{k=0}^{n} b_k t^k, \quad t \in \Omega.$$

Let α be real. It is obvious that $\alpha f \in P_n(\Omega)$ and $f + g \in P_n(\Omega)$, i.e. $P_n(\Omega)$ is closed under addition and scalar multiplication. In addition, $P_n(\Omega)$ contains the zero function. It follows that $P_n(\Omega)$ is a vector space. $\square$

Proposition 3.2. *$P_n(\Omega)$ is a subspace of $C(\Omega)$ for all $n \in \mathbb{N}$.*

Proof. Every function in $P_n(\Omega)$ is continuous and thus is in $C(\Omega)$. On the other hand, transcendental functions on Ω are continuous but are not in $P_n(\Omega)$. Thus, $P_n(\Omega)$ is a proper subset of $C(\Omega)$. It follows from Proposition 3.1 that $P_n(\Omega)$ is a subspace of $C(\Omega)$. $\square$

Note $B = \{t^0, t^1, t^2, \cdots, t^n\}$ is a set of linearly independent functions on Ω, and so B is a natural basis of $P_n(\Omega)$. Thus, $P_n(\Omega) = \operatorname{span} B$ and $\dim P_n(\Omega) = n + 1$. The maximal degree of a polynomial in $P_n(\Omega)$ is n.

Proposition 3.3. *Let $f \in P_n(\Omega)$, $n \geq 1$, be a non-constant function. Then f is unbounded.*

Proof. By contradiction. Consider

$$f(t) = \sum_{k=0}^{n} c_k t^k,$$

where the c_k are real and $c_n \neq 0$. Assume $f(t)$ is bounded by $M > 0$, i.e. $|f(t)| < M$. Then, for all $t \in \Omega$, $t > 0$, we have

$$\left| \sum_{k=0}^{n} c_k t^k \right| < M,$$

or

$$\left| \frac{c_0}{t^n} + \frac{c_1}{t^{n-1}} + \cdots + c_n \right| < \frac{M}{|t^n|}.$$

Taking the limit on both sides, we have

$$\lim_{t\to\infty} |c_n| < 0,$$

a contradiction. Thus, $f(t)$ is unbounded. □

It follows immediately that $P_n(\Omega)$ is not a subset of $B(\Omega)$.

Proposition 3.4. *Let $f \in P_n(\Omega)$ be a non-zero function. Then f is integral-divergent on Ω.*

Proof. Let $f \in P_n(\Omega)$ be a non-zero function. Clearly, if f is a constant function, then it is integral-divergent on Ω. If f is a non-constant polynomial, then f is unbounded on Ω. It is necessary for a continuous function on Ω to be integral-convergent if it is bounded and vanishing at infinity. Thus, by Proposition 2.10, f is integral-divergent. □

It follows immediately that $P_n(\Omega)$ is not a subset of $D(\Omega)$ or $V(\Omega)$.

Proposition 3.5. *$P_k(\Omega)$ is a subspace of $P_n(\Omega)$ if and only if $0 \leq k \leq n,\ k, n \in \mathbb{N}$.*

Proof. Since $P_k(\Omega) = \operatorname{span}\{1, t, \cdots, t^k\}$ and $P_n(\Omega) = \operatorname{span}\{1, t, \cdots, t^n\}$, the proof is trivial. □

This subspace structure can be compared to that of the Euclidean space $\mathbb{R}^n$, the space of vectors in n-tuples of real numbers, the principle of which is similar.

Proposition 3.6. *The usual L_1, L_2, and L_∞ norms are not defined for $P_n(\Omega)$.*

Proof. Consider a non-constant $f \in P_n(\Omega)$. Since f is unbounded on Ω, the sup norm $\|\cdot\|_\infty$ is unbounded. Since f is integral-divergent on Ω, the norm $\|\cdot\|_1$ is divergent. Since f^2 is also a polynomial, it follows that the norm $\|\cdot\|_2$ is divergent. Thus, $P_n(\Omega)$ is neither a normed space nor a metric space induced by a usual norm. □

The above statements should not be confused with the case of the polynomial spaces on a closed interval $[a, b]$, $0 \leq a \leq b < \infty$, denoted by $P_n(a, b)$, which is a normed space. For example, the sup norm for $P_n(a, b)$ can be defined as

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|$$

for every $f \in P_n(a, b)$. Then $(P_n(a, b), \|\cdot\|_\infty)$ is a normed space or a metric space induced by the norm. Similarly, $(P_n(a, b), \|\cdot\|_1)$ and $(P_n(a, b), \|\cdot\|_2)$ are also normed spaces or metric spaces.

$(P_n(a, b), \|\cdot\|_\infty)$ is useful for function approximations on $[a, b]$ by polynomials.

Theorem 3.7. *(Stone-Weierstrass approximation theorem). Any continuous real-valued function f on a closed interval $[a, b]$ can be approximated by polynomials. That is for every $\epsilon > 0$, there exists a polynomial p such that for all $t \in [a, b]$, we have*

$$|f(t) - p(t)| < \epsilon.$$

Proof. Omitted. There are many proofs of this theorem. For example, Bernstein has given a constructive proof using the so-called Bernstein polynomials. □

3.2. **The exponential decaying function $e^{-\lambda t}$.** The *exponential decaying function*, or *decaying function*, on Ω, denoted by $e^{-\lambda t}, \lambda > 0$, is an important real-valued function. It is a solution to a first order differential equation

$$\frac{df}{dt} = -\lambda f, \quad f(0) = 1,$$

i.e. it is a basis function of the solution function space of the above differential equation. The normalized decaying function is

$$\lambda e^{-\lambda t} = \frac{1}{\tau} e^{-\frac{t}{\tau}},$$

where $\tau = \frac{1}{\lambda}$ is called the *time constant* of the function, and its integral on Ω is unity.

A decaying function $e^{-\lambda t}$ is characterized by its *decaying parameter* λ or its time constant τ. The function is positive and bounded on Ω. At $t = 0$, it reaches its maximum value. Then, it decreases monotonically as t gets large and vanishes at infinity. This means for any sufficiently small $\epsilon > 0$, we can find a T such that $e^{-\lambda T} = \epsilon$ and when $t > T$, $e^{-\lambda t}$ is strictly less than ϵ. In addition, $e^{-\lambda t}$ has the following properties.

Proposition 3.8. *For any $p(t) \in P_n(\Omega)$, $n \in \mathbb{N}$, and $\lambda > 0$, we have*

$$\lim_{t\to\infty} p(t)e^{-\lambda t} = 0.$$

Proof. By induction. Firstly,

$$\lim_{t\to\infty} \frac{1}{e^{\lambda t}} = 0.$$

Assume for any $k \in \mathbb{N}$,

$$\lim_{t\to\infty} \frac{t^k}{e^{\lambda t}} = 0.$$

Then, by L'Hopital's rule,

$$\lim_{t\to\infty} \frac{t^{k+1}}{e^{\lambda t}} = \frac{k+1}{\lambda} \lim_{t\to\infty} \frac{t^k}{e^{\lambda t}} = 0.$$

Since $p(t)$ is a linear combination of basis functions, $p(t)e^{-\lambda t}$ vanishes at infinity. Consequently, the proposition is proved. □

3.3. **Decaying polynomial spaces $P_n^\lambda(\Omega)$.** Now, we formally introduce a new finite dimensional function space on Ω. The new space is spanned by a finite basis with a non-negative real decaying parameter λ. It actually represents a continuum family of function spaces. Moreover, the new space is naturally isomorphic to the polynomial space or the Euclidean space.

A *decaying polynomial* is a function of the form $p(t)e^{-\lambda t}$ on Ω, where $p(t)$ is a polynomial and $\lambda > 0$. A decaying polynomial is not a polynomial but a transcendental function.

A *decaying polynomial space* of degree $n \geq 0$, $n \in \mathbb{N}$, denoted by $P_n^\lambda(\Omega)$, is a collection of decaying polynomials on Ω of the form $p(t)e^{-\lambda t}$, where $p(t) \in P_n(\Omega)$ and $\lambda > 0$.

Theorem 3.9. *$P_n^\lambda(\Omega)$ is a vector space.*

Proof. Let $f, g \in P_n^\lambda(\Omega)$. Then f and g can be represented by $f(t) = p(t)e^{-\lambda t}$ and $g(t) = q(t)e^{-\lambda t}$ for some $p(t), q(t) \in P_n(\Omega)$. It is obvious

$$f(t) + g(t) = (p(t) + q(t))\, e^{-\lambda t} \in P_n^\lambda(\Omega),$$

and

$$\alpha f(t) = \alpha p(t)e^{-\lambda t} \in P_n^\lambda(\Omega), \quad \alpha \in \mathbb{R}.$$

Thus, $P_n^\lambda(\Omega)$ is closed under addition and scalar multiplication. In addition, the zero function is in $P_n^\lambda(\Omega)$. Therefore, $P_n^\lambda(\Omega)$ is a vector space. □

For example, the function $(2 + 3t + 4t^2)e^{-2.5t}$ on Ω is in $P_n^\lambda(\Omega)$ with $n = 2$ and $\lambda = 2.5$, or $P_2^{2.5}(\Omega)$.

We can also express $P_n^\lambda(\Omega)$ as

$$P_n^\lambda(\Omega) = \left\{ \sum_{k=0}^{n} c_k t^k e^{-\lambda t} \;\middle|\; c_0, c_1, \cdots, c_n \in \mathbb{R}, t \in \Omega \right\}$$

for every $\lambda > 0$ and each $n \geq 0, n \in \mathbb{N}$.

Define the *natural basis functions* of $P_n^\lambda(\Omega)$ as

$$\phi_k(t) = t^k e^{-\lambda t}, \quad k = 0, 1, \cdots, n.$$

Then a natural basis of $P_n^\lambda(\Omega)$ is

$$B_n = \{e^{-\lambda t}, te^{-\lambda t}, \cdots, t^n e^{-\lambda t}\},$$

and

$$P_n^\lambda(\Omega) = \operatorname{span} B_n.$$

The $(n + 1)$-tuples $(c_0, c_1, \cdots, c_n)$ are called the *coefficients*, *components*, or *coordinates* of a function in $P_n^\lambda(\Omega)$ with regard to the natural basis B_n. Clearly,

$$\dim P_n^\lambda(\Omega) = n + 1.$$

Define the *time constant* τ of $P_n^\lambda(\Omega)$ as

$$\tau = \frac{1}{\lambda}.$$

The *time span* of a basis function $t^k e^{-\lambda t}$ is the length of the *time span interval* defined as

$$\left[\frac{k-0.5}{\lambda}, \frac{k+0.5}{\lambda}\right] = \left[(k-0.5)\tau, (k+0.5)\tau\right], \quad k = 1, 2, \cdots, n.$$

The time span of $e^{-\lambda t}$ is 0.5τ, and that of the zero function is 0. Thus, the maximum time span of a function in $P_n^\lambda(\Omega)$ of degree n is

$$\frac{n+0.5}{\lambda} = (n+0.5)\tau,$$

or approximately

$$\frac{n}{\lambda} = n\tau,$$

when n is large.

3.4. **Algebraic properties of $P_n^\lambda(\Omega)$.**

Proposition 3.10. *For every $\lambda > 0$, let*

$$f(t) = \left(c_0 + c_1 t + \cdots + c_n t^n\right) e^{-\lambda t}$$

on Ω. Then

$$f(0) = c_0.$$

Proof. Trivial. □

Proposition 3.11. *Let $f(t) = p(t)e^{-\lambda t}$, $\lambda > 0$, $t \in \mathbb{R}$. Then $f(t)$ and $p(t)$ have the same zeros.*

Proof. Firstly, $e^{-\lambda t}$ is positive on $\mathbb{R}$. For each zero $t = t_0$ of $f(t)$, we have $p(t_0)e^{-\lambda t_0} = 0$. Dividing both sides by $e^{-\lambda t_0}$, we have $p(t_0) = 0$. Conversely, for each zero $t = t_0$ of $p(t)$, we have $p(t_0) = 0$. Multiplying both sides by $e^{-\lambda t_0}$, we have $p(t_0)e^{-\lambda t_0} = 0$, i.e. $f(t_0) = 0$. Note the proposition is true whether $p(t)$ is a polynomial or not. □

Proposition 3.12. *Let $f(t) \in P_n^\lambda(\Omega)$, $\lambda > 0$. If $f(t)$ has m zeros on Ω, then $m \leq n$.*

Proof. Consider $f(t) = p(t)e^{-\lambda t} \in P_n^\lambda(\Omega)$. Then, by Proposition 3.11, $f(t)$ and $p(t)$ have the same number of zeros. Since $p(t)$ is a polynomial of degree up to n, it has at most n zeros on Ω. By hypothesis, $f(t)$ has m zeros on Ω. Clearly, $m \leq n$. □

Proposition 3.13. *Each natural basis function $t^k e^{-\lambda t}$, $k = 0, 1, \cdots, n$, of $P_n^\lambda(\Omega)$ is positive on Ω.*

Proof. It follows from that both t^k and $e^{-\lambda t}$, $k = 0, 1, \cdots, n$, are positive on Ω. $\square$

Proposition 3.14. *Consider the natural basis functions $\phi_k(t) = t^k e^{-\lambda t}$, $k = 0, 1, \cdots, n$, of $P_n^\lambda(\Omega)$. For each k, $\phi_k(t)$ has a maximum value at $t = \frac{k}{\lambda}$ and is monotonically increasing on $[0, \frac{k}{\lambda}]$ and monotonically decreasing on $[\frac{k}{\lambda}, \infty)$.*

Proof. The case for $k = 0$ is trivial. For each $k = 1, 2, \cdots, n$, setting $\phi_k'(t) = 0$ and solving the equation

$$(t^k e^{-\lambda t})' = (k - \lambda t)t^{k-1}e^{-\lambda t} = 0,$$

we have a critical point

$$t = \frac{k}{\lambda}.$$

Evaluating

$$\left(t^k e^{-\lambda t}\right)''\Big|_{t=\frac{k}{\lambda}} = -\frac{k^{k-1}}{\lambda^{k-2}}e^{-k} < 0,$$

we conclude $\phi_k(t)$ has a maximum value at $t = \frac{k}{\lambda}$.

When $0 \le t < \frac{k}{\lambda}$,

$$\phi_k'(t) = (k - \lambda t)t^{k-1}e^{-\lambda t} > 0$$

and $\phi_k(t)$ is monotonically increasing on $[0, \frac{k}{\lambda}]$. Similarly, when $t > \frac{k}{\lambda}$,

$$\phi_k'(t) = (k - \lambda t)t^{k-1}e^{-\lambda t} < 0$$

and $\phi_k(t)$ is monotonically decreasing on $[\frac{k}{\lambda}, \infty)$. $\square$

Proposition 3.15. *Each natural basis function of $P_n^\lambda(\Omega)$ is bounded.*

Proof. This follows from Proposition 3.14 but can also be proved as follows. Consider the natural basis functions $\phi_k(t) = t^k e^{-\lambda t}, k = 0, 1, \cdots, n$, of $P_n^\lambda(\Omega)$. By Proposition 3.8, for each k, we have

$$\lim_{t\to\infty} \phi_k(t) = \lim_{t\to\infty} t^k e^{-\lambda t} = 0.$$

By proposition 2.4, for any $\epsilon > 0$, there exists a compact set $[0, T] \subset \Omega$, $T > 0$, such that $\phi_k(t) < \epsilon$ on $\Omega \backslash [0, T]$, i.e. $\phi_k(t)$ is bounded on $[T, \infty)$. Since $\phi_k(t)$ is continuous on the closed interval $[0, T]$, $\phi_k(t)$ is bounded on $[0, T]$. Therefore, $\phi_k(t)$ is bounded on Ω, a special case of the global boundedness in Proposition 2.5. $\square$

Proposition 3.16. *Every function in $P_n^\lambda(\Omega)$ is bounded, i.e. $P_n^\lambda(\Omega) \subset B(\Omega)$.*

Proof. By Proposition 3.15, each natural basis function $\phi_k(t)$ of $P_n^\lambda(\Omega)$ is bounded on Ω. Then, by the linear properties of the bounded functions, every linear combination of $\phi_k(t)$, $k = 0, 1, \cdots, n$, is bounded on Ω. Thus, $P_n^\lambda(\Omega) \subseteq B(\Omega)$. Obviously, $P_n^\lambda(\Omega) \subset B(\Omega)$. $\square$

Proposition 3.17. *Each natural basis function in $P_n^\lambda(\Omega)$ vanishes at infinity.*

Proof. By Proposition 3.8, for each $k = 0, 1, \cdots, n$, we have

$$\lim_{t\to\infty} t^k e^{-\lambda t} = 0.$$

□

Proposition 3.18. *Every function in $P_n^\lambda(\Omega)$ vanishes at infinity.*

Proof. This is a direct result of Proposition 3.8. Another proof is based on linear properties of the continuous vanishing at infinity functions. Proposition 3.17 implies each basis function $\phi_k(t)$, $k = 0, 1, \cdots, n$, of $P_n^\lambda(\Omega)$ vanishes at infinity. It follows that every linear combination of $\phi_k(t)$ vanishes at infinity. Thus, $P_n^\lambda(\Omega) \subseteq V(\Omega)$. Obviously, $P_n^\lambda(\Omega) \subset V(\Omega)$. □

Proposition 3.19. *Let*

$$f(t) = p(t)e^{-\lambda t} = \sum_{k=0}^{n} c_k t^k e^{-\lambda t}.$$

For $t > 1$ sufficiently large, $|f(t)|$ is in the same order as $|c_n t^n e^{-\lambda t}|$.

Proof. Firstly,

$$\begin{aligned} |p(t)| &= |p(t) - c_n t^n + c_n t^n| \\ &\le \left|\sum_{k=0}^{n-1} c_k t^k\right| + |c_n t^n|. \end{aligned}$$

For $t > 1$ sufficiently large,

$$\lim_{t\to\infty} \frac{\left|\sum_{k=0}^{n-1} c_k t^k\right|}{|c_n t^n|} = 0.$$

Therefore,

$$\lim_{t\to\infty} |p(t)| = |c_n t^n|,$$

or

$$p(t) = c_n t^n + \mathcal{O}(t^n), \quad t \to \infty.$$

This implies $f(t)$ is in the same order as $c_n t^n e^{-\lambda t}$ for large t. □

Proposition 3.20. *For any function $f(t) \in P_n^\lambda(\Omega)$, there exists a T such that $|f(t)|$ is bounded, decreasing monotonically, and vanishing at infinity on $[T, \infty)$.*

Proof. We begin by finding all the critical points of a function $f(t) = p_n(t)e^{-\lambda t} \in P_n^\lambda(\Omega)$. Setting the first derivative of $f(t)$ to 0, we have

$$f'(t) = \left[p_n(t)e^{-\lambda t}\right]' = \left(p_n'(t) - \lambda p_n(t)\right) e^{-\lambda t} = 0.$$

This implies

$$p_n'(t) - \lambda p_n(t) = 0.$$

This is an equation involving a polynomial of degree up to n and there are n real or complex roots. Let T be the largest positive real root (critical point) or zero (boundary point). Since $f(t)$ is bounded and vanishes at infinity on Ω, we have $|f(t)| < |f(T)|$ for all $t > T$. $\square$

Proposition 3.21. *Each basis function $\phi_k(t) = t^k e^{-\lambda t}$, $k = 0, 1, \cdots, n$, of $P_n^\lambda(\Omega)$ is absolutely integral-convergent, i.e.*

$$\int_0^\infty |\phi_k(t)| dt = \frac{k!}{\lambda^{k+1}} < \infty$$

for all $k = 0, 1, \cdots, n$.

Proof. By induction. Firstly,

$$\int_0^\infty |t^0 e^{-\lambda t}| dt = \frac{1}{\lambda}.$$

Assume

$$\int_0^\infty |\phi_k(t)| dt = \frac{k!}{\lambda^k} < \infty.$$

Then

$$\begin{aligned}
\int_0^\infty |\phi_{k+1}(t)| dt &= \int_0^\infty t^{k+1} e^{-\lambda t} dt \\
&= -\frac{1}{\lambda} \int_0^\infty t^{k+1} d e^{-\lambda t} \\
&= -\frac{1}{\lambda} t^{k+1} e^{-\lambda t} \Big|_0^\infty + \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} d t^{k+1} \\
&= \frac{(k+1)!}{\lambda^{k+1}} < \infty.
\end{aligned}$$

This concludes

$$\int_0^\infty |\phi_k(t)| dt = \int_0^\infty t^k e^{-\lambda t} dt = \frac{k!}{\lambda^{k+1}}$$

for all $k = 0, 1, \cdots, n$. $\square$

Proposition 3.22. *Every function in $P_n^\lambda(\Omega)$ is absolutely integral-convergent.*

Proof. Denote $f(t) \in P_n^\lambda(\Omega)$ as

$$f(t) = \sum_{k=0}^n c_k t^k e^{-\lambda t}.$$

It follows from Proposition 3.21 that

$$\begin{aligned}\int_0^\infty |f(t)|dt &\leq \int_0^\infty \sum_{k=0}^n |c_k| t^k e^{-\lambda t} dt \\ &= \sum_{k=0}^n |c_k| \int_0^\infty t^k e^{-\lambda t} dt \\ &= \sum_{k=0}^n |c_k| \frac{k!}{\lambda^{k+1}} < \infty.\end{aligned}$$

□

Proposition 3.23. *Each basis function $\phi_k(t) = t^k e^{-\lambda t}$, $k = 0, 1, \cdots, n$, of $P_n^\lambda(\Omega)$ is analytic (or has a power series) about the origin. In fact, $\phi_k(t)$ is analytic about any point in Ω with an infinite radius of convergence.*

Proof. Firstly,

$$e^{-\lambda t} = 1 - \lambda t + \frac{\lambda^2}{2!} t^2 + \cdots$$

with an infinite radius of convergence. Next, for each $k = 1, 2, \cdots, n$,

$$\begin{aligned}t^k e^{-\lambda t} &= t^k \left(1 - \lambda t + \frac{\lambda^2}{2!} t^2 + \cdots\right) \\ &= 0 + 0t + \cdots + 0t^{k-1} + t^k - \lambda t^{k+1} + \frac{\lambda^2}{2!} t^{k+2} + \cdots,\end{aligned}$$

also with an infinite radius of convergence.

Similarly, we can show that $\phi_k(t)$ is analytic about any point in Ω. □

Proposition 3.24. *Every function in $P_n^\lambda(\Omega)$ is analytic (or has a power series) about the origin or about any point in Ω with an infinite radius of convergence.*

Proof. By Proposition 3.23, each basis function is analytic about the origin. It follows that every linear combination of the basis is analytic about the origin with an infinite radius of convergence. Similarly, the proposition is also true when the analytic center is any point in Ω. □

Proposition 3.25. *Each natural basis function $\phi_k(t) = t^k e^{-\lambda t}$, $k = 0, 1, \cdots, n$, of $P_n^\lambda(\Omega)$ has the following asymptotic properties:*

(1) $\phi_{k+1}(t) = \mathcal{O}(\phi_k(t)),\ t \to 0$*; and*

(2) $\phi_k(t) = \mathcal{O}(\phi_{k+1}(t)),\ t \to \infty$.

Proof. For each $k = 0, 1, \cdots, n-1$, we have

$$\lim_{t\to 0} \left|\frac{\phi_{k+1}(t)}{\phi_k(t)}\right| = \lim_{t\to 0} \left|\frac{t^{k+1} e^{-\lambda t}}{t^k e^{-\lambda t}}\right| = \lim_{t\to 0} |t| = 0,$$

and

$$\lim_{t\to\infty}\left|\frac{\phi_k(t)}{\phi_{k+1}(t)}\right| = \lim_{t\to\infty}\left|\frac{t^k e^{-\lambda t}}{t^{k+1}e^{-\lambda t}}\right| = \lim_{t\to\infty}\frac{1}{|t|} = 0.$$

It follows that

$$\phi_{k+1}(t) = \mathcal{O}(\phi_k(t)), \quad t \to 0,$$

and

$$\phi_k(t) = \mathcal{O}(\phi_{k+1}(t)), \quad t \to \infty.$$

□

The asymptotic structure of $P_n^\lambda(\Omega)$ is inherited from, and thus is very similar to, that of $P_n(\Omega)$. This is due to the existence of an isomorphism between $P_n(\Omega)$ and $P_n^\lambda(\Omega)$, which implies a linear continuous transformation between the two spaces mapping unbounded functions to the ones bounded and vanishing at infinity.

Proposition 3.26. *Each natural basis function $t^k e^{-\lambda t}$, $k = 0, 1, \cdots, n$, of $P_n^\lambda(\Omega)$ has a rational Laplace transform as*

$$\bar{\phi}_k(s) = \mathscr{L}\left\{t^k e^{-\lambda t}\right\} = \frac{k!}{(s+\lambda)^{k+1}}.$$

Proof. By induction. Firstly,

$$\mathscr{L}\left\{e^{-\lambda t}\right\} = \int_0^\infty e^{-\lambda t}e^{-st}dt = \frac{1}{s+\lambda}.$$

Assume for each $k = 0, 1, \cdots, n-1$,

$$\mathscr{L}\left\{t^k e^{-\lambda t}\right\} = \frac{k!}{(s+\lambda)^{k+1}}.$$

Then

$$\begin{aligned}
\mathscr{L}\left\{t^{k+1}e^{-\lambda t}\right\} &= \int_0^\infty t^{k+1}e^{-\lambda t}e^{-st}dt \\
&= \frac{-1}{s+\lambda}\int_0^\infty t^{k+1}de^{-(s+\lambda)t} \\
&= \frac{-1}{s+\lambda}t^{k+1}e^{-(s+\lambda)t}\Big|_0^\infty + \frac{k+1}{s+\lambda}\int_0^\infty t^k e^{-(s+\lambda)t}dt \\
&= \frac{(k+1)!}{(s+\lambda)^{k+2}}.
\end{aligned}$$

□

Proposition 3.27. *Every function $f(t) \in P_n^\lambda(\Omega)$ has a rational Laplace transform with an expression*

$$\mathscr{L}\left\{f(t)\right\} = \frac{N(s)}{(s+\lambda)^{n+1}},$$

where $N(s)$ is a polynomial of degree up to n. The converse is also true.

Proof. By Proposition 3.26, each basis function in $P_n^\lambda(\Omega)$ has a Laplace transform

$$\mathscr{L}\{t^k e^{-\lambda t}\} = \frac{k!}{(s+\lambda)^{k+1}}, \quad 0 \le k \le n.$$

Write $f(t) \in P_n^\lambda(\Omega)$ as

$$f(t) = \sum_{k=0}^{n} c_k t^k e^{-\lambda t}, \quad c_k \in \mathbb{R}.$$

Then

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \sum_{k=0}^{n} c_k \mathscr{L}\left\{t^k e^{-\lambda t}\right\} \\
&= \sum_{k=0}^{n} \frac{c_k k!}{(s+\lambda)^{k+1}} \\
&= \frac{\sum_{k=0}^{n} c_k k!(s+\lambda)^{n-k}}{(s+\lambda)^{n+1}} \\
&= \frac{N(s)}{(s+\lambda)^{n+1}},
\end{aligned}$$

where $N(s) = \sum_{k=0}^{n} c_k k!(s+\lambda)^{n-k}$ is a polynomial of degree up to n.

The converse is also true since $\frac{N(s)}{(s+\lambda)^{n+1}}$ can be inverted by partial fractions. □

Proposition 3.28. *Each basis function's Laplace transform $\bar{\phi}_k(s)$, $k = 0, 1, \cdots$, $n-1$, of $P_n^\lambda(\Omega)$ has the following property:*

$$\bar{\phi}_{k+1}(s) = -\frac{d}{ds}\bar{\phi}_k(s).$$

Proof. By Proposition 3.26,

$$\bar{\phi}_k(s) = \mathscr{L}\left\{t^k e^{-\lambda t}\right\} = \frac{k!}{(s+\lambda)^{k+1}}.$$

Then

$$-\frac{d}{ds}\bar{\phi}_k(s) = -\frac{d}{ds}\frac{k!}{(s+\lambda)^{k+1}} = \frac{(k+1)k!}{(s+\lambda)^{k+2}} = \frac{(k+1)!}{(s+\lambda)^{k+2}} = \bar{\phi}_{k+1}(s).$$

□

Proposition 3.29. *Each basis function $\phi_k(t) = t^k e^{-\lambda t}$, $k = 0, 1, \cdots, n-1$, of $P_n^\lambda(\Omega)$ has the following derivative property:*

$$\phi_k(t) = \frac{1}{k+1}\left(\frac{d}{dt} + \lambda\right)\phi_{k+1}(t).$$

Proof.

$$\begin{aligned}\frac{d}{dt}\phi_{k+1}(t) &= \frac{d}{dt}t^{k+1}e^{-\lambda t}\\ &= -\lambda t^{k+1}e^{-\lambda t} + (k+1)t^k e^{-\lambda t}\\ &= -\lambda\phi_{k+1}(t) + (k+1)\phi_k(t).\end{aligned}$$

Re-organizing the above equation proves the proposition. □

Proposition 3.30. *Each basis function's Laplace transform $\bar{\phi}_k(s)$, $k = 0, 1, \cdots$, n, of $P_n^\lambda(\Omega)$ has a power series about the origin with the radius of convergence λ.*

Proof. Clearly, by Proposition 3.26, each basis function's Laplace transform

$$\bar{\phi}_k(s) = \frac{k!}{(s+\lambda)^{k+1}}$$

can be expanded as a power series about the origin with a radius of convergence λ. □

Proposition 3.31. *Every function in $P_n^\lambda(\Omega)$ has a Laplace transform which can be expanded as a power series about the origin with a radius of convergence λ.*

Proof. This follows directly from Proposition 3.27 or from Proposition 3.30 and the linear properties of Laplace transformations. □

3.5. **Five representations of $P_n^\lambda(\Omega)$ functions.** Each $g(t) \in P_n^\lambda(\Omega)$ has the following five representations:

(1) The standard form

$$\begin{aligned}g(t) &= \sum_{k=0}^{n} c_k t^k e^{-\lambda t}\\ &= \sum_{k=0}^{n} c_k \phi_k(t), \quad c_0, c_1, \cdots, c_n \in \mathbb{R},\end{aligned}$$

where $\phi_k(t) = t^k e^{-\lambda t}$, $k = 0, 1, \cdots, n$, are the basis functions of $P_n^\lambda(\Omega)$. Thus, $g(t)$ is a linear combination of $\phi_k(t)$. Sometimes, it is convenient to write

$$g(t) = \sum_{k=0}^{n} \frac{c'_k}{k!} t^k e^{-\lambda t},$$

where $c'_k = c_k k!$, $k = 0, 1, \cdots, n$.

(2) The product form

$$g(t) = \left(\sum_{k=0}^{n} c_k t^k\right) e^{-\lambda t} = p(t)e^{-\lambda t},$$

where

$$p(t) = \sum_{k=0}^{n} c_k t^k, \quad c_0, c_1, \cdots, c_n \in \mathbb{R},$$

is a polynomial of degree up to n.

(3) The power series or the Taylor's series form

$$g(t) = \sum_{k=0}^{\infty} a_k t^k, \quad a_0, a_1, \cdots \in \mathbb{R},$$

with an infinite radius of convergence.

(4) The Laplace transform standard form

$$\bar{g}(s) = \mathscr{L}\{g(t)\} = \sum_{k=0}^{n} c_k \frac{k!}{(s+\lambda)^{k+1}} = \frac{N(s)}{(s+\lambda)^{n+1}}, \quad c_0, c_1, \cdots, c_n \in \mathbb{R},$$

where

$$N(s) = \sum_{k=0}^{n} c_k k!(s+\lambda)^{n-k} = \sum_{k=0}^{n} d_k s^k, \quad d_0, d_1, \cdots, d_n \in \mathbb{R},$$

is a polynomial of degree up to n. Thus, $\bar{g}(s)$ is rational.

(5) The Laplace transform series form (moment form)

$$\bar{g}(s) = \mathscr{L}\{g(t)\} = \sum_{k=0}^{\infty} b_k s^k, \quad b_0, b_1, \cdots \in \mathbb{R}, = \sum_{k=0}^{\infty} (-1)^k \frac{M_k}{k!} s^k,$$

where $M_0, M_1, \cdots$ are moments of $g(t)$, the expansion center is the origin, and the radius of convergence is λ.

The five forms of $g(t) \in P_n^{\lambda}(\Omega)$ are equivalent and equally useful in function approximations. There is a linear transformation between any two of the five forms, which can be represented by simple algebraic operations such as matrix multiplications.

3.6. **Topological properties of $P_n^\lambda(\Omega)$.** We shall define norms for $P_n^\lambda(\Omega)$ so we can discuss topological concepts such as distance, limit, and convergence for the space.

Let $f(t) \in P_n^\lambda(\Omega)$. Define the L_1 norm, the L_2 norm, and the L_∞ norm (the essential norm or the sup norm) for $P_n^\lambda(\Omega)$ as

$$\|f\|_1 = \int_0^\infty |f(t)|\, dt,$$

$$\|f\|_2 = \left[\int_0^\infty f^2(t)dt\right]^{\frac{1}{2}},$$

and

$$\|f\|_\infty = \sup_{t\in\Omega} |f(t)|,$$

respectively.

We shall show the above norms are well-defined. Obviously, $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms by the absolute convergence property in Proposition 3.22 and by the boundedness in Proposition 3.16 with the corresponding triangle inequality, respectively. To show $\|\cdot\|_2$ is a norm, we have the following propositions.

Proposition 3.32. *Let $f(t), g(t) \in P_n^\lambda(\Omega)$. Then*

$$\int_0^\infty f(t)g(t)dt, \quad \int_0^\infty f^2(t)dt, \quad \text{and} \quad \int_0^\infty g^2(t)dt$$

converge.

Proof. Let

$$f(t) = p(t)e^{-\lambda t} \quad \text{and} \quad g(t) = q(t)e^{-\lambda t},$$

where $p(t)$ and $q(t)$ are polynomials of degree up to n. Then

$$f(t)g(t) = p(t)q(t)e^{-2\lambda t}$$

is a function in $P_{2n}^{2\lambda}$. By Proposition 3.22,

$$\int_0^\infty f(t)g(t)dt$$

is convergent.

It follows immediately that $\int_0^\infty f^2(t)dt$ and $\int_0^\infty g^2(t)dt$ are also convergent. □

Proposition 3.33. *Let $f(t), g(t) \in P_n^\lambda(\Omega)$. Then*

$$\left(\int_0^\infty f^2(t)dt\right)^{\frac{1}{2}} \left(\int_0^\infty g^2(t)dt\right)^{\frac{1}{2}} \geq \int_0^\infty f(t)g(t)dt.$$

Proof. For any $X \in \mathbb{R}$,

$$(f(t)X + g(t))^2 = f^2(t)X^2 + 2f(t)g(t)X + g^2(t) \geq 0.$$

Then, by Proposition 3.32, we can write

$$X^2 \int_0^\infty f^2(t)dt + X \int_0^\infty 2f(t)g(t)dt + \int_0^\infty g^2(t)dt \geq 0.$$

This implies

$$\left(\int_0^\infty 2f(t)g(t)dt\right)^2 - 4\left(\int_0^\infty f^2(t)dt\right)\left(\int_0^\infty g^2(t)dt\right) \leq 0,$$

which is

$$\left|\int_0^\infty f(t)g(t)dt\right| \leq \left(\int_0^\infty f^2(t)dt\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(t)dt\right)^{\frac{1}{2}}.$$

□

Proposition 3.34. *(Cauchy-Bunyakovsky-Schwarz inequality). Let $f(t), g(t) \in P_n^\lambda(\Omega)$. Then*

$$\|f\|_2 + \|g\|_2 \geq \|f+g\|_2.$$

Proof.

$$\begin{aligned}
& (\|f\|_2 + \|g\|_2)^2 \\
= & \left(\left(\int_0^\infty f^2(t)dt\right)^{\frac{1}{2}} + \left(\int_0^\infty g^2(t)dt\right)^{\frac{1}{2}}\right)^2 \\
= & \int_0^\infty f^2(t)dt + 2\left(\int_0^\infty f^2(t)dt\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(t)dt\right)^{\frac{1}{2}} + \int_0^\infty g^2(t)dt \\
\geq & \int_0^\infty f^2(t)dt + 2\int_0^\infty f(t)g(t)dt + \int_0^\infty g^2(t)dt \\
= & \int_0^\infty f^2(t) + 2f(t)g(t) + g^2(t)dt \\
= & \|f+g\|_2^2,
\end{aligned}$$

where the inequality follows from Proposition 3.33.

Thus,

$$\|f\|_2 + \|g\|_2 \geq \|f+g\|_2$$

and $\|\cdot\|_2$ satisfies the triangle inequality. It follows that $\|\cdot\|_2$ is a norm. □

Let $p > 1$. We may generalize the L_2 norm to the L_p norm for $P_n^\lambda(\Omega)$ as

$$\|f\|_p = \left(\int_0^\infty |f(t)|^p\, dt\right)^{\frac{1}{p}}.$$

The proof of the existence of the L_p norm needs the Minkowski inequality (Hermann Minkowski, 1864-1909) and the Hölder's inequality (Otto Hölder, 1859-1937). Then $(P_n^\lambda(\Omega), \|\cdot\|_p)$ is a subspace of the $L_p(\Omega)$ space. This will be left for future research.

Each norm on $P_n^\lambda(\Omega)$ induces a topology on $P_n^\lambda(\Omega)$. The above three norms induce three normed topologies on $P_n^\lambda(\Omega)$. By Proposition 2.23, the three topologies are equivalent. In this monograph, we use a generic norm notation $\|\cdot\|$ for $P_n^\lambda(\Omega)$ unless a specific norm is involved. Thus, $(P_n^\lambda(\Omega), \|\cdot\|)$, or simply $P_n^\lambda(\Omega)$, is a normed space.

For any two functions $f, g \in P_n^\lambda(\Omega)$, we define the three metric functions induced by the indicated norms as

$$\rho_1(f,g) = \|f-g\|_1 = \int_0^\infty |f(t)-g(t)|\,dt,$$

$$\rho_2(f,g) = \|f-g\|_2 = \left[\int_0^\infty (f(t)-g(t))^2\,dt\right]^{\frac{1}{2}},$$

and

$$\rho_\infty(f,g) = \|f-g\|_\infty = \sup_{t\in[0,\infty)} |f(t)-g(t)|.$$

In this monograph, we use a generic metric (distance or error) function $\rho(\cdot,\cdot) : P_n^\lambda(\Omega) \times P_n^\lambda(\Omega) \to [0,\infty)$ defined as

$$\rho(f,g) = \|f-g\|.$$

Then $(P_n^\lambda(\Omega), \rho)$ is a metric space. By ρ, we usually mean the essential metric ρ_∞, which implies the uniform convergence in the space.

Let an *open ball* in $P_n^\lambda(\Omega)$ be the set of all functions in $P_n^\lambda(\Omega)$ having a distance to a center function $f \in P_n^\lambda(\Omega)$ less than a radius $r > 0$, denoted by

$$B_r(f) = \left\{ g \in P_n^\lambda(\Omega) \mid \|g-f\| < r \right\}.$$

The corresponding *closed ball* and *sphere* are defined as

$$\bar{B}_r(f) = \left\{ g \in P_n^\lambda(\Omega) \mid \|g-f\| \le r \right\}$$

and

$$S_r(f) = \left\{ g \in P_n^\lambda(\Omega) \mid \|g-f\| = r \right\},$$

respectively.

Using the concept of open balls, we can define open sets, closed sets, and an induced topology for $P_n^\lambda(\Omega)$.

A function sequence $\{f_m\}$ in $P_n^\lambda(\Omega)$ is said to *converge* to a *limit function* $f \in P_n^\lambda(\Omega)$, denoted by

$$\lim_{m\to\infty} f_m(t) = f(t),$$

if and only if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\|f_m - f\| < \epsilon, \quad \text{for all } m > N,$$

or

$$f_m \in B_\epsilon(f), \quad \text{for all } m > N.$$

The convergence defined above is the uniform convergence. In some cases, we may use the L_1- or the L_2-convergence in $P_n^\lambda(\Omega)$, which is topologically equivalent to the uniform convergence.

Proposition 3.35. *Let*

$$f_m(t) = \sum_{k=0}^{n} c_{k,m} t^k e^{-\lambda t}$$

and

$$f(t) = \sum_{k=0}^{n} c_k t^k e^{-\lambda t}$$

be functions in $P_n^\lambda(\Omega)$. Then

$$\lim_{m\to\infty} f_m(t) = f(t)$$

if and only if

$$\lim_{m\to\infty} c_{k,m} = c_k, \quad k = 0, 1, \cdots, n.$$

Proof. Let

$$M_k = \left\| t^k e^{-\lambda t} \right\| = \max_{t\in\Omega} \left| t^k e^{-\lambda t} \right|, \quad k = 0, 1, \cdots, n.$$

For any $\epsilon > 0$, if

$$\lim_{m\to\infty} c_{k,m} = c_k,$$

for $k = 0, 1, \cdots, n$, then there is an $N_0 \in \mathbb{N}$ such that when $m > N_0$,

$$|c_{0,m} - c_0| < \frac{\epsilon}{n+1} \frac{1}{M_0}.$$

It follows there is an $N_k \in \mathbb{N}$ such that when $m > N_k$,

$$|c_{k,m} - c_k| < \frac{\epsilon}{n+1} \frac{1}{M_k}$$

for $k = 1, 2, \cdots, n$.

Thus, whenever $m > \max(N_0, N_1, \cdots, N_n)$, we have

$$\begin{aligned}
\left\| \sum_{k=0}^{n} (c_{k,m} - c_k) t^k e^{-\lambda t} \right\| &\leq \sum_{k=0}^{n} |c_{k,m} - c_0| \left\| t^k e^{-\lambda t} \right\| \\
&\leq \sum_{k=0}^{n} \frac{\epsilon}{n+1} \frac{1}{M_k} M_k \\
&= \epsilon.
\end{aligned}$$

Therefore,

$$\lim_{m \to \infty} f_m(t) = f(t).$$

The proof for the converse statement is omitted. □

A function sequence $\{f_m\}$ in $P_n^\lambda(\Omega)$ is said to be a *Cauchy* sequence if, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, k > N$, we have

$$\|f_m - f_k\| < \epsilon.$$

Proposition 3.36. *Every Cauchy sequence in $P_n^\lambda(\Omega)$ converges to a limit in $P_n^\lambda(\Omega)$.*

Proof. Consider a Cauchy sequence $f_m \in P_n^\lambda$. Then for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that whenever $m, k > N$, we have

$$\|f_m - f_k\| < \epsilon.$$

Let $K_N = \{f_{N+1}, f_{N+2}, \cdots\}$. Then $\operatorname{diam} K_N < \epsilon$ and K_N is bounded. Thus, there is a closed (or compact) set $\overline{K}_N$, the closure of K_N, such that

$$\operatorname{diam} \overline{K}_N = \operatorname{diam} K_N < \epsilon.$$

Consider the sequence of numbers $\{\epsilon_i = \frac{\epsilon}{2^i}, i = 0, 1, \cdots\}$ converging to 0. Then there is a sequence of compact sets $\{\overline{K}_{N_i}, i = 0, 1, \cdots\}$ such that

$$\overline{K}_{N_{i+1}} \subset \overline{K}_{N_i}$$

and

$$\operatorname{diam} \overline{K}_{N_i} < \frac{\epsilon}{2^i},$$

for $i = 0, 1, \cdots$, which is

$$\lim_{i \to \infty} \operatorname{diam} \overline{K}_{N_i} = 0.$$

This implies there exists a point

$$f \in \cap_{i=0}^{\infty} \overline{K}_{N_i}.$$

Clearly, when $m > N$, we have

$$\|f_m - f\| < \epsilon.$$

Thus, the Cauchy sequence $\{f_m\}$ converges to f in $P_n^\lambda(\Omega)$. □

Proposition 3.37. *$P_n^\lambda(\Omega)$ is complete and is a Banach space.*

Proof. By Proposition 3.36, every Cauchy sequence in $P_n^\lambda(\Omega)$ converges to a limit in the space. It follows that $P_n^\lambda(\Omega)$ is complete and is a Banach space. $\square$

3.7. **Isomorphisms for $P_n^\lambda(\Omega)$.** Let $T : P_n(\Omega) \to P_n^\lambda(\Omega)$ be a transformation which maps each $p \in P_n(\Omega)$ to an $f \in P_n^\lambda(\Omega)$ for every fixed $\lambda > 0$ as

$$f(t) = T\{p(t)\} = p(t)e^{-\lambda t}, \quad t \in \Omega.$$

Proposition 3.38. *T is an isomorphism from $P_n(\Omega)$ to $P_n^\lambda(\Omega)$.*

Proof. Clearly, T is surjective (or onto) since for every element $f \in P_n^\lambda(\Omega)$, there exists an element $p \in P_n(\Omega)$ such that $f(t) = p(t)e^{-\lambda t}$, $t \in \Omega$. T is also injective (one-to-one) since $T(p_1) = T(p_2)$ implies $p_1 = p_2$. Thus, T is bijective. Therefore, T is an isomorphism. $\square$

Proposition 3.39. *The above transformation T has the following properties:*

(1) T maps the zero element in $P_n(\Omega)$ to the zero element in $P_n^\lambda(\Omega)$.
(2) T maps the natural basis in $P_n(\Omega)$ to the natural basis in $P_n^\lambda(\Omega)$.

Proof. Trivial. $\square$

Proposition 3.40. *For each fixed $n \in \mathbb{N}$ and any $\lambda_1, \lambda_2 \geq 0$, $P_n^{\lambda_1}(\Omega)$ is isomorphic to $P_n^{\lambda_2}(\Omega)$.*

Proof. For each fixed $n \in \mathbb{N}$ and any $\lambda_1, \lambda_2 > 0$, Proposition 3.38 implies that both $P_n^{\lambda_1}(\Omega)$ and $P_n^{\lambda_2}(\Omega)$ are isomorphic to $P_n(\Omega)$. It follows that $P_n^{\lambda_1}(\Omega)$ is isomorphic to $P_n^{\lambda_2}(\Omega)$. $\square$

Proposition 3.41. *$P_n^\lambda(\Omega)$ is isomorphic to $\mathbb{R}^{n+1}$.*

Proof. For every vector $\vec{a} = [a_0, a_1, \cdots, a_n]^T \in \mathbb{R}^{n+1}$, there exists a continuous mapping A defined as

$$f(t) = A(\vec{a}) = \left(\sum_{k=0}^{n} a_k t^k\right) e^{-\lambda t},$$

which maps $\vec{a} \in \mathbb{R}^{n+1}$ to $f(t) \in P_n^\lambda(\Omega)$. Conversely, for every $f(t) \in P_n^\lambda$, there exists a continuous mapping which extracts the coefficients of $f(t)$ and forms a vector in $\mathbb{R}^{n+1}$. Thus, A is bijective or one-to-one. Therefore, A is an isomorphism between the two spaces. $\square$

The isomorphism between any two of $P_n^\lambda(\Omega)$, $P_n(\Omega)$, and $\mathbb{R}^{n+1}$ is a linear, one-to-one, and invertible transformation. Therefore, the topological properties of the three spaces are equivalent. It may allow us to convert a difficult mathematical problem from one space to another, where the problem may be easily solved, and to convert the solution back to the original space.

3.8. **Subspace structures and expansions of $P_n^\lambda(\Omega)$.**

Proposition 3.42. *For every $\lambda > 0$, $P_n^\lambda(\Omega)$ is a subspace of $P_m^\lambda(\Omega)$ if and only if $0 \le n \le m,\ n, m \in \mathbb{N}$.*

Proof. It is trivial to show the natural basis of $P_n^\lambda(\Omega)$ is a subset of that of $P_m^\lambda(\Omega)$ for $0 \le n \le m$. Thus, $P_n^\lambda(\Omega)$ is a subspace of $P_m^\lambda(\Omega)$. □

It follows that for any $\lambda > 0$

$$P_0^\lambda(\Omega) \subset P_1^\lambda(\Omega) \subset \cdots \subset P_n^\lambda(\Omega) \subset \cdots .$$

Clearly, this is a countable family of partially ordered spaces $\{P_k(\Omega)\}_{k=0}^\infty$. Let $B = \{e^{-\lambda t}, te^{-\lambda t}, t^2 e^{-\lambda t}, \cdots\}$ on Ω and denote

$$P_\infty^\lambda(\Omega) = \bigcup_{n=0}^\infty P_n^\lambda(\Omega) = \operatorname{span} B.$$

Then, for any $\lambda > 0$ and $n \in \mathbb{N}$,

$$P_n^\lambda(\Omega) \subset P_\infty^\lambda(\Omega).$$

It also follows from Proposition 3.42 that for any $\lambda > 0$, by adding more basis functions, we can expand $P_n^\lambda(\Omega)$ to $P_m^\lambda(\Omega)$ for $0 \le n \le m$.

Denote

$$P_{m\backslash n}^\lambda(\Omega) = \operatorname{span}\{t^{n+1}e^{-\lambda t}, t^{n+2}e^{-\lambda t}, \cdots, t^m e^{-\lambda t}\}.$$

Clearly,

$$\dim P_{m\backslash n}^\lambda(\Omega) = m - n.$$

If $f_{m\backslash n}(t) \in P_{m\backslash n}^\lambda(\Omega)$, then $f_{m\backslash n}(t)$ is also in $P_m^\lambda(\Omega)$.

It is obvious that for any $\lambda > 0$, we have

$$P_n^\lambda(\Omega) \oplus P_{m\backslash n}^\lambda(\Omega) = P_m^\lambda(\Omega).$$

Let $n \in \mathbb{N}$ be fixed and $\lambda > 0$ vary as a parameter. Then $P_n^\lambda(\Omega)$ denotes a continuum family of spaces, or a set of parametric spaces with a parameter λ. This is called the parametrization of $P_n^\lambda(\Omega)$.

When λ tends to 0, we have

$$\lim_{\lambda \to 0} P_n^\lambda(\Omega) = P_n(\Omega).$$

When λ tends to infinity, $p_n(t)e^{-\lambda t}$ tends to a discontinuous function at the origin. From this, we define a generalized Dirac delta function space as

$$P_n^\infty(\Omega) = \lim_{\lambda \to \infty} P_n^\lambda(\Omega).$$

Then $P_n^\infty(\Omega)$ contains all the generalized Dirac delta functions of degree up to n with finite magnitudes and various shapes. The underlying domain Ω of $P_n^\infty(\Omega)$ shrinks to the support $[0, 0^+]$. We will leave the analysis of $P_n^\infty(\Omega)$ for future research.

Let $n \in \mathbb{N}$ and $p_n(t) \in P_n(\Omega)$ be fixed. Then $p_n(t)e^{-\lambda t}$ is a family of parametric continuous functions in λ, each of which belongs to a particular $P_n^\lambda(\Omega)$ space. $p_n(t)e^{-\lambda t}$ can also be viewed as the function $p_n(t)$ going through a continuous shape changing (or an attenuation) as λ increases from 0 to the current value.

We can expand $P_n^\lambda(\Omega)$ spaces by direct sums. Let

$$P_{n_1}^{\lambda_1}(\Omega) = \operatorname{span}\{t^k e^{-\lambda_1 t}\}_{k=0}^{n_1}$$

and

$$P_{n_2}^{\lambda_2}(\Omega) = \operatorname{span}\{t^k e^{-\lambda_2 t}\}_{k=0}^{n_2}.$$

Then the direct sum of $P_{n_1}^{\lambda_1}(\Omega)$ and $P_{n_2}^{\lambda_2}(\Omega)$ is

$$P_{n_1}^{\lambda_1}(\Omega) \oplus P_{n_2}^{\lambda_2}(\Omega) = \operatorname{span}\{t^k e^{-\lambda_1 t}\}_{k=0}^{n_1} \cup \{t^j e^{-\lambda_2 t}\}_{j=0}^{n_2}.$$

Thus, the natural basis of the above direct sum is the union of the two bases

$$\{t^k e^{-\lambda_1 t}\}_{k=0}^{n_1} \cup \{t^j e^{-\lambda_2 t}\}_{j=0}^{n_2}$$

and the dimension of the new space is

$$\dim P_{n_1}^{\lambda_1}(\Omega) \oplus P_{n_2}^{\lambda_2}(\Omega) = n_1 + n_2 + 2.$$

In general, for any finite N, we have

$$\bigoplus_{i=1}^{N} P_{n_i}^{\lambda_i}(\Omega) = P_{n_1}^{\lambda_1}(\Omega) \bigoplus P_{n_2}^{\lambda_2}(\Omega) \bigoplus \cdots \bigoplus P_{n_N}^{\lambda_N}(\Omega),$$

with the basis

$$\cup_{i=1}^{N}\{t^k e^{-\lambda_i t}\}_{k=0}^{n_i}$$

and the dimension

$$\dim \bigoplus_{i=1}^{N} P_{n_i}^{\lambda_i}(\Omega) = \sum_{i=1}^{N} n_i + N.$$

This is a class of functions whose Laplace transforms are meromorphic functions with real poles.

Let c be a constant representing a constant function on Ω. The set $c \oplus P_n^\lambda(\Omega)$ denotes a *linear manifold of translation* c, which is a non-empty set satisfying the following property: for every function $f(t) \in c \oplus P_n^\lambda(\Omega)$, there exists a function $g(t) \in P_n^\lambda(\Omega)$ such that $f(t) = g(t) + c$. The definition of the linear manifold implies the space $P_n^\lambda(\Omega)$ is *translated* by c units.

In addition, we may define

$$\dim c \oplus P_n^\lambda(\Omega) = \dim P_n^\lambda(\Omega) = n + 1.$$

Proposition 3.43. *Let $f(t) \in c \oplus P_n^\lambda(\Omega)$. Then*

$$\lim_{t\to\infty} f(t) = c.$$

Proof. Let $f(t) = g(t) + c \in c \oplus P_n^\lambda(\Omega)$ such that $g(t) \in P_n^\lambda(\Omega)$. Then

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) + \lim_{t\to\infty} c = c,$$

since

$$\lim_{t\to\infty} g(t) = 0.$$

□

Proposition 3.44. *For each fixed c, $c \oplus P_n^\lambda(\Omega)$ is isomorphic to $P_n^\lambda(\Omega)$.*

Proof. This is trivial, since translation is one-to-one. □

A linear manifold $c \oplus P_n^\lambda(\Omega)$ with variable c can be viewed as the set of the indefinite integral on the space $P_n^\lambda(\Omega)$. Another view of a manifold is that $P_n^\lambda(\Omega)$ is homogeneous and $c \oplus P_n^\lambda(\Omega)$ is non-homogeneous. We shall leave the analysis of $c \oplus P_n^\lambda(\Omega)$ for future research.

4. Approximations in $P_n^\lambda(\Omega)$ spaces

This section discusses one of the most important applications of $P_n^\lambda(\Omega)$ spaces: function approximations. A modern view of function approximation is to create an approximation function space with some desirable properties and then to choose a function in the space which is "close" to the target function. In this view, there is only one function space to choose the approximation function from. However, in our new approximation theory, we search for the approximation function from a continuum family of function spaces.

In this section, we shall first discuss the general theory of approximations to a special class of target functions and then develop three specific kinds of $P_n^\lambda(\Omega)$ based approximation methods with illustrative examples.

4.1. The General Theory of Approximations in $P_n^\lambda(\Omega)$ Spaces.

4.1.1. *Target functions.* Initially, we aim to approximate a class of continuous probability density functions defined on Ω, which are continuous, bounded, non-negative, and integral-convergent. After a careful study of the above properties, we conclude a more general class of target functions should be considered. This new target function class can be formed, by omitting the non-negativity and integral-convergence requirements of the former class and adding the vanishing at infinity, as

$$C_0^B(\Omega) = \left\{ f \in C(\Omega) \cap B(\Omega) \;\middle|\; \lim_{t\to\infty} f(t) = 0 \right\}.$$

In fact, $C_0^B(\Omega)$ is $V(\Omega)$ and the boundedness in the definition is not necessary. In addition, this definition may be further generalized to allow any finite limit at infinity.

Proposition 4.1. *Every continuous integral-convergent function on Ω belongs to $C_0^B(\Omega)$.*

Proof. By Proposition 2.10, every continuous integral-convergent function on Ω vanishes at infinity and therefore belongs to $C_0^B(\Omega)$. □

It follows immediately that $C_0^B(\Omega)$ contains all the continuous probability density functions on Ω. In addition, not all vanishing at infinity functions are integral-convergent, e.g. $\frac{1}{1+t}$. Thus, $C_0^B(\Omega)$ also contains some integral-divergent functions, e.g. some rational functions, which are useful. The new approximation theory is mainly developed for "smooth", infinitely differentiable, or transcendental functions. It may also be used for approximations to piecewise continuous or even discontinuous functions on Ω.

4.1.2. *Notations and conventions.* In order to better describe $P_n^\lambda(\Omega)$ space approximation theories and methods, we shall use the following notations and conventions throughout this chapter.

We always denote $f(t) \in C_0^B(\Omega)$ as the target function and may assume $f(t)$ is non-negative on Ω without loss of generality. We consider $P_n^\lambda(\Omega)$ as the approximation function space and denote the approximation function $g(t)$ or $g_n(t)$ in $P_n^\lambda(\Omega)$ as

$$\begin{aligned} g(t) = g_n(t) &= c_0 e^{-\lambda t} + c_1 t e^{-\lambda t} + \cdots + c_n t^n e^{-\lambda t} \\ &= \left(c_0 + c_1 t + \cdots + c_n t^n\right) e^{-\lambda t} \\ &= p_n(t) e^{-\lambda t}, \end{aligned}$$

where

$$p_n(t) = \sum_{k=0}^{n} c_k t^k$$

is a polynomial in t of degree up to n, and $\lambda > 0$ is the decaying parameter. In this standard form, $g_n(t)$ can be viewed either as the product of the polynomial $p_n(t)$ and the exponential decaying function $e^{-\lambda t}$ or as a linear combination of basis functions $t^k e^{-\lambda t}$, $k = 0, 1, \cdots, n$.

We sometimes use $g(t;\lambda)$ or $g_n(t;\lambda)$ to emphasize the approximation function has an undetermined parameter λ or to express a particular approximation function for a particular value of λ. In many situations, we treat λ as a constant to avoid taking partial derivatives of a double variable function $g(t;\lambda)$. This allows us to talk about the Taylor series or the Laplace transforms of $g(t;\lambda)$ in a simple way.

Technically, an approximation problem is solved if a unique function $g_n(t;\lambda)$ on Ω is found, i.e. the degree n, the coefficients c_k, and the decaying parameter λ of $g_n(t;\lambda)$ are all solved for. The degree n can be arbitrarily chosen in advance and is usually set to $10 \sim 20$ for many practical approximation problems. Sometimes, the parity of n, even or odd, may be considered by a particular approximation method. The *shape coefficients* c_k, $k = 0, 1, \cdots, n$, determine the curve shape of the approximation function. So we name each of our approximation methods by the way the shape coefficients are solved. The decaying parameter λ also affects the curve shape of the approximation function but not in the same manner as the shape coefficients. It can be determined by imposing an additional condition or restriction of various kinds on $g_n(t;\lambda)$. If the restriction results in multiple values for λ, we choose the one that minimizes the approximation error.

On obtaining the final approximation function $g_n(t)$, we shall calculate its integration function

$$G_n(t) = \int_0^t g_n(\tau) d\tau$$

and compare it with the target integration function

$$F(t) = \int_0^t f(\tau)d\tau.$$

4.1.3. *Main interval and tail interval.* One of the key ideas in $P_n^\lambda(\Omega)$ space approximation theory is to partition the domain Ω of the target function. Generally in an approximation problem, the approximation function is required to be "close" to the target function on the entire domain. Such problems become difficult when the domain is an unbounded interval such as Ω, and currently there seems to be no effective resolution for them. The difficulty is mainly because there is no easy way to define a usual norm on a non-compact interval for generic functions without worrying about convergence. For example, can we properly define a norm for a polynomial, which is unbounded and integral-divergent, on Ω?

On the other hand, it may not be necessary to approximate a target function in $C_0^B(\Omega)$ on the entire domain Ω because by vanishing at infinity, the function value is too small to be useful as the independent variable gets sufficiently large. This implies a single uniform error (or maximum error) may not properly measure the approximation error for the tail of the function.

A reasonable treatment is to split the approximation problem into two parts and deal with each part individually. This can be done by partitioning the entire domain Ω into a compact interval and an unbounded interval. We can use existing methods for the approximation on the compact interval to achieve good approximation properties such as uniform convergence. For the approximation on the unbounded interval, we reduce the problem and only require the approximation function to be bounded and vanishing at infinity on the interval.

The principle of partitioning Ω is based on the tail boundedness property of the target function space. By Proposition 2.4, for any $f(t) \in C_0^B(\Omega)$ and $\epsilon > 0$, there exists a T such that $|f(t)|$ is bounded by ϵ on $[T, \infty)$. Thus, we can partition Ω into the *main interval* $[0, T]$ and the *tail interval* $[T, \infty)$, where T is the *partition point* or the *cut-off point*. The new approximation method is to find an approximation function $g(t)$ that is "close" to $f(t)$ on $[0, T]$ and is bounded by ϵ (or maybe a little larger) and vanishing at infinity on $[T, \infty)$. Fortunately, the boundedness and vanishing at infinity are the intrinsic properties of the approximation function space $P_n^\lambda(\Omega)$. Thus, we only need to control the maximum value of the approximation function on $[T, \infty)$. Although T can be chosen arbitrarily, it should be chosen so the target function has its major, dominant, and essential aspect on the main interval and its secondary, insignificant, and trivial aspect on the tail interval. For example, a probability density function on Ω can be partitioned so that its tail probability is less than 5%.

4.1.4. *From $P_n(\Omega)$ to $P_n^\lambda(\Omega)$.* Before we introduce $P_n^\lambda(\Omega)$ based approximation methods, let us digress to discuss why polynomials are not appropriate for approximations to continuous probability density functions on Ω. In fact, our new approximation method is motivated by, and is an improvement of, the former.

Polynomials have many good properties. They are simple, continuous, bounded on any closed intervals, and related to power series as partial sums. By Theorem 3.7 (Stone-Weierstrass approximation theorem), polynomials can approximate any continuous function on any closed or compact intervals. It is natural to attempt to use polynomials to approximate continuous probability density functions.

However, polynomials are unbounded at infinity and their integrals on Ω are divergent. These fundamental disadvantages make it impossible to approximate any function about infinity. For example, in a polynomial interpolation problem, the approximation polynomial may only approximate the target function inside the interpolation interval and will not do so elsewhere.

A polynomial may be attenuated by another polynomial so that the resulting function may approach zero at infinity. The collection of such functions is a small subset of rational functions. But rational functions are integral-divergent even if they vanish at infinity on Ω. It is concluded the set of rational functions is also not a good approximation function space for our problem.

By Proposition 3.8, any polynomial of finite degree can be attenuated by an exponential decaying function to vanish at infinity. This property leads to the introduction of the decaying polynomial space $P_n^\lambda(\Omega)$. By Propositions 3.16, 3.18, and 3.22, all functions in $P_n^\lambda(\Omega)$ are bounded, vanishing at infinity, and integral-convergent. By choosing particular values of the shape coefficients and the decaying parameter of a decaying polynomial function, we can control the bound of the function on the tail interval. All these desirable properties make $P_n^\lambda(\Omega)$ an ideal approximation function space to $C_0^B(\Omega)$ functions.

4.1.5. *Three kinds of approximation methods in $P_n^\lambda(\Omega)$.* In this chapter, we shall discuss three kinds of constructive approximation methods to our approximation problem. They are method I, asymptotic (or Taylor, power) series expansion, in Section 4.2, method II, Laplace transform moment matching, in Section 4.3, and method III, interpolation, in Section 4.4.

These three methods apply to different situations but they share a common process for finding the final approximation function. This is not surprising since they are all based on linear structures of $P_n^\lambda(\Omega)$ spaces. Each method sets up a specific system of equations to solve for the shape coefficients of the approximation function according to a specific linear structure of the space. This process of finding the shape coefficients itself implies the existence of the approximation function. Which method to use depends on how the approximation problem is linearized.

The process of each method can be described in two steps. In the first step, we assume λ is a fixed number and set up a system of $n+1$ equations by imposing $n+1$ independent conditions on the approximation function. In method I, initial or boundary conditions from the power series expansion of the target function are used to construct the system of equations; in method II, the Laplace transform power series coefficients of the target function are used; and in method III, the interpolation nodes are used to set up the system of equations. By solving the system of equations in each method, we obtain the shape coefficients and the corresponding approximation function $g_n(t;\lambda)$. Now, $g_n(t;\lambda)$ is a function of λ. Let λ vary. We obtain a *one-parameter family* of approximation functions.

The second step is to determine the optimal value for λ in $g_n(t;\lambda)$ so that the approximation error on the main interval is somewhat minimum. This step is an optimization problem and varies from one method to another. When λ changes continuously, both the basis functions of $P_n^\lambda(\Omega)$ and the shape coefficients are also changing continuously. Thus, the curve shape of $g_n(t;\lambda)$ is continuously going through deformation while still maintaining the restrictions set for the approximation function in the first step. Imposing an additional condition on $g_n(t;\lambda)$, we can find the optimal value(s) for λ and then the final approximation function.

It should be noted that various characteristics of the target function can be used to determine λ, e.g. the target function value at a specific point, the definite integral or the curve length of the target function on the main interval. The validity of these methods is based on the idea these characteristics of the target function have been preserved under uniform convergence. We do not intend to prove this general statement. But if the characteristic is a linear operator, e.g. a definite integral, then we are confident it is preserved by uniform convergence. By varying λ, we can either match the characteristics of the target and the approximation functions or make them very close.

4.1.6. *Error analysis.* Error patterns are like "fingerprints" of the approximation methods. Each approximation method has a unique error pattern reflecting the nature of its mathematical principle. By observing the error pattern, we can learn not only how large it is but also what approximation method generated it.

In our new approximation methods, we only consider the approximation error function on the main interval, which corresponds to and gives more information than the weak error or the weak metric induced by the weak norm. Let $g_n(t;\lambda)$ be an approximation function to $f(t)$. Then the approximation error function is

$$e_n(t) = e_n(t;\lambda) = f(t) - g_n(t;\lambda)$$

and the absolute error function is

$$|e_n(t)| = |f(t) - g_n(t;\lambda)|.$$

There are two major types of main interval errors, the *asymptotic type error* and the *Chebyshev type error*, in our approximation methods. Each type error reflects the nature of each approximation method applied. Asymptotic type errors appear in method I, where there is no error at the expansion center and the absolute error function is monotonically increasing as the independent variable gets further away from the expansion center inside its convergence interval. This clearly shows asymptotic series expansion is somewhat a local approximation method. Chebyshev type errors appear in method II and III, where the error function oscillates in finite times on the main interval. The name Chebyshev type error is in memory of the Russian mathematician Pafnuty Chebyshev (1821-1894), who first introduced the concept of distribution of interpolation nodes (or Chebyshev nodes) in order to achieve uniform approximation in polynomial interpolation problems. In method II, due to the global property of moment integrals, the approximation error is somewhat uniformly distributed on the main interval and is distinctive from the asymptotic type error. Because method III forces the approximation error to zero at interpolation nodes, the resulting error also shows bounded oscillation patterns. However, the error in method III also has an asymptotic pattern. The magnitude of the error oscillation around the interpolation center is very small but increases dramatically near both end-points of the interpolation interval. Thus, the error pattern in method III is in fact a mixture of the asymptotic type and the Chebyshev type.

Besides studying graphs of approximation error functions, we sometimes use a single number to measure the approximation error. We define the *weak maximum absolute error* (WMAE) on $[0,T]$ as

$$\mathrm{WMAE}_{[0,T]} = \|e_n(t)\|^w_{\infty,[0,T]} = \max_{t\in[0,T]} |f(t) - g_n(t)|$$

and the *weak mean square error* (WMSE) on $[0,T]$ as

$$\mathrm{WMSE}_{[0,T]} = \|e_n(t)\|^w_{2,[0,T]} = \int_{[0,T]} (f(t) - g_n(t))^2 \, dt.$$

It should be pointed out that studying error functions or using weak measures on the main interval must be accompanied by checking the boundedness of an approximation function on the tail interval.

4.2. **Method I: asymptotic/Taylor series expansion in $P_n^\lambda(\Omega)$.** The first approximation method (method I) is the asymptotic series expansion in $P_n^\lambda(\Omega)$ spaces. The idea is to use initial or boundary conditions, the derivatives of the target function at the origin or other points, to construct an approximation function in $P_n^\lambda(\Omega)$ which is "close" to the target function. The method is motivated by the polynomial approximation method from the power series (or Taylor series) expansion of the target function, which fails in approximation of $C_0^B(\Omega)$ functions.

Since method I needs higher order derivatives of the target function at the origin or other points, we assume the target function is analytic on Ω. We shall mainly approximate two types of analytic functions on Ω: the *finite-radius-convergent* and the *infinite-radius-convergent* analytic functions, according to their power series expansion about the origin. We further assume a finite-radius-convergent function has its largest singularity in the negative real axis, i.e. a finite-radius-convergent function has a half-convergent half-divergent power series about the origin. To effectively approximate each type of target function, we introduce some variants (Ia, Ib, and Ic) to the asymptotic series expansion method.

4.2.1. *Method* I*a: asymptotic series expansion about the origin.* Let the target function $f(t) \in C_0^B(\Omega)$ be analytic on Ω. Assume $f(t)$ has its largest singularity at $t = -R$, $R > 0$. Then the power series expansion of $f(t)$ about the origin converges on interval $[0, R)$. Let $[0, T]$, $T > 0$, be the main interval. We wish to find a function $g(t) = g_n(t; \lambda) \in P_n^\lambda(\Omega)$ for some $\lambda > 0$ satisfying $n + 1$ initial conditions of $f(t)$ at the origin, i.e.

$$g(0) = f(0), \quad g'(0) = f'(0), \quad \cdots, \quad g^{(n)}(0) = f^{(n)}(0).$$

Then we shall call $g(t)$ an approximation function to $f(t)$ if the approximation error is sufficiently small on $[0, T]$ and $g(t)$ is satisfactorily bounded on $[T, \infty)$.

Equivalently, the above initial conditions can be rephrased in the form of power series or Taylor series expansion. Let $f(t)$ be expanded into a power series about the origin as

$$f(t) = \sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k, \quad t \in [0, R),$$

and $g(t)$ be expanded into a power series about the origin as

$$g(t) = \sum_{k=0}^{\infty} a'_k t^k, \quad t \in \Omega$$

(This is a property of $P_n^\lambda(\Omega)$). Then the initial conditions are

$$a'_k = a_k, \quad k = 0, 1, \cdots, n. \tag{4.1}$$

We shall show such a $g(t)$ exists.

First, we cannot directly use power series expansion as an approximation method because the power series of $f(t)$ about the origin is only convergent on $[0, R)$ and divergent on $[R, \infty)$. In order to treat such half-convergent half-divergent series, we introduce the concept of asymptotic series expansion. Let us expand $f(t)$ asymptotically into a power series about the origin with respect to a basis $\{t^k\}_{k=0}^{\infty}$ as

$$f(t) \sim \sum_{k=0}^{\infty} a_k t^k, \quad t \to 0.$$

Then this asymptotic power series coincides with the MacLaurin series of $f(t)$ on $[0, R)$. An asymptotic power series is only a formal power series and is still divergent on $[R, \infty)$. However, this allows us to treat a power series, regardless of its convergence, as if it has a "limit function" on Ω. Thus, an asymptotic series may be operated on the same way as a convergent series or a usual real function on Ω.

Now let us expand $f(t)$ asymptotically into another series about the origin on Ω with respect to a basis $\{t^k e^{-\lambda t}\}_{k=0}^{\infty}$ satisfying the initial conditions (4.1). Then we may write

$$f(t) \sim \sum_{k=0}^{\infty} c_k t^k e^{-\lambda t}, \quad t \to 0, \tag{4.2}$$

where c_k, $k = 0, 1, \cdots$, are undetermined real coefficients. The main difference between the asymptotic power series and the new asymptotic series is the basis for the former is unbounded while for the latter it is bounded and vanishing at infinity. This can be easily shown by Propositions 3.15 and 3.17. Although the new asymptotic series is still divergent outside its convergence interval, we impose a strong restriction there so its partial sums are bounded and vanishing at infinity.

Define the n-th partial sum of the new asymptotic series as

$$g_n(t) = \sum_{k=0}^{n} c_k t^k e^{-\lambda t}, \tag{4.3}$$

which can also be written as

$$g_n(t) = p_n(t) e^{-\lambda t},$$

where

$$p_n(t) = \sum_{k=0}^{n} c_k t^k.$$

The coefficients c_k of the new asymptotic series in (4.2) satisfying the initial conditions (4.1) are in fact the coefficients of a Cauchy product defined as follows.

For any $\lambda > 0$,

$$e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} t^k$$

absolutely on $\mathbb{R}$. Define the Cauchy product

$$\left(\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k\right)\left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} t^k\right) = \sum_{k=0}^{\infty} c_k t^k, \tag{4.4}$$

where the coefficient sequence

$$\begin{aligned} c_k &= \sum_{i=0}^{k} \frac{f^{(i)}(0)}{i!} \frac{\lambda^{k-i}}{(k-i)!} \\ &= \sum_{i=0}^{k} a_i \frac{\lambda^{k-i}}{(k-i)!}, \quad k = 0, 1, \cdots, \end{aligned}$$

is the convolution of the coefficients of the two factor series.

Theorem 4.2. *For every $\lambda > 0$, the Cauchy product in (4.4) converges to $f(t)e^{\lambda t}$ absolutely for every $t \in [0, R)$.*

Proof. By Proposition 2.36, the power series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k$ converges to $f(t)$ absolutely for every $t \in [0, R)$ and $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} t^k$ converges to $e^{\lambda t}$ absolutely for every $t \in \Omega$. Then, by Theorem 2.30, the Cauchy product in (4.4) converges to $f(t)e^{\lambda t}$ absolutely for every $t \in [0, R)$. $\square$

The above theorem implies

$$f(t)e^{\lambda t} = \sum_{k=0}^{\infty} c_k t^k, \quad t \in [0, R).$$

Multiplying $e^{-\lambda t}$ on both sides, we have

$$f(t) = \left(\sum_{k=0}^{\infty} c_k t^k\right) e^{-\lambda t} = \sum_{k=0}^{\infty} c_k t^k e^{-\lambda t}, \quad t \in [0, R), \tag{4.5}$$

absolutely.

If $f(t)$ is infinite-radius-convergent about the origin ($R = \infty$), then Equation (4.5) holds for every $t \in \Omega$. If $f(t)$ is finite-radius-convergent about the origin ($R < \infty$), then (4.5) holds only for $t \in [0, R)$ and the function series $\sum_{k=0}^{\infty} c_k t^k e^{-\lambda t}$ does not converge to $f(t)$ on $[R, \infty)$.

The above process may be viewed as a transformation from one asymptotic series to another, which is equivalent to an infinite-dimensional matrix which transforms the coefficients of one series into those of another. This transformation is not linear in nature. However, the transformation between the n-th partial sums of the two series is linear for each $n \in \mathbb{N}$. Thus, the method of undetermined coefficients can be used in the following derivations.

Theorem 4.3. *(Existence theorem for method* I*a). Let $f(t)$ be expanded asymptotically into a series in (4.2). Then, for every $\lambda > 0$ and $n \in \mathbb{N}$, the partial sum $g_n(t)$ in (4.3) of the above asymptotic series satisfies the initial conditions (4.1).*

Proof. (Constructive proof). Let $\lambda > 0$ and $n \in \mathbb{N}$ fixed. Let $\sum_{k=0}^{n} a_k t^k$ and $\sum_{k=0}^{n} a'_k t^k$ be the partial sums of the power series expansions about the origin of $f(t)$ and $g_n(t)$, respectively. Consider $p_n(t) = \sum_{k=0}^{n} c_k t^k$. Then, the first $n+1$ coefficients c_k are related to the a_k by the following equations:

$$\begin{aligned}
c_0 &= a_0, \\
c_1 &= a_0\lambda + a_1, \\
c_2 &= a_0\frac{\lambda^2}{2!} + a_1\lambda + a_2, \\
&\vdots \\
c_n &= a_0\frac{\lambda^n}{n!} + a_1\frac{\lambda^{n-1}}{(n-1)!} + \cdots + a_{n-1}\lambda + a_n,
\end{aligned}$$

with the general row equation as

$$c_k = \sum_{i=0}^{k} a_i \frac{\lambda^{k-i}}{(k-i)!}, \quad k = 0, 1, \cdots, n,$$

or in the matrix form as

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \lambda & 1 & 0 & \cdots & 0 \\ \frac{\lambda^2}{2!} & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\lambda^n}{n!} & \frac{\lambda^{n-1}}{(n-1)!} & \frac{\lambda^{n-2}}{(n-2)!} & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

The coefficients a'_k can be calculated from the c_k by

$$\begin{aligned}
a'_0 &= c_0, \\
a'_1 &= c_0(-\lambda) + c_1, \\
a'_2 &= c_0\frac{(-\lambda)^2}{2!} + c_1(-\lambda) + c_2, \\
&\cdots \\
a'_n &= c_0\frac{(-\lambda)^n}{n!} + \cdots + c_{n-1}(-\lambda) + c_n,
\end{aligned}$$

with the general term as

$$a'_k = \sum_{i=0}^{n} c_k \frac{(-\lambda)^{k-i}}{(k-i)!}, \quad k = 0, 1, \cdots, n,$$

or in the matrix form as

$$\begin{pmatrix} a'_0 \\ a'_1 \\ a'_2 \\ \vdots \\ a'_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ (-\lambda) & 1 & 0 & \cdots & 0 \\ \frac{(-\lambda)^2}{2!} & (-\lambda) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(-\lambda)^n}{n!} & \frac{(-\lambda)^{n-1}}{(n-1)!} & \frac{(-\lambda)^{n-2}}{(n-2)!} & \cdots & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

Thus, the a'_k are related to the a_k by

$$\begin{pmatrix} a'_0 \\ a'_1 \\ a'_2 \\ \vdots \\ a'_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ (-\lambda) & 1 & \cdots & 0 \\ \frac{(-\lambda)^2}{2!} & (-\lambda) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{(-\lambda)^n}{n!} & \frac{(-\lambda)^{n-1}}{(n-1)!} & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \lambda & 1 & \cdots & 0 \\ \frac{\lambda^2}{2!} & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\lambda^n}{n!} & \frac{\lambda^{n-1}}{(n-1)!} & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

which is

$$\begin{pmatrix} a'_0 \\ a'_1 \\ a'_2 \\ \vdots \\ a'_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

This implies $a'_k = a_k$, $k = 0, 1, \cdots, n$. $\square$

Theorem 4.4. *(Uniform convergence theorem for method* I*a). In the above theorem, for every $\lambda > 0$, $g_n(t)$ converges to $f(t)$ uniformly on $[0, R)$.*

Proof. By Theorem 4.2, for any $\lambda > 0$ and any closed interval $[0, T] \subset [0, R)$,

$$\sum_{k=0}^{\infty} c_k t^k = f(t) e^{\lambda t}$$

absolutely for any $t \in [0, T]$. It follows from Proposition 2.37 that $\sum_{k=0}^{\infty} c_k t^k$ converges to $f(t)e^{\lambda t}$ uniformly on $[0, T]$. This implies for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > N$,

$$\left| f(t)e^{\lambda t} - \sum_{k=0}^{n} c_k t^k \right| = \left| \sum_{k=n+1}^{\infty} c_k t^k \right| < \epsilon$$

for all $t \in [0, T]$. It follows that for all $t \in [0, T]$ when $n > N$,

$$\left| \left(f(t)e^{\lambda t} - \sum_{k=0}^{n} c_k t^k \right) e^{-\lambda t} \right| = \left| \sum_{k=n+1}^{\infty} c_k t^k \right| \left| e^{-\lambda t} \right| < \epsilon,$$

since $\left| e^{-\lambda t} \right| \leq 1$. Therefore,

$$\left| f(t) - \sum_{k=0}^{n} c_k t^k e^{-\lambda t} \right| < \epsilon,$$

and

$$\lim_{n\to\infty} g_n(t;\lambda) = \lim_{n\to\infty} \sum_{k=0}^{n} c_k t^k e^{-\lambda t} = f(t)$$

uniformly on $[0, T]$. □

It follows that for any fixed $n \in \mathbb{N}$, there is a family of functions $g_n(t;\lambda)$ in $P_n^{\lambda}(\Omega)$ with a parameter $\lambda > 0$ satisfying the initial conditions (4.1), and for every $\lambda > 0$, $g_n(t;\lambda)$ converges to $f(t)$ uniformly on $[0, R)$. It is obvious $g_n(t;\lambda)$ is continuous, bounded, integral-convergent, and vanishing at infinity on Ω.

The shape coefficients c_k of $g_n(t;\lambda)$ are actually functions in λ and have the following properties.

Proposition 4.5.

$$\frac{d}{d\lambda} c_{k+1}(\lambda) = c_k(\lambda), \quad k = 0, 1, \cdots, n-1. \tag{4.6}$$

Proof. For every $k = 0, 1, \cdots, n-1$, we have

$$\begin{aligned} \frac{d}{d\lambda} c_{k+1}(\lambda) &= \frac{d}{d\lambda} \left(a_0 \frac{\lambda^{k+1}}{(k+1)!} + a_1 \frac{\lambda^k}{k!} + \cdots + a_k \lambda + a_{k+1} \right) \\ &= a_0 \frac{\lambda^k}{k!} + a_1 \frac{\lambda^{k-1}}{(k-1)!} + \cdots + a_{k-1}\lambda + a_k \\ &= c_k(\lambda). \end{aligned}$$

□

Proposition 4.6.

$$c_{k+1}(\lambda) = \int_0^{\lambda} c_k(\alpha) d\alpha + a_{k+1}, \quad k = 0, 1, \cdots, n-1. \tag{4.7}$$

Proof. For every $k = 0, 1, \cdots, n-1$, we have

$$\begin{aligned}
& \int_0^\lambda c_k(\alpha)d\alpha + a_{k+1} \\
= & \int_0^\lambda \left(a_0 \frac{\alpha^k}{k!} + a_1 \frac{\alpha^{k-1}}{(k-1)!} + \cdots + a_k \right) d\alpha + a_{k+1} \\
= & \left(a_0 \frac{\alpha^{k+1}}{(k+1)!} + a_1 \frac{\alpha^k}{k!} + \cdots + a_k \alpha \right) \Big|_0^\lambda + a_{k+1} \\
= & \; c_{k+1}(\lambda).
\end{aligned}$$

□

The above two propositions imply the coefficients c_k of $g_n(t;\lambda)$ satisfying initial conditions (4.1) are not independent. They are connected and restricted by Equations (4.6) and (4.7). The expression of $c_n(\lambda)$ contains all the coefficients $c_k(\lambda)$, $k = 0, 1, \cdots, n-1$, and is another form of initial conditions.

For any fixed n and λ, define the approximation error function as

$$e_n(t;\lambda) = f(t) - g_n(t;\lambda).$$

Proposition 4.7.

$$\lim_{t \to 0} e_n(t;\lambda) = 0.$$

Proof. Trivial, since the origin is the expansion center. □

Proposition 4.8.

$$e_n(t;\lambda) = \mathcal{O}(t^{n+1}), \quad t \to 0.$$

Proof. By the initial conditions (4.1),

$$\begin{aligned}
e_n(t;\lambda) &= \sum_{k=0}^{\infty} a_k t^k - \sum_{k=0}^{\infty} a'_k t^k \\
&= \sum_{k=n+1}^{\infty} (a_k - a'_k) t^k \\
&= \mathcal{O}(t^{n+1}), \quad t \to 0.
\end{aligned}$$

□

It follows from Propositions 4.7 and 4.8 that $|e_n(t)|$ has a funnel-like shape, which starts with no error at the origin and increases as t increases on some neighborhood of the origin.

It follows from Theorem 4.4 that if we choose $T < R$, then $g_n(t)$ converges to $f(t)$ uniformly on the main interval $[0, T] \subset [0, R)$. In particular, if $f(t)$ is infinite-radius-convergent on Ω, i.e. $R = \infty$, then T can be any positive value. This is the approximation method Ia.

If $f(t)$ is finite-radius-convergent on Ω and $T > R$, then the conditions for uniform convergence do not hold and the approximation method Ia may not be appropriate. In this case, we have to use the variants Ib and Ic of the asymptotic series expansion method, which are discussed in Sections 4.2.2, 4.2.3, and 4.2.4.

Next, we shall determine the optimal value of λ and obtain the final approximation function. For many λ values, $g_n(t;\lambda)$ may not "look" like $f(t)$ at all, as can be demonstrated by numerical experiments. To find the optimal λ value, we need to impose an additional condition on $g_n(t;\lambda)$, which may be some simple property of $f(t)$, such as its value at $t = T$, its definite integral or curve length on $[0, T]$. If the additional condition results in multiple values of λ, we can always use the minimum mean square error criterion to choose the optimal one. After checking the tail boundedness of $g_n(t;\lambda)$ with the optimal λ value, we obtain the final approximation function.

In matching function values at $t = T$, we wish to solve

$$g_n(T;\lambda) = f(T),$$

or

$$\sum_{k=0}^{n} c_k(\lambda)T^k - f(T)e^{\lambda T} = 0.$$

The existence of λ such that the two functions match at $t = T$ is generally difficult to discuss. In future research, we shall give some general conditions for the existence.

If the integration function of $f(t)$ is important, we can find the optimal λ by matching the definite integrals of $f(t)$ and $g_n(t;\lambda)$ on $[0, T]$ as

$$\int_0^T f(t)dt = \int_0^T g_n(t;\lambda)dt,$$

or

$$\int_0^T e_n(t;\lambda)dt = \int_0^T f(t) - g_n(t;\lambda)dt = 0.$$

The existence of such λ is also difficult to discuss. Practically, the equation of matching the definite integrals may not have real solution(s). We can choose some particular n to ensure there exists at least one positive real solution, or we can change the condition to minimize the difference of the two definite integrals.

Other additional conditions for optimal λ, such as matching the curve length on $[0, T]$, can be dealt with similarly.

On obtaining the final approximation function $g_n(t;\lambda)$, we may calculate its integration function as

$$G_n(t;\lambda) = \int_0^t g_n(\tau;\lambda)d\tau.$$

By Proposition 3.22, $G_n(t;\lambda)$ is finite for any $t > 0$ or $t = \infty$, and is thus bounded.

Example 4.9. *Asymptotic series expansion about the origin for infinite-radius-convergent functions.*

Consider an analytic target probability density function

$$f(t) = \frac{\sqrt{2}}{\sqrt{\pi}+\sqrt{2}}(1+t)e^{-\frac{t^2}{2}},$$

whose definite integral on Ω is unity. $f(t)$ is infinite-radius-convergent, since it has a power series expansion about the origin as

$$\begin{aligned} f(t) &= 0.4438 + 0.4438t - 0.2219t^2 - 0.2219t^3 \\ &\quad + 0.05547t^4 + 0.05547t^5 - 0.009246t^6 - 0.009246t^7 \\ &\quad + 0.001156t^8 + 0.001156t^9 - 0.0001156t^{10} + \cdots \end{aligned}$$

with an infinite radius of convergence.

Let $n = 10$. By Proposition 4.3, there is a family of functions $g_{10}(t;\lambda)$ denoted by Equation (4.3) such that the initial conditions (4.1) are satisfied.

Let $T = 3$ and $[0,3]$ the main interval. In this case, we will match the definite integrals of $f(t)$ and $g_n(t;\lambda)$ on $[0,3]$ to determine the optimal λ. Solving the equation

$$\int_0^3 g_{10}(t;\lambda)dt = \int_0^3 f(t)dt,$$

we obtain five positive roots (after ignoring five negative roots), and list them in Table 1 as well as the corresponding WMSE's on $[0,3]$.

TABLE 1. The five positive roots from method of matching definite integrals and the corresponding WMSE's on $[0,3]$.

i	λ_i	$\text{WMSE}_{[0,3]}$
1	0.7986	1.1036×10^{-3}
2	1.7918	4.2496×10^{-5}
3	2.8232	5.0442×10^{-6}
4	3.9321	2.1646×10^{-6}
5	5.2144	5.2434×10^{-6}

Practically, apart from λ_1, all other λ_i are acceptable. With λ_4, the resulting $g_{10}(t;\lambda_4)$ has the minimum WMSE on $[0,3]$. With λ_5, the resulting $g_{10}(t;\lambda_5)$ is positive on Ω, which is a desirable property. Since the target function is a probability density function, we choose the largest root $\lambda_5 = 5.2144$ as the optimal value.

Substituting λ_5 into $g_{10}(t;\lambda)$, we obtain the final approximation function

$$\begin{aligned} g_{10}(t) &= (0.4438 + 2.7579t + 8.1256t^2 \\ &\quad + 15.1412t^3 + 20.0392t^4 + 20.0122t^5 \\ &\quad + 15.6026t^6 + 9.6637t^7 + 4.7690t^8 \\ &\quad + 1.8454t^9 + 0.5294t^{10})e^{-5.2144t}. \end{aligned}$$

The corresponding WMAE on $[0, 3]$ is 2.5630×10^{-3}. In addition, $g_{10}(t)$ is nicely bounded on $[3, \infty)$.

The integration function of $g_{10}(t)$ is

$$\begin{aligned} G_{10}(t) &= 1.0019 - (1.0019 + 4.7804t + 11.0845t^2 \\ &\quad + 16.5579t^3 + 17.7996t^4 + 14.5550t^5 \\ &\quad + 9.3140t^6 + 4.7092t^7 + 1.8615t^8 \\ &\quad + 0.5486t^9 + 0.1015t^{10})e^{-5.2144t}. \end{aligned}$$

Figure 1 shows the approximation results on $[0,6]$, twice the length of the main interval $[0,3]$, since we cannot plot the overall graph of $f(t)$ on the unbounded interval Ω. The left figure shows the overall graphs of the approximation function $g_{10}(t)$ (blue line) and the target function $f(t)$ (red line). The right figure shows the overall graphs of $G_{10}(t)$ (blue line) and $F(t)$ (red line), the integration functions of $g_{10}(t)$ and $f(t)$, respectively. In each figure, the graphs of the target function and the approximation function almost completely coincide with each other and are indistinguishable. The right figure shows there is a very small difference between the two asymptotes of $G_{10}(t)$ and $F(t)$. In addition, $g_{10}(t)$ is nicely bounded on $[3,\infty)$ and vanishes at infinity.

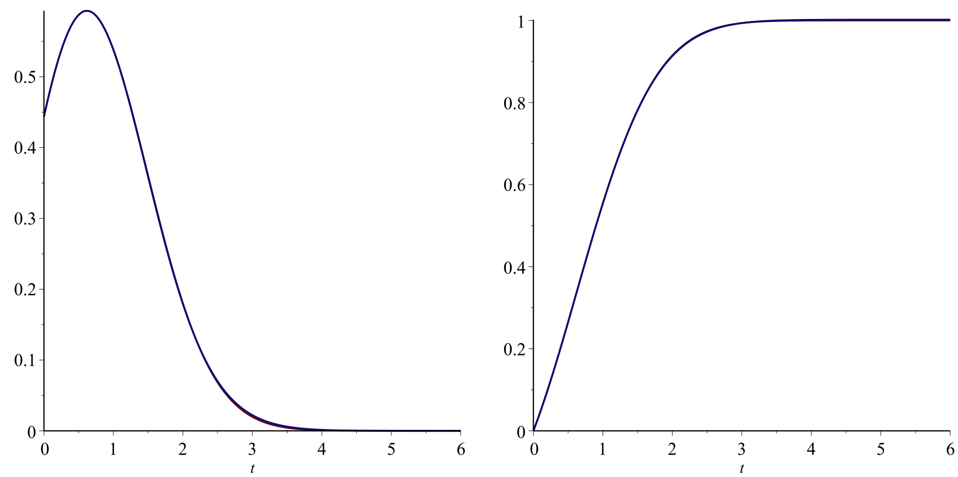

FIGURE 1. Method Ia overall approximations on $[0,6]$ ($n=10$): Left – the overall graphs of the approximation function $g_{10}(t)$ (blue line) and the target function $f(t)$ (red line); Right – the overall graphs of $G_{10}(t)$ (blue line) and $F(t)$ (red line), the integration functions of $g_{10}(t)$ and $f(t)$, respectively.

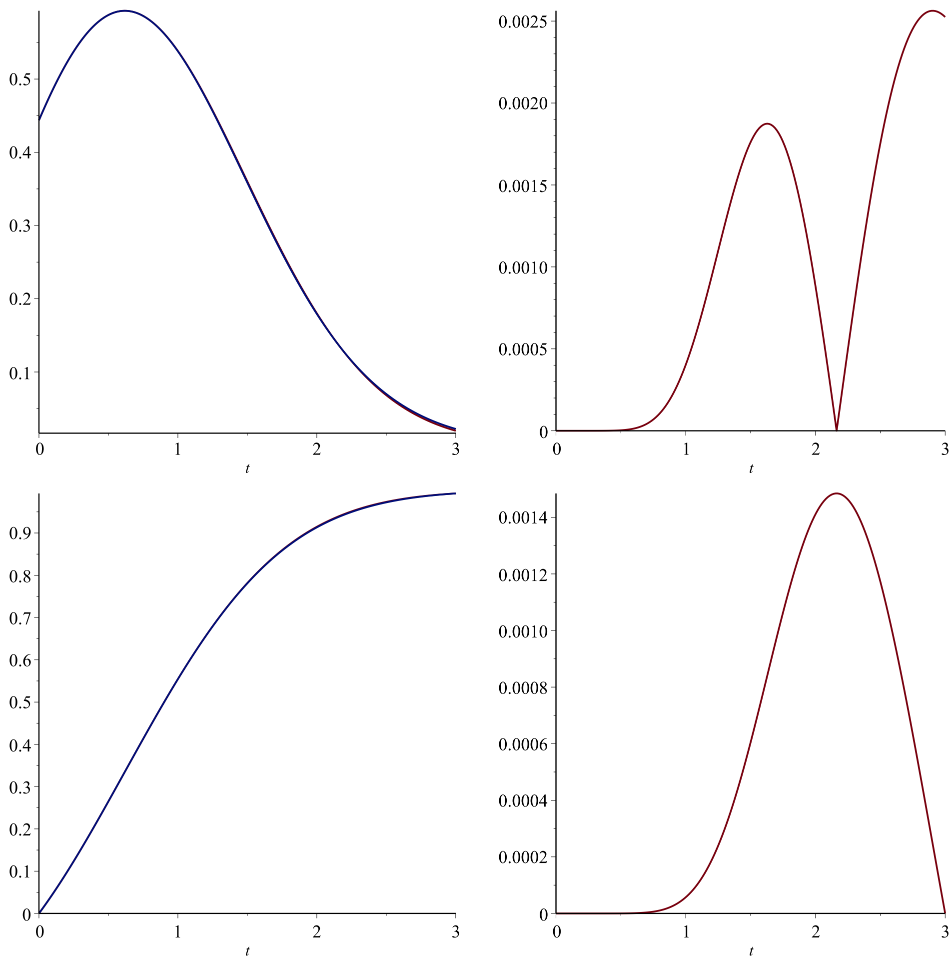

FIGURE 2. Method Ia approximations on the main interval $[0, 3]$ $(n = 10)$: Top-left – $g_{10}(t)$ (blue line) and $f(t)$ (red line) almost coincide with each other; Top-right – the absolute error function $|e_{10}(t)|$ of the approximation error function $e_{10}(t) = f(t) - g_{10}(t)$; Bottom-left – $G_{10}(t)$ (blue line) and $F(t)$ (red line) almost coincide with each other; Bottom-right – the absolute error function $|F(t) - G_{10}(t)|$.

Figure 2 shows the approximation results on the main interval $[0, 3]$. The top-left figure shows $g_{10}(t)$ (blue line) and $f(t)$ (red line) almost coincide with each other on $[0, 3]$. The top-right figure is the absolute error function $|e_{10}(t)|$ of the approximation error function $e_{10}(t) = f(t) - g_{10}(t)$. $e_{10}(t)$ is zero at the origin and asymptotically increases as t increases on $[0, 3]$. Then $e_{10}(t)$ crosses the t-axis at $t = 2.1632$ in $[0, 3]$. This point is the result of matching the definite integrals of $f(t)$ and $g_{10}(t)$ on $[0, 3]$, i.e. the area under the curve of $|e_{10}(t)|$ on $[0, 2.1632]$ is equal to that on $[2.1632, 3]$. The bottom-left figure shows the integration functions $G_{10}(t)$ (blue line) and $F(t)$ (red line) almost coincide with each other on $[0, 3]$. The bottom-right figure shows the absolute error function $|F(t) - G_{10}(t)|$. Clearly, $G_{10}(T) = F(T)$.

In order to demonstrate the uniform convergence on the main interval for the approximation method Ia, we repeat the above approximation experiment with n = 20, 30, and 40, and record the corresponding optimal λ values, WMAE's and WMSE's on $[0, 3]$ in Table 2. Figure 3 shows the graphs of the absolute error functions $|e_n(t)|$ on the main interval $[0, 3]$ for n = 10, 20, 30, and 40.

Table 2. Method Ia multiple approximation experiments: the optimal λ values, WMAE's and WMSE's on $[0, 3]$ for n = 10, 20, 30, and 40.

n	λ	$\mathrm{WMAE}_{[0,3]}$	$\mathrm{WMSE}_{[0,3]}$	$\frac{n}{\lambda}$
10	5.2144	2.5630×10^{-3}	5.2434×10^{-6}	1.9
20	7.8416	1.4235×10^{-4}	5.2073×10^{-9}	2.6
30	9.8668	6.1359×10^{-6}	4.5149×10^{-12}	3.0
40	11.5772	1.8011×10^{-7}	2.2966×10^{-15}	3.5

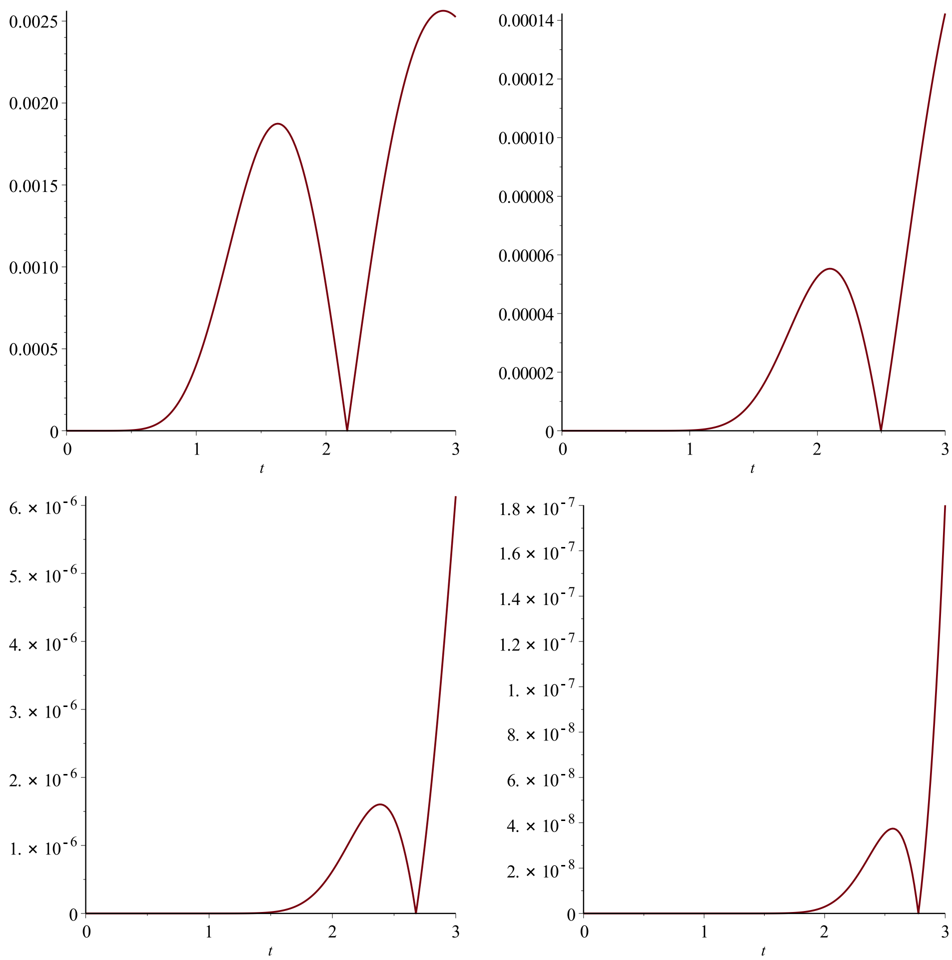

FIGURE 3. Method Ia multiple approximation experiments: The graphs of the absolute error functions $|e_n(t)|$ on the main interval $[0, 3]$ for $n = 10$ (top-left), $n = 20$ (top-right), $n = 30$ (bottom-left), and $n = 40$ (bottom-right).

4.2.2. *Limitations of method* I*a and critically damped functions.* The approximation method Ia has essential limitations for finite-radius-convergent target functions. Suppose the radius of convergence of the target function power series expansion about the origin is R. Then we can only define the main interval $[0, T]$ such that $T < R$ to achieve the uniform convergence on $[0, T]$. If $T > R$, the approximation by the method Ia on $[R, T]$ is divergent and we cannot find an appropriate combination of λ and n so the approximation error on $[R, T]$ is sufficiently small.

The complete solutions to the above problem will be discussed in the approximation methods Ib and Ic. In this section, we shall introduce the concept of critically damped approximation functions in $P_n^\lambda(\Omega)$ spaces to the method Ia and investigate its tendency as n increases.

Consider the approximation problem in Section 4.2.1 that the target function $f(t) \in C_0^B(\Omega)$ is analytic and has its largest singularity at $t = -R$, $R > 0$. Let $T = R$ and $[0, R]$ the main interval. Then, by Proposition 4.3, we can find a family of functions $g_n(t; \lambda)$ in $P_n^\lambda(\Omega)$ denoted by (4.3) with a parameter $\lambda > 0$ such that the $n + 1$ initial conditions (4.1) are satisfied. The family represents all the possible trajectories of $g_n(t; \lambda)$ tending to zero as t tends to infinity on $[R, \infty)$.

Let λ vary. We shall investigate the tail behavior of $g_n(t; \lambda)$ on the tail interval $[T, \infty)$. Since $g_n(t; \lambda)$ vanishes at infinity, its tail behavior is similar to a finite energy damping physical system, whose amplitude, which is a function of time, is returning to its steady state zero.

We shall introduce the concept of critically damping for the above physical system with a continuous parameter controlling the damping. The system is said to be *overdamped*, if its amplitude decreases monotonically and vanishes at infinity on $[T, \infty)$ for some $T > 0$; The system is said to be *underdamped*, if its amplitude crosses the t-axis at least once and vanishes at infinity on $[T, \infty)$. Suppose the system assumes all the possible overdamped and underdamped states continuously. Then there exists a *critically damped* amplitude function that decreases slower than all the overdamped functions but does not oscillate or cross the t-axis. In other words, a critically damped amplitude function separates the overdamped and underdamped states.

A function $g_n(t; \lambda)$ in a decaying polynomial space $P_n^\lambda(\Omega)$ may be used to describe the above physical system, where t represents time and λ controls the damping. Assume $g_n(t; \lambda)$ is positive at $t = T$ on $[T, \infty)$. For a given $\lambda > 0$, $g_n(t; \lambda)$ is said to be overdamped, if it decreases monotonically and vanishes at infinity on $[T, \infty)$; it is underdamped, if it crosses the t-axis at least once and vanishes at infinity on $[T, \infty)$; and it is critically damped, if it decreases slower than all other overdamped functions but does not oscillate or cross the t-axis. Denote the *critical value* of λ as λ^*.

By Proposition 3.19, the tail behavior of $g_n(t;\lambda)$ on the tail interval $[T,\infty)$ is mostly affected by its last term $c_n t^n e^{-\lambda t}$. In particular, $c_n t^n e^{-\lambda t}$ affects the function crossing the t-axis and the tail decaying rate. Thus, we may find the critical value λ^* for $g_n(t;\lambda)$ by minimizing $|c_n(\lambda)|$.

The degree n for $g_n(t;\lambda)$ may affect the above minimization process. It is obvious by Proposition 3.11 and the theory of polynomials that a function $g_n(t;\lambda)$ with n even may not be able to cross the t-axis once in $\mathbb{R}$, but with n odd, it will definitely cross the t-axis at least once in $\mathbb{R}$, although the crossing point may not be in Ω. In this respect, the parity of n also affects how λ^* is found.

Let n be odd. If $c_n(\lambda)=0$ has a positive real root λ^*, then

$$g_n(t;\lambda^*)=\sum_{k=0}^{n-1} c_k t^k e^{-\lambda^* t} + 0\cdot t^n e^{-\lambda^* t} = g_{n-1}(t;\lambda^*).$$

Since its n-th term has a coefficient zero and has no contribution to the function value, $g_n(t;\lambda^*)$ is actually in $P^{\lambda}_{n-1}(\Omega)$ and may be considered as the projection of a function of degree n onto a function of degree $n-1$. Although we are unable to discuss the existence of λ^* in general, Proposition 4.10 in the following shows that such λ^* may exist under certain conditions.

If n is even, then $c_n(\lambda)=0$ may not have a positive real root. In this case, however, we may still find the critical value λ^* by minimizing the absolute value $|c_n(\lambda)|$. By Proposition 4.5,

$$c_n'(\lambda)=c_{n-1}(\lambda).$$

Thus, the positive λ value such that $|c_n(\lambda)|$ is minimum is a positive real root of $c_{n-1}(\lambda)=0$, where $n-1$ is odd.

Proposition 4.10. *(Conditions for the existence of the critical value of λ). Let $f(t)=\sum_{k=0}^{\infty} a_k t^k$ with the a_k having alternating signs. Then we can always find a real λ such that*

$$c_n(\lambda)=\sum_{k=0}^{n} a_k \frac{\lambda^{n-k}}{(n-k)!}=0$$

for some $n\in\mathbb{N}$. The proposition is also true if the a_k have random signs.

Proof. Without loss of generality, assume $a_0>0$. For every $n\in\mathbb{N}$,

$$\lim_{\lambda\to 0} c_n(\lambda)=a_n$$

and

$$\lim_{\lambda\to\infty} c_n(\lambda)=\lim_{\lambda\to\infty} a_0\frac{\lambda^n}{n!}+\mathcal{O}(\lambda^{n-1})=\infty.$$

By hypothesis the a_k have alternating or random signs, we can always find an n such that a_0 and a_n have different signs. Therefore, there must be at least one $\lambda>0$ such that $c_n(\lambda)=0$. $\square$

In this section, we are interested in a particular critically damped approximation function in $P^{\lambda}_{n+1}(\Omega)$ defined as

$$g_n(t;\lambda^*) = g_{n+1}(t;\lambda^*) = \sum_{k=0}^{n} c_k t^k e^{-\lambda^* t} + 0 \cdot t^{n+1} e^{-\lambda^* t} \tag{4.8}$$

such that the $n+2$ initial conditions are satisfied. Define the approximation error function as

$$e_n(t;\lambda^*) = f(t) - g_n(t;\lambda^*).$$

Proposition 4.11.

$$e_n(t;\lambda^*) = \mathcal{O}\left(t^{n+2}\right), \quad t \to 0.$$

Proof. This is trivial, since the zero to $(n+1)$-st terms of the power series of $f(t)$ and $g_n(t)$ are matching. $\square$

$e_n(t;\lambda^*)$ has a funnel-like shape on its convergence interval $[0, R)$ with no error at the origin and the maximum error at $t \to R^-$. Overall, $e_n(t;\lambda^*)$ has a spindle shape on Ω which does not cross the t-axis other than at the origin or at infinity.

Example 4.12. *Asymptotic series expansion critically damped approximations to finite-radius-convergent analytic functions.*

Consider a target function

$$f(t) = \frac{1}{1+t},$$

which has a geometric power series about the origin as

$$f(t) = \sum_{k=0}^{\infty} (-1)^k t^k$$

with a convergence interval $[0, 1)$. Clearly, the a_k are alternating with $a_0 > 0$ and $f(t)$ is positive on Ω.

Let $n = 10$ (even). Solving the equation $c_{11}(\lambda) = 0$, we have the critical value $\lambda^* = 3.9055$. Substituting $\lambda^* = 3.9055$ into $g_{10}(t;\lambda^*)$ in (4.8), we obtain the critically damped approximation function

$$\begin{aligned} g_{10}(t) \quad = \quad & (1.0000 + 2.9055t + 4.7208t^2 \\ & + 5.2072t^3 + 4.4862t^4 + 3.0852t^5 \\ & + 1.8431t^6 + 0.9065t^7 + 0.4358t^8 \\ & + 0.1467t^9 + 0.0808t^{10})e^{-3.9055t} \end{aligned}$$

such that the $n+2 = 12$ initial conditions are satisfied. In addition, the coefficients c_k, $k = 0, 1, \cdots, 10$, are all positive.

We cannot arbitrarily choose the main interval, since the convergence interval is $[0, 1)$. The largest possible main interval is $[0, T]$ for some T close to 1 or simply for $T = 1$. Then we can only approximate $f(t)$ on $[0, 1]$. For $g_{10}(t; \lambda^* = 3.9055)$, we have WMAE $= 3.7385 \times 10^{-4}$ and WMSE $= 8.0427 \times 10^{-9}$ on $[0, 1]$.

Figure 4 shows the overall graphs of the critically damped approximation function $g_{10}(t)$ (blue line) and the target function $f(t)$ (red line) on $[0, 6]$, which is six times the length of the main interval $[0, 1]$, since we cannot plot the overall graph of $f(t)$ on the unbounded interval Ω.

Figure 5 shows the approximation by the critically damped function on the main interval $[0, 1]$. The left figure shows the critically damped function $g_{10}(t)$ (blue line) and the target function $f(t)$ (red line) almost coincide with each other and are indistinguishable on $[0, 1]$. The right figure shows the absolute error function $|e_{10}(t)| = |f(t) - g_{10}(t)|$ on $[0, 1]$, which has a funnel-like shape with no error at the origin and the maximum error at the right end-point $T = 1$.

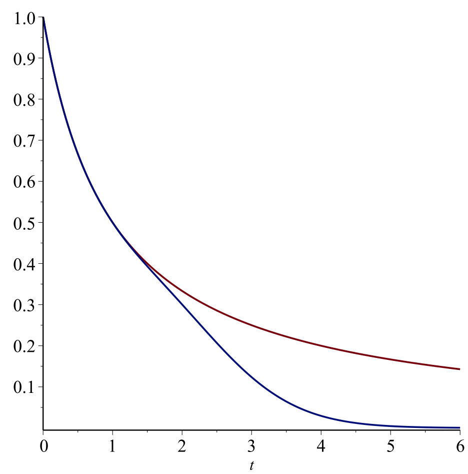

FIGURE 4. Method Ia overall approximation: the critically damped approximation function $g_{10}(t)$ (blue line) and the target function $f(t)$ (red line) on $[0, 6]$.

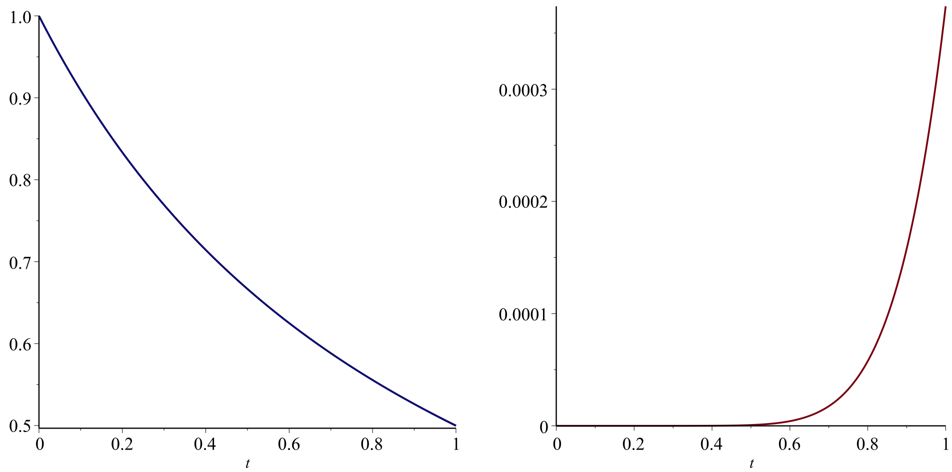

Figure 5. Method Ia approximation by the critically damped function on the main interval $[0, 1]$ ($n = 10$): Left – the critically damped approximation function $g_{10}(t)$ (blue line) and the target function $f(t)$ (red line) almost coincide with each other; Right – the absolute error function $|e_{10}(t)| = |f(t) - g_{10}(t)|$.

If we set $\lambda = 3.4055$ or $\lambda = 6.9055$ in $g_{n+1}(t;\lambda) \in P^{\lambda}_{n+1}(\Omega)$ in (4.8), we obtain the underdamped or the overdamped approximation function, $g_{11}(t;3.4055)$ or $g_{11}(t;6.9055)$, respectively. Note the coefficients for the $t^{n+1}e^{-\lambda t}$ terms in $g_{11}(t;3.4055)$ and $g_{11}(t;6.9055)$ are not zeros, and the two approximation functions match up to 11-th derivatives of $f(t)$ at the origin. Figure 6 shows the graphs of the target function $f(t)$ (red line), the absolute underdamped, critically damped, and overdamped approximation functions $|g_{11}(t;3.4055)|$ (green line), $g_{10}(t;3.9055)$ (blue line), and $g_{11}(t;6.9055)$ (azure line) on $[0,6]$. Clearly, for sufficiently large t, e.g. $t > 4$, The overdamped approximation function $g_{11}(t;6.9055)$ vanishes faster than the critically damped function $g_{10}(t;3.9055)$, which vanishes faster than the absolute underdamped function $g_{11}(t;3.4055)$, as t approaches infinity.

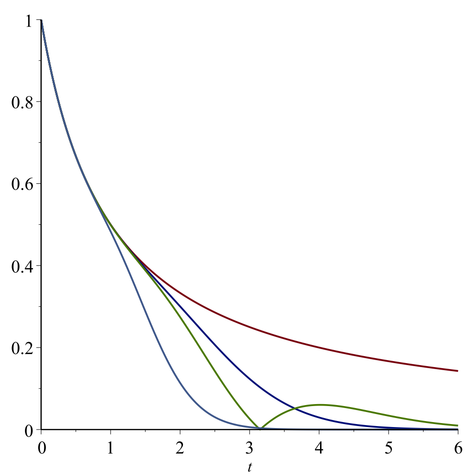

FIGURE 6. Method Ia critically damped approximation experiment: the target function $f(t)$ (red line) and its approximation functions (matching up to 11-th derivatives of $f(t)$ at the origin): the absolute underdamped, critically damped, and overdamped approximation functions $|g_{11}(t;3.4055)|$ (green line), $g_{10}(t;3.9055)$ (blue line), and $g_{11}(t;6.9055)$ (azure line) on $[0,6]$.

We repeat the above approximation experiment for $n = 50$, 100, 500, and 1000, and obtain the corresponding critically damped approximation functions $g_{50}(t)$, $g_{100}(t)$, $g_{500}(t)$, and $g_{1000}(t)$. We calculate the approximate time span for each approximation function in Table 3. It is interesting the time span of each approximation function slowly expands as n increases. Figure 7 shows the graphs of $g_{10}(t)$, $g_{50}(t)$, $g_{100}(t)$, $g_{500}(t)$, $g_{1000}(t)$, and $f(t)$. Clearly, the approximation to $f(t)$ by critically damped functions is uniformly convergent on the convergence interval $[0, 1)$. Although each function approximates $f(t)$ well outside $[0, 1]$ but within its time span, it does poorly for any sufficiently large t outside its time span.

TABLE 3. Method Ia multiple approximation experiments by critically damped approximation functions: the critical values λ^* and the approximate time spans for $n = 10$, 50, 100, 500, and 1000.

n	λ^*	$\frac{n}{\lambda^*}$
10	3.9055	2.56
50	15.179	3.29
100	29.170	3.43
500	140.72	3.55
1000	280.03	3.57

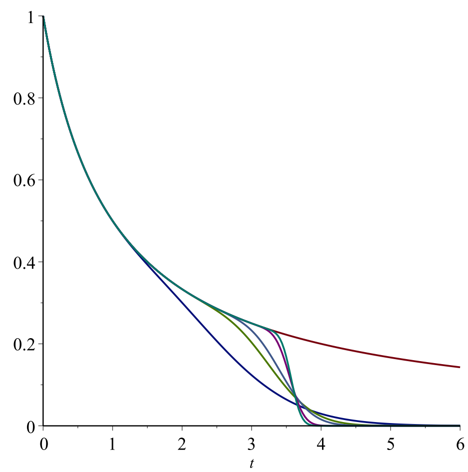

Figure 7. Method Ia multiple experiments by critically damped approximation functions $g_{10}(t)$ (blue line), $g_{50}(t)$ (green line), $g_{100}(t)$ (azure line), $g_{500}(t)$ (purple line), $g_{1000}(t)$ (bluegreen line), and the target function $f(t)$ (red line) on $[0, 6]$.

4.2.3. *Method* I*b: asymptotic series expansion about an arbitrary point.* In this section, we shall develop the approximation method Ib, asymptotic series expansion about an arbitrary center, to address the problem of a target function being finite-radius-convergent about the origin and the main interval right end-point being outside the convergence interval. The idea is to expand the target function into a power series about another center such that the new convergence interval completely contains the main interval.

Without loss of generality, assume $f(t)$ is analytic with a finite radius of convergence on Ω and has its largest singularity at $t = -R$. Let the main interval be $[0, T]$ and $T > 0$ the *second expansion center*. Then $f(t)$ can be expanded into a power series about $t = T$ with the radius of convergence $T + R$. This implies the *second convergence interval* $(-R, 2T + R) \supseteq [0, T]$. Thus, the *second asymptotic series* is not half-convergent half-divergent but rather convergent on $[0, T]$.

Let $\lambda > 0$ and $n \in \mathbb{N}$ fixed. The new approximation problem is to find a function $g(t) = g_n(t) \in P_n^\lambda(\Omega)$ which satisfies the boundary conditions at $t = T$, i.e. $g(t)$ matches $f(t)$ up to the n-th derivatives at $t = T$, denoted by

$$g(T) = f(T), \quad g'(T) = f'(T), \quad \cdots, \quad g^{(n)}(T) = f^{(n)}(T).$$

Then we shall call $g(t)$ an approximation to $f(t)$ if it is sufficiently "close" to the latter on $[0, T]$ and is satisfactorily bounded on $[T, \infty)$.

Equivalently, the above boundary conditions can be rephrased in the form of power series expansion. Expand $f(t)$ into a power series about $t = T$ as

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(T)}{k!}(t-T)^k \\ &= \sum_{k=0}^{\infty} d_k (t-T)^k, \quad t \in (-R, 2T+R), \end{aligned} \tag{4.9}$$

where

$$d_k = \frac{f^{(k)}(T)}{k!}, \quad k = 0, 1, \cdots.$$

Consider the power series expansion of $g(t)$ about $t = T$ as

$$g(t) = \sum_{k=0}^{\infty} d'_k (t-T)^k$$

on Ω (infinite radius of convergence). Then the boundary conditions are

$$d'_k = d_k, \quad k = 0, 1, \cdots, n. \tag{4.10}$$

We shall show such a $g(t)$ exists and uniformly converges to $f(t)$ on $[0, T] \subset (-R, 2T + R)$.

By Theorem 2.32, the power series in (4.9) converges absolutely on $(-R, 2T+R)$. And for every $\lambda > 0$,

$$e^{\lambda(t-T)} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}(t-T)^k$$

absolutely on $\mathbb{R}$. Define a Cauchy product

$$\left(\sum_{k=0}^{\infty} d_k (t-T)^k\right)\left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}(t-T)^k\right) = \sum_{k=0}^{\infty} b_k (t-T)^k, \tag{4.11}$$

where

$$\begin{aligned} b_k &= \sum_{i=0}^{k} d_i \frac{\lambda^{k-i}}{(k-i)!} \\ &= \sum_{i=0}^{k} \frac{f^{(i)}(T)}{i!}\frac{\lambda^{k-i}}{(k-i)!}, \quad k = 0, 1, \cdots . \end{aligned}$$

By Theorem 2.30, $\sum_{k=0}^{\infty} b_k (t-T)^k$ converges to $f(t)e^{\lambda(t-T)}$ absolutely for every $t \in (-R, 2T+R)$.

Since $\sum_{k=0}^{\infty} b_k (t-T)^k$ is half-convergent half-divergent on Ω, we consider it as an asymptotic series expansion about $t = T$. Let

$$\begin{aligned} g_n(t) &= \sum_{k=0}^{n} b_k (t-T)^k e^{-\lambda(t-T)} \\ &= \sum_{k=0}^{n} c_k t^k e^{-\lambda t} \\ &= p_n(t) e^{-\lambda t}, \end{aligned} \tag{4.12}$$

where the c_k can be calculated from the b_k and

$$p_n(t) = \sum_{k=0}^{n} c_k t^k = \sum_{k=0}^{n} b_k (t-T)^k e^{\lambda T}. \tag{4.13}$$

Then for every $\lambda > 0$, we can write

$$f(t) \sim \lim_{n\to\infty} g_n(t;\lambda), \quad t \to T. \tag{4.14}$$

Theorem 4.13. *(Existence theorem for method* I*b). Expand* $f(t)$ *asymptotically into a series about* $t = T$ *as in (4.14). Then for every* $\lambda > 0$ *and* $n \in \mathbb{N}$*, the partial sum* $g_n(t)$ *in (4.12) of the new asymptotic series satisfies the boundary conditions in (4.10).*

Proof. (Constructive proof). For every fixed $\lambda > 0$, consider the asymptotic power series $f(t) = \sum_{k=0}^{\infty} d_k(t-T)^k$ in (4.9) and the Cauchy product $\sum_{k=0}^{\infty} b_k(t-T)^k$ in (4.11). The coefficients b_k and d_k, $k = 0, 1, \cdots, n$, are related by the following equations:

$$\begin{aligned}
b_0 &= d_0, \\
b_1 &= d_0\lambda + d_1, \\
b_2 &= d_0\frac{\lambda^2}{2!} + d_1\lambda + d_2, \\
&\vdots \\
b_n &= d_0\frac{\lambda^n}{n!} + d_1\frac{\lambda^{n-1}}{(n-1)!} + \cdots + d_{n-1}\lambda + d_n,
\end{aligned}$$

with the general row equation as

$$b_k = \sum_{i=0}^{k} d_i \frac{\lambda^{k-i}}{(k-i)!}, \quad k = 0, 1, \cdots, n,$$

or in the matrix form as

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \lambda & 1 & 0 & \cdots & 0 \\ \frac{\lambda^2}{2!} & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\lambda^n}{n!} & \frac{\lambda^{n-1}}{(n-1)!} & \frac{\lambda^{n-2}}{(n-2)!} & \cdots & 1 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}.$$

Let this matrix be B; then B is $(n+1) \times (n+1)$.

Next, rewrite $p_n(t)$ in (4.13) as

$$p_n(t) = \left(\sum_{k=0}^{n} c'_k t^k\right) e^{\lambda T}.$$

Then the c'_k can be obtained from the b_k as

$$\begin{aligned}
c'_0 &= b_0\binom{0}{0} + b_1\binom{1}{0}(-T) + \cdots + b_n\binom{n}{0}(-T)^n,\\
\vdots &= \vdots\\
c'_{n-2} &= b_{n-2}\binom{n-2}{n-2} + b_{n-1}\binom{n-1}{n-2}(-T) + b_n\binom{n}{n-2}(-T)^2,\\
c'_{n-1} &= b_{n-1}\binom{n-1}{n-1} + b_n\binom{n}{n-1}(-T),\\
c'_n &= b_n\binom{n}{n},
\end{aligned}$$

with the general term as

$$c'_k = \sum_{i=k}^{n} b_i \binom{i}{k}(-T)^{i-k}, \quad k = 0, 1, \cdots, n,$$

or in the matrix form as

$$\begin{pmatrix} c'_0 \\ \vdots \\ c'_{n-2} \\ c'_{n-1} \\ c'_n \end{pmatrix} = \begin{pmatrix} \binom{0}{0} & \binom{1}{0}(-T) & \binom{2}{0}(-T)^2 & \cdots & \binom{n}{0}(-T)^n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \binom{n-2}{n-2} & \binom{n-1}{n-2}(-T) & \binom{n}{n-2}(-T)^2 \\ 0 & \cdots & 0 & \binom{n-1}{n-1} & \binom{n}{n-1}(-T) \\ 0 & \cdots & 0 & 0 & \binom{n}{n} \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_{n-2} \\ b_{n-1} \\ b_n \end{pmatrix}.$$

Let the above matrix be A; then A is $(n+1)\times(n+1)$. For $n = 4$, we have

$$A = \begin{pmatrix} 1 & -T & T^2 & -T^3 & T^4 \\ 0 & 1 & -2T & 3T^2 & -4T^3 \\ 0 & 0 & 1 & -3T & 6T^2 \\ 0 & 0 & 0 & 1 & -4T \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, the d_k are transformed into the c_k' by the matrix AB in λ and T. For $n = 4$, we have

$$AB = \begin{pmatrix} 1 & -T & T^2 & -T^3 & T^4 \\ 0 & 1 & -2T & 3T^2 & -4T^3 \\ 0 & 0 & 1 & -3T & 6T^2 \\ 0 & 0 & 0 & 1 & -4T \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 & 0 \\ \frac{\lambda^2}{2!} & \lambda & 1 & 0 & 0 \\ \frac{\lambda^3}{3!} & \frac{\lambda^2}{2!} & \lambda & 1 & 0 \\ \frac{\lambda^4}{4!} & \frac{\lambda^3}{3!} & \frac{\lambda^2}{2!} & \lambda & 1 \end{pmatrix},$$

or

$$\begin{pmatrix} c_0' \\ c_1' \\ c_2' \\ c_3' \\ c_4' \end{pmatrix} = \begin{pmatrix} 1 & -T & T^2 & -T^3 & T^4 \\ 0 & 1 & -2T & 3T^2 & -4T^3 \\ 0 & 0 & 1 & -3T & 6T^2 \\ 0 & 0 & 0 & 1 & -4T \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 & 0 \\ \frac{\lambda^2}{2!} & \lambda & 1 & 0 & 0 \\ \frac{\lambda^3}{3!} & \frac{\lambda^2}{2!} & \lambda & 1 & 0 \\ \frac{\lambda^4}{4!} & \frac{\lambda^3}{3!} & \frac{\lambda^2}{2!} & \lambda & 1 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}.$$

Note that $AB \neq BA$ and the matrix multiplication does not commute.

Clearly, both A and B are triangular and invertible. Thus, we conclude

$$g_n(t) = \sum_{k=0}^{n} p_n(t)e^{-\lambda t} = \sum_{k=0}^{n} \left(c_k' e^{\lambda T}\right) t^k e^{-\lambda t}$$

is in $P_n^\lambda(\Omega)$ and satisfies the boundary conditions (4.10). □

Theorem 4.14. *(Uniform convergence theorem for method* I*b). In the above theorem, for every $\lambda > 0$, $g_n(t)$ converges to $f(t)$ uniformly on $[0,T] \subset (-R, 2T+R)$.*

Proof. Let $\lambda > 0$ be fixed. Since the Cauchy product $\sum_{k=0}^{\infty} b_k(t-T)^k$ in (4.11) converges absolutely for every $t \in (-R, 2T+R)$, by Proposition 2.37, it converges to $f(t)e^{\lambda(t-T)}$ uniformly on $(-R, 2T+R)$.

Then for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > N$, we have

$$\left| f(t)e^{\lambda(t-T)} - \sum_{k=0}^{n} b_k(t-T)^k \right| = \left| \sum_{k=n+1}^{\infty} b_k(t-T)^k \right| < \epsilon$$

for all $t \in (-R, 2T+R)$. It follows that for all $t \in (-R, 2T+R)$ when $n > N$

$$\left| \left(f(t)e^{\lambda(t-T)} - \sum_{k=0}^{n} b_k(t-T)^k \right) e^{-\lambda(t-T)} \right| = \left| \sum_{k=n+1}^{\infty} b_k(t-T)^k \right| \left| e^{-\lambda(t-T)} \right| < \epsilon,$$

since $\left|e^{-\lambda(t-T)}\right| \leq 1$. Therefore,

$$\left|f(t) - \sum_{k=0}^{n} b_k(t-T)^k e^{-\lambda(t-T)}\right| < \epsilon,$$

and

$$\lim_{n\to\infty} g_n(t;\lambda) = \lim_{n\to\infty} \sum_{k=0}^{n} b_k(t-T)^k e^{-\lambda(t-T)} = f(t)$$

uniformly on $(-R, 2T+R) \supset [0,T]$. □

For any fixed $n \in \mathbb{N}$, there is a family of approximation functions $g_n(t;\lambda)$ with a parameter $\lambda > 0$ satisfying the boundary conditions (4.10), and for every $\lambda > 0$, $g_n(t;\lambda)$ converges to $f(t)$ uniformly on $[0, 2T+R)$. It is obvious for every $\lambda > 0$, $g_n(t;\lambda)$ obtained above is continuous, bounded, integral-convergent, and vanishing at infinity on Ω. In addition, $g_n(t;\lambda)$ has the following properties.

Proposition 4.15. *For every $\lambda > 0$,*

$$g_n(T;\lambda) = f(T).$$

Proof.

$$\begin{aligned} g_n(T;\lambda) &= \left. \sum_{k=0}^{n} b_k(t-T)^k e^{-\lambda(t-T)} \right|_{t=T} \\ &= b_0 \\ &= f(T). \end{aligned}$$

□

Proposition 4.16. *For every $\lambda > 0$,*

$$\frac{d}{d\lambda} b_k(\lambda) = b_{k-1}(\lambda), \quad k = 1, 2, \cdots, n.$$

Proof. By definition,

$$b_k(\lambda) = \sum_{i=0}^{k} d_i \frac{\lambda^{k-i}}{(k-i)!}.$$

Then

$$\begin{aligned}\frac{d}{d\lambda}b_k(\lambda) &= \frac{d}{d\lambda}\left(\sum_{i=0}^{k-1} d_i \frac{\lambda^{k-i}}{(k-i)!} + d_k\right) \\ &= \sum_{i=0}^{k-1} d_i \frac{(k-i)\lambda^{k-i-1}}{(k-i)!} \\ &= \sum_{i=0}^{k-1} d_i \frac{\lambda^{k-1-i}}{(k-1-i)!} \\ &= b_{k-1}(\lambda).\end{aligned}$$

□

Next, we shall find the optimal λ and the final approximation function. There are various methods to find the optimal λ, e.g. by matching the definite integrals on the main interval, which has been discussed in method Ia.

Another obvious method to obtain the optimal λ is by matching the function values at the origin, i.e. to solve the equation

$$g_n(0; \lambda) = f(0),$$

which is

$$c_0(\lambda) = f(0).$$

Since $c_0(\lambda)$ is a polynomial in λ of degree up to n, the above equation may have multiple roots. We shall choose the optimal value of λ for which the corresponding WMSE is minimum.

For every $\lambda > 0$ and $n \in \mathbb{N}$, define the approximation error function as

$$e_n(t; \lambda) = f(t) - g_n(t; \lambda).$$

Proposition 4.17.

$$\lim_{t \to T} e_n(t; \lambda) = 0.$$

Proof. This is the direct result of Proposition 4.15. □

Similarly to method Ia, the approximation error function $e_n(t; \lambda)$ in method Ib has a funnel-like shape on $[0, T]$. It has no error at $t = T$ and increases as t decreases. The error is forced to zero at the origin if we match the function values at the origin. In this case, the maximum error occurs somewhere near $t = \frac{1}{\lambda}$.

Example 4.18. *Asymptotic series expansion about an arbitrary point for finite-radius-convergent analytic functions.*

Let the target function be $f(t) = \frac{1}{1+t}$, same as in Example 4.12. Let $T = 6$ and the main interval $[0, 6]$. Method Ia fails because $f(t)$ is finite-radius-convergent about the origin and is convergent only on $[0, 1)$, which does not contain the main interval $[0, 6]$. Let $n = 10$. Expand $f(t)$ asymptotically into a power series about $t = T$ as

$$\begin{aligned} f(t) \quad \sim \quad & 0.1429 - 2.0408 \times 10^{-2}(t-6) + 2.9155 \times 10^{-3}(t-6)^2 \\ & - 4.1649 \times 10^{-4}(t-6)^3 + 5.9499 \times 10^{-5}(t-6)^4 \\ & - 8.4999 \times 10^{-6}(t-6)^5 + 1.2143 \times 10^{-6}(t-6)^6 \\ & - 1.7347 \times 10^{-7}(t-6)^7 + 2.4781 \times 10^{-8}(t-6)^8 \\ & - 3.5401 \times 10^{-9}(t-6)^9 + 5.0573 \times 10^{-10}(t-6)^{10} \\ & + \mathcal{O}((t-6)^{11}), \quad t \to 6, \end{aligned}$$

on convergence interval $(-1, 13) \supset [0, 6]$. By Theorem 4.13, there is a family of functions $g_{10}(t; \lambda)$ in (4.12) satisfying the boundary conditions (4.10).

We match the function values at the origin to find the optimal λ. Solving equation $c_0(\lambda) = f(0)$ and choosing the root with the minimum WMSE on $[0, 6]$, we obtain the optimal $\lambda = 0.6410$, and the final approximation function is

$$\begin{aligned} g_{10}(t) \quad = \quad & (1.0000 - 0.2585t + 0.2837t^2 - 9.3431 \times 10^{-2}t^3 \\ & + 2.9525 \times 10^{-2}t^4 - 5.8604 \times 10^{-3}t^5 + 8.9427 \times 10^{-4}t^6 \\ & - 9.2713 \times 10^{-5}t^7 + 6.7291 \times 10^{-6}t^8 - 2.9374 \times 10^{-7}t^9 \\ & + 6.6410 \times 10^{-9}t^{10})e^{-0.6410t}. \end{aligned}$$

The corresponding WMSE is 6.8538×10^{-5} and WMAE is 1.1014×10^{-2} on $[0, 6]$.

The integration function of $g_{10}(t)$ is

$$\begin{aligned} G_{10}(t) \quad = \quad & 3.2190 - (3.2190 + 1.0635t + 0.4701t^2 \\ & + 5.8898 \times 10^{-3}t^3 + 2.4302 \times 10^{-2}t^4 - 2.7894 \times 10^{-3}t^5 \\ & + 6.7872 \times 10^{-4}t^6 - 6.5597 \times 10^{-5}t^7 + 6.3328 \times 10^{-6}t^8 \\ & - 2.9661 \times 10^{-7}t^9 + 1.0360 \times 10^{-8}t^{10})e^{-0.6410t}. \end{aligned}$$

Figure 8 shows the overall approximation results on $[0, 30]$, which is five times the length of the main interval $[0, 6]$. The left figure shows the overall graphs of the approximation function $g_{10}(t)$ (blue line) and the target function $f(t)$ (red line). The right figure shows the overall graphs of $G_{10}(t)$ (blue line) and $F(t)$ (red line), the integration functions of $g_{10}(t)$ and $f(t)$, respectively. In each figure, the graphs of the target function and the approximation function almost coincide with each other and are indistinguishable on $[0, 6]$. In addition, $g_{10}(t)$ is nicely bounded on $[6, \infty)$ and vanishes at infinity, and $G_{10}(t)$ grows slowly on $[6, \infty)$ and approaches a horizontal asymptote $y = 3.2190$ as t tends to infinity. Notice that $F(t)$ does not approach any horizontal asymptote because $f(t)$ is integral-divergent on Ω.

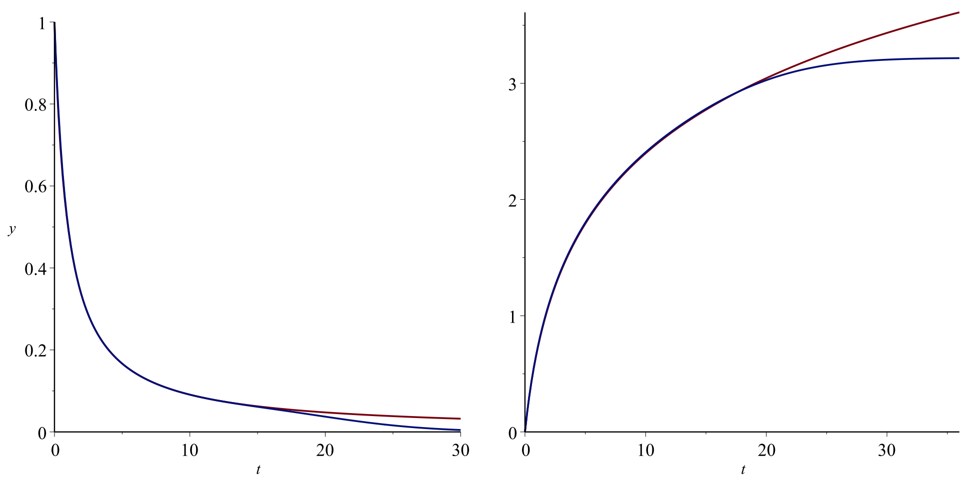

FIGURE 8. Method Ib overall approximations on $[0, 30]$ $(n = 10)$: Left – the overall graphs of the approximation function $g_{10}(t)$ (blue line) and the target function $f(t)$ (red line); Right – the overall graphs of $G_{10}(t)$ (blue line) and $F(t)$ (red line), the integration functions of $g_{10}(t)$ and $f(t)$, respectively.

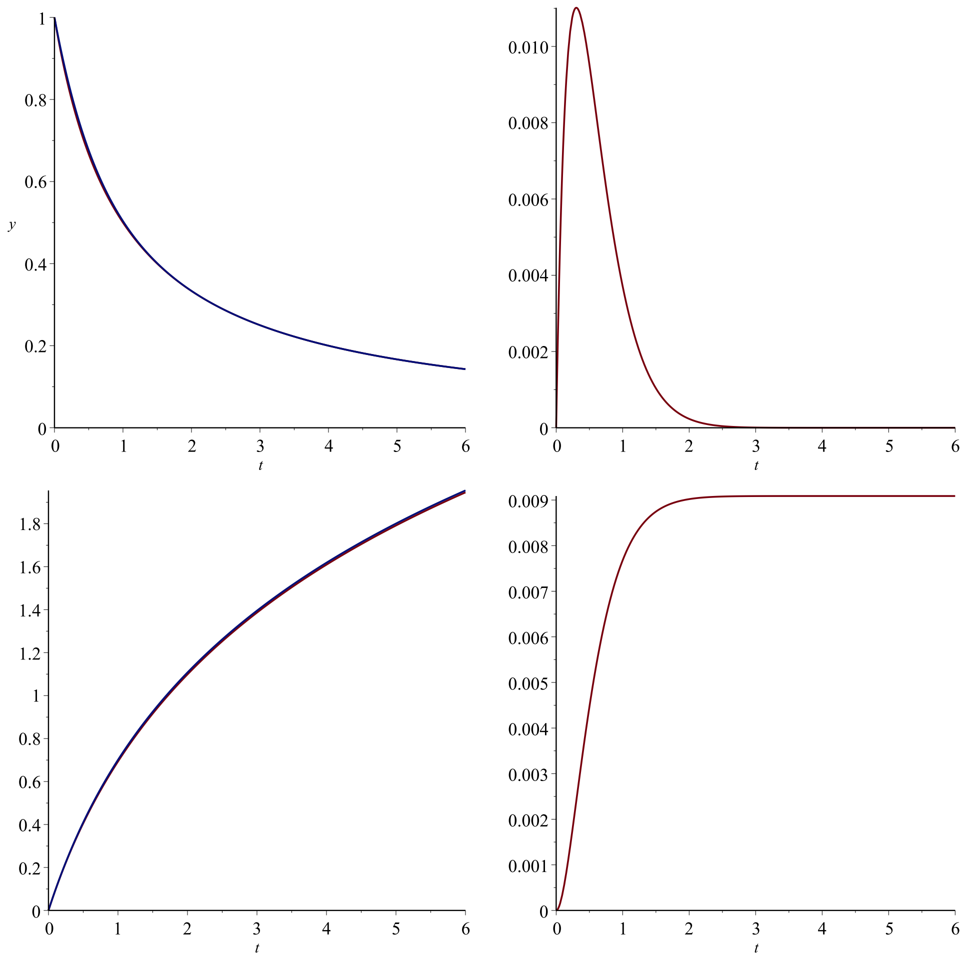

Figure 9. Method Ib approximations on the main interval $[0,6]$ ($n = 10$): Top-left – $g_{10}(t)$ (blue line) and $f(t)$ (red line) almost coincide with each other; Top-right – the absolute error function $|e_{10}(t)|$ of the approximation error function $e_{10}(t) = f(t) - g_{10}(t)$; Bottom-left – $G_{10}(t)$ (blue line) and $F(t)$ (red line) almost coincide with each other; Bottom-right – the absolute error function $|F(t) - G_{10}(t)|$.

Figure 9 shows the approximation results on the main interval $[0, 6]$. The top-left figure shows $g_{10}(t)$ (blue line) and $f(t)$ (red line) almost coincide with each other on $[0, 6]$. The top-right figure is the absolute error function $|e_{10}(t)|$ of the approximation error function $e_{10}(t) = f(t) - g_{10}(t)$. $|e_{10}(t)|$ starts with no error at $t = T$ and asymptotically increases as t decreases to zero. Since we match the function values at the origin, $e_{10}(t)$ also crosses the t-axis at the origin inside $[0, 6]$. The bottom-left figure shows $G_{10}(t)$ (blue line) and $F(t)$ (red line) almost coincide with each other on $[0, 6]$. The bottom-right figure shows the absolute error function $|F(t) - G_{10}(t)|$.

In order to demonstrate the uniform convergence on the main interval for the approximation method Ib, we repeat the above approximation experiment with $n = 20$, 30, 40, and 50, and record the corresponding optimal λ values, WMAE's and WMSE's on $[0, 6]$ in Table 4. Figure 10 shows the graphs of the absolute error functions $|e_n(t)|$ for $n = 20$, 30, 40, and 50.

Table 4. Method Ib multiple approximation experiments: the optimal λ values, WMAE's and WMSE's on $[0, 6]$ for $n = 10$, 20, 30, 40, and 50.

n	λ	$\text{WMAE}_{[0,6]}$	$\text{WMSE}_{[0,6]}$	$\frac{n}{\lambda}$
10	0.6410	1.1014×10^{-2}	6.8538×10^{-5}	15.6
20	1.0395	9.6703×10^{-4}	3.1602×10^{-7}	19.2
30	1.4396	1.0026×10^{-4}	2.4424×10^{-9}	20.8
40	1.8398	1.1274×10^{-5}	2.4173×10^{-11}	21.7
50	2.2397	1.3308×10^{-6}	2.7703×10^{-13}	22.3

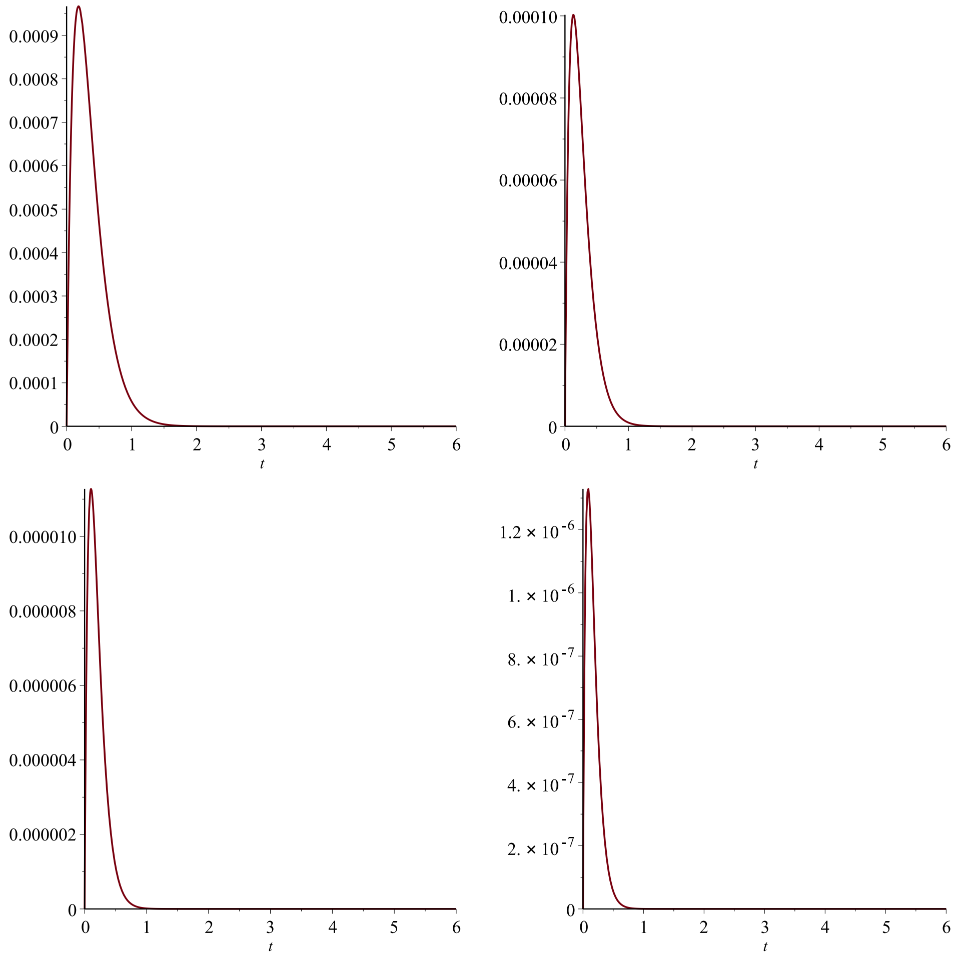

FIGURE 10. Method Ib multiple approximation experiments: The graphs of the absolute error functions $|e_n(t)|$ on the main interval $[0, 6]$ for $n = 20$ (top-left), $n = 30$ (top-right), $n = 40$ (bottom-left), and $n = 50$ (bottom-right).

4.2.4. *Method* Ic*: double asymptotic series expansions.* In this section, we shall develop the approximation method Ic, double asymptotic series expansions, as another solution to approximate finite-radius-convergent analytic functions. The method can be described in the following steps. First, we asymptotically expand the target function about the origin and obtain the left or first approximation function. The difference function between the target function and the left approximation function is also obtained. Next, we asymptotically expand the difference function about the right end-point of the main interval and obtain the right or second approximation function. Finally, we merge the left and right approximation functions to form the final approximation function.

Without loss of generality, we assume $f(t)$ is finite-radius-convergent analytic on Ω, and has its largest singularity at $t = -R$, $R > 0$. Let the main interval be $[0, T]$ and $T > R$ the second expansion center. We wish to find a function $g_m(t; \lambda) \in P_m^\lambda(\Omega)$ for some m and λ such that $g_m(t; \lambda)$ matches up to the n-th derivatives of $f(t)$ at the origin for some $n < m$ and also matches or approximates the $(n+1)$-st to m-th derivatives at $t = T$. Then we shall call $g_m(t; \lambda)$ an approximation to $f(t)$ if it is sufficiently "close" to the latter on $[0, T]$ and is nicely bounded on $[T, \infty)$.

By Theorem 4.10, we can find a critical λ and a critically damped function $g_n(t; \lambda) \in P_n^\lambda(\Omega)$ such that the $n + 1$ initial conditions are satisfied. Denote $g_n(t; \lambda)$ the *first* or *left* approximation function.

Let

$$r_n(t) = f(t) - g_n(t; \lambda)$$

and an integer

$$m = [\lambda(2T + R)] - 1.$$

Expand $r_n(t)$ asymptotically as a power series about $t = T$ as

$$r_n(t) \sim \sum_{k=0}^{m} d_k (t - T)^k + \mathcal{O}\left((t - T)^{m+1}\right), \quad t \to T \tag{4.15}$$

on convergence interval $(-R, 2T + R) \supset [0, T]$.

Theorem 4.19. *(Existence theorem for the second approximation function). In the above problem, for the critical λ and m, there exists a function in $P_m^\lambda(\Omega)$ which matches up to the m-th derivatives of $r_n(t)$ at $t = T$.*

Proof. (Constructive proof). For the critical λ and $m = [\lambda(2T + R)] - 1 \in \mathbb{N}$, Theorem 4.13 implies there exists a function in $P_m^\lambda(\Omega)$ matching up to the m-th derivatives of $r_n(t)$ at $t = T$. The following constructive proof will reveal more details.

Rewrite the partial sum in Equation (4.15) in ascending order of t as

$$\sum_{k=0}^{m} d_k (t - T)^k = \sum_{k=0}^{m} b_k t^k.$$

The b_k can be obtained from the d_k as

$$\begin{aligned}
b_0 &= d_0\binom{0}{0} + d_1\binom{1}{0}(-T) + \cdots + d_m\binom{m}{0}(-T)^m,\\
\vdots &= \quad \vdots\\
b_{m-2} &= d_{m-2}\binom{m-2}{m-2} + d_{m-1}\binom{m-1}{m-2}(-T) + d_m\binom{m}{m-2}(-T)^2,\\
b_{m-1} &= d_{m-1}\binom{m-1}{m-1} + d_m\binom{m}{m-1}(-T),\\
b_m &= d_m\binom{m}{m},
\end{aligned}$$

with the general term as

$$b_k = \sum_{i=k}^{m} d_i \binom{i}{k}(-T)^{i-k}, \quad k = 0, 1, \cdots, m,$$

or in the matrix form as

$$\begin{pmatrix} b_0 \\ \vdots \\ b_{m-2} \\ b_{m-1} \\ b_m \end{pmatrix} = \begin{pmatrix} \binom{0}{0} & \binom{1}{0}(-T) & \binom{2}{0}(-T)^2 & \cdots & \binom{m}{0}(-T)^m \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \binom{m-2}{m-2} & \binom{m-1}{m-2}(-T) & \binom{m}{m-2}(-T)^2 \\ 0 & \cdots & 0 & \binom{m-1}{m-1} & \binom{m}{m-1}(-T) \\ 0 & \cdots & 0 & 0 & \binom{m}{m} \end{pmatrix} \begin{pmatrix} d_0 \\ \vdots \\ d_{m-2} \\ d_{m-1} \\ d_m \end{pmatrix}.$$

Let this matrix be A; then A is $(m+1)\times(m+1)$. For $m = 4$, we have

$$A = \begin{pmatrix} 1 & -T & T^2 & -T^3 & T^4 \\ 0 & 1 & -2T & 3T^2 & -4T^3 \\ 0 & 0 & 1 & -3T & 6T^2 \\ 0 & 0 & 0 & 1 & -4T \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Next, consider the m-th partial sum of the Cauchy product

$$\left(\sum_{k=0}^{\infty} b_k t^k\right)\left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} t^k\right) = \left(\sum_{k=0}^{\infty} c'_k t^k\right).$$

Then the c'_k can be calculated from the b_k as

$$\begin{aligned}
c'_0 &= b_0 \\
c'_1 &= b_0\lambda + b_1 \\
c'_2 &= b_0\frac{\lambda^2}{2!} + b_1\lambda + b_2 \\
\vdots &= \qquad \vdots \\
c'_m &= b_0\frac{\lambda^m}{m!} + b_1\frac{\lambda^{m-1}}{(m-1)!} + \cdots + b_{m-1}\lambda + b_m,
\end{aligned}$$

with the general term as

$$c'_k = b_0\frac{\lambda^k}{k!} + b_1\frac{\lambda^{k-1}}{(k-1)!} + \cdots + b_{k-1}\lambda + b_k, \quad k = 0, 1, \cdots, m,$$

or in the matrix form as

$$\begin{pmatrix} c'_0 \\ c'_1 \\ c'_2 \\ \vdots \\ c'_m \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \lambda & 1 & 0 & \cdots & 0 \\ \frac{\lambda^2}{2!} & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\lambda^m}{m!} & \frac{\lambda^{m-1}}{(m-1)!} & \frac{\lambda^{m-2}}{(m-2)!} & \cdots & 1 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

where λ is the critical value in the first critically damped approximation function $g_n(t;\lambda)$. Let this matrix be B; then B is $(m+1) \times (m+1)$.

Thus, the d_k are transformed into the c'_k by matrix BA. For $m = 4$, we have

$$BA = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 & 0 \\ \frac{\lambda^2}{2!} & \lambda & 1 & 0 & 0 \\ \frac{\lambda^3}{3!} & \frac{\lambda^2}{2!} & \lambda & 1 & 0 \\ \frac{\lambda^4}{4!} & \frac{\lambda^3}{3!} & \frac{\lambda^2}{2!} & \lambda & 1 \end{pmatrix} \begin{pmatrix} 1 & -T & T^2 & -T^3 & T^4 \\ 0 & 1 & -2T & 3T^2 & -4T^3 \\ 0 & 0 & 1 & -3T & 6T^2 \\ 0 & 0 & 0 & 1 & -4T \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where λ and T are known, and

$$\begin{pmatrix} c_0' \\ c_1' \\ c_2' \\ c_3' \\ c_4' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 & 0 \\ \frac{\lambda^2}{2!} & \lambda & 1 & 0 & 0 \\ \frac{\lambda^3}{3!} & \frac{\lambda^2}{2!} & \lambda & 1 & 0 \\ \frac{\lambda^4}{4!} & \frac{\lambda^3}{3!} & \frac{\lambda^2}{2!} & \lambda & 1 \end{pmatrix} \begin{pmatrix} 1 & -T & T^2 & -T^3 & T^4 \\ 0 & 1 & -2T & 3T^2 & -4T^3 \\ 0 & 0 & 1 & -3T & 6T^2 \\ 0 & 0 & 0 & 1 & -4T \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}.$$

Let

$$s_m(t) = \sum_{k=0}^{m} c_k' t^k e^{-\lambda t}.$$

Clearly $s_m(t) \in P_m^\lambda(\Omega)$. Expand $s_m(t)$ asymptotically into a power series about $t = T$ as

$$s_m(t) = \sum_{k=0}^{m} d_k'(t-T)^k + \mathcal{O}\left((t-T)^{m+1}\right), \quad t \to T.$$

It is easily verified the vector $[d_0' \, d_1' \, \cdots \, d_m']^T$ can be obtained from $[c_0' \, c_1' \, \cdots \, c_m']^T$ by left-multiplication with $A^{-1}B^{-1}$, or from $[d_0 \, d_1 \, \cdots \, d_m]^T$ by left-multiplication with $A^{-1}B^{-1}BA = I$, an identity matrix, which implies

$$d_k' = d_k, \quad k = 0, 1, \cdots, m,$$

i.e. $s_m(t)$ matches up to the m-th derivatives of $r_n(t)$ at $t = T$.

Since the zero to n-th terms of $r_n(t)$ are zeros,

$$r_n(t) \sim \mathcal{O}\left(t^{n+1}\right), \quad t \to 0.$$

The fact the approximation function $s_m(t)$ matches up to the m-th order derivatives at $t = T$ implies $s_m(t)$ should be in a space of dimension $m - n$. We can simply drop the zero to n-th terms of $s_m(t)$ and obtain the *second* or *right* approximation function

$$g_{m\backslash n}(t) = g_{m\backslash n}(t; \lambda) = \sum_{k=n+1}^{m} c_k' t^k e^{-\lambda t}. \tag{4.16}$$

This results in an approximation error function

$$\sum_{k=0}^{n} c_k' t^k e^{-\lambda t},$$

which uniformly converges to the limit function

$$0 = 0 + 0t + \cdots + 0t^n$$

on $[0, T]$.

Thus, the matrix B above is actually $(m-n)\times(m+1)$ and the general row equation can be rewritten as

$$c'_k = b_0\frac{\lambda^k}{k!} + b_1\frac{\lambda^{k-1}}{(k-1)!} + \cdots + b_{k-1}\lambda + b_k, \quad k = n+1, n+2, \cdots, m.$$

In conclusion, for the critical value of λ and n, $g_{m\backslash n}(t)$ matches or approximates the $(n+1)$-st to m-th derivatives of $r_n(t)$ at $t = T$. □

Theorem 4.20. *(Merging and uniform convergence theorem for method* I*c). In the above problem, for the critical value of λ and $n < m$, let*

$$g_m(t) = g_m(t;\lambda) = g_n(t;\lambda) + g_{m\backslash n}(t;\lambda). \tag{4.17}$$

Then $g_m(t)$ converges to $f(t)$ on $[0, T]$ uniformly.

Proof. For the critical value of λ, as n tends to infinity, $m = [\lambda(2T+R)]-1$ also tends to infinity. By Theorem 4.14, as both n, m tend to infinities, $g_{m\backslash n}(t;\lambda)$ converges to $r_n(t)$, i.e.

$$\lim_{n,m\to\infty} g_{m\backslash n}(t;\lambda) = r_n(t)$$

on $(-R, 2T+R)$ uniformly.

For every $t \in (-R, 2T+R)$, we have

$$\begin{aligned} |f(t) - g_m(t)| &= \left|f(t) - \big(g_n(t) + g_{m\backslash n}(t)\big)\right| \\ &= \left|(f(t) - g_n(t)) - g_{m\backslash n}(t)\right| \\ &= \left|r_n(t) - g_{m\backslash n}(t)\right|. \end{aligned}$$

It follows that, as both n, m tend to infinities, $g_m(t)$ converges to $f(t)$, i.e.

$$\lim_{n,m\to\infty} g_m(t) = f(t)$$

on $(-R, 2T+R) \supset [0, T]$ uniformly. □

In the above problem, for the critical value of λ and n, m, define the approximation error function as

$$e_m(t) = e_m(t;\lambda) = f(t) - g_m(t;\lambda).$$

Proposition 4.21. *$e_m(t)$ has two asymptotic series expansion centers, the origin and $t = T$.*

Proof. Trivial. Since $g_n(t)$ and $g_{m\backslash n}(t)$ are the partial sums of two asymptotic series centered at the origin and $t = T$, respectively, the approximation error function $e_m(t)$ has two asymptotic series expansion centers. □

Proposition 4.22. *$|e_m(t)|$ has a maximum value at a point near either $\frac{n}{\lambda}$ or $\frac{n+1}{\lambda}$ on $[0, T]$.*

Proof. There are basically two cases of the approximation error function $e_m(t)$ in the above problem. The first case is the error function has different signs on $[0, \frac{n}{\lambda}]$ and on $[\frac{n+1}{\lambda}, T]$. Then, by continuity, it must cross the t-axis between $\frac{n}{\lambda}$ and $\frac{n+1}{\lambda}$, and the maximum absolute error occurs near either point because of the asymptotically increasing properties.

The second case is the error function has the same sign on both intervals. Thus, by continuity and the asymptotically increasing properties, the maximum absolute error occurs between the above two points. □

It follows that $|e_m(t)|$ has two types of curve shapes on the main interval $[0, T]$, a bell (spindle) shape or an M-shape, which may be affected by the parities of n and m.

Theorem 4.23. *(Degree extension or time span expansion theorem). In the above problem, for the critical λ, the first critically damped approximation function $g_n(t; \lambda)$ can be extended to the final approximation function $g_m(t; \lambda)$ with the maximum degree*

$$m = [\lambda(2T + R)] - 1$$

such that the main interval $[0, T]$ and the time span interval of $g_m(t; \lambda)$ are completely contained inside the convergence interval $(-R, 2T + R)$.

Proof. Trivial. Approximately,

$$\frac{m}{\lambda} = \frac{[\lambda(2T + R)] - 1}{\lambda} < 2T + R$$

implies

$$[0, \frac{m}{\lambda}] \subset [0, 2T + R).$$

□

Example 4.24. *Double asymptotic series expansions for finite-radius-convergent analytic functions.*

Let the target function be $f(t) = \frac{1}{1+t}$, same as in Example 4.12 and 4.18. We apply method Ic to this approximation problem.

Let $n = 4$, $T = 6$, and the main interval $[0, 6]$. By Theorem 4.10, we have a critical $\lambda = 2.1806$ and a critically damped first approximation function

$$g_4(t) = (1 + 1.1806t + 1.1969t^2 + 0.5312t^3 + 0.4109t^4)e^{-2.1806t}$$

satisfying $n + 2 = 6$ initial conditions.

The first approximation function only converges to $f(t)$ on $[0, 1)$ and the approximation cannot be improved by increasing n. We pursue the second approximation function which matches the boundary conditions at $t = 6$.

The remainder function is

$$r_4(t) = f(t) - g_4(t).$$

By Theorem 4.23, we set

$$m = [\lambda(2T + R)] - 1 = 27.$$

Then we shall find the second approximation function satisfying the $m - n = 23$ boundary conditions for $r_4(t)$ at $t = 6$.

We expand $r_4(t)$ asymptotically into a power series about $t = 6$ and obtain

$$\begin{aligned} r_4(t) \quad \sim \quad & 0.1414 - 1.8132 \times 10^{-2}(t-6) + 1.1964 \times 10^{-3}(t-6)^2 \\ & + 4.0715 \times 10^{-4}(t-6)^3 - 2.1661 \times 10^{-4}(t-6)^4 \\ & + 5.7970 \times 10^{-5}(t-6)^5 - 9.6748 \times 10^{-6}(t-6)^6 \\ & + 6.4686 \times 10^{-7}(t-6)^7 + 1.7758 \times 10^{-7}(t-6)^8 \\ & - 7.4803 \times 10^{-8}(t-6)^9 + 1.5216 \times 10^{-8}(t-6)^{10} \\ & - 1.8761 \times 10^{-9}(t-6)^{11} + 6.8582 \times 10^{-11}(t-6)^{12} \\ & + 3.1373 \times 10^{-11}(t-6)^{13} - 9.5462 \times 10^{-12}(t-6)^{14} \\ & + 1.6239 \times 10^{-12}(t-6)^{15} - 1.8446 \times 10^{-13}(t-6)^{16} \\ & + 1.0382 \times 10^{-14}(t-6)^{17} + 1.1749 \times 10^{-15}(t-6)^{18} \\ & - 4.6418 \times 10^{-16}(t-6)^{19} + 8.4969 \times 10^{-17}(t-6)^{20} \\ & - 1.1276 \times 10^{-17}(t-6)^{21} + 1.1340 \times 10^{-18}(t-6)^{22} \\ & - 7.4490 \times 10^{-20}(t-6)^{23} - 7.3244 \times 10^{-22}(t-6)^{24} \\ & + 1.2424 \times 10^{-21}(t-6)^{25} - 2.5887 \times 10^{-22}(t-6)^{26} \\ & + 3.8874 \times 10^{-23}(t-6)^{27} + \mathcal{O}\left((t-6)^{28}\right), \quad t \to 6. \end{aligned}$$

By Theorem 4.19, the coefficients $\{c'_k\}_{k=5}^{27}$ of $g_{27\backslash 4}(t;\lambda)$ in (4.16) are obtained, and we have

$$\begin{aligned} g_{27\backslash 4}(t) \quad = \quad & (8.6935 \times 10^{-2}t^5 + 4.0355 \times 10^{-2}t^6 + 1.0132 \times 10^{-2}t^7 \\ & + 2.4110 \times 10^{-3}t^8 + 5.6881 \times 10^{-4}t^9 + 1.1416 \times 10^{-4}t^{10} \\ & + 2.0226 \times 10^{-5}t^{11} + 3.4645 \times 10^{-6}t^{12} + 5.6053 \times 10^{-7}t^{13} \\ & + 8.1200 \times 10^{-8}t^{14} + 1.0834 \times 10^{-8}t^{15} + 1.4143 \times 10^{-9}t^{16} \\ & + 1.7826 \times 10^{-10}t^{17} + 2.0439 \times 10^{-11}t^{18} + 2.1364 \times 10^{-12}t^{19} \\ & + 2.2095 \times 10^{-13}t^{20} + 2.3477 \times 10^{-14}t^{21} + 2.3345 \times 10^{-15}t^{22} \\ & + 1.9716 \times 10^{-16}t^{23} + 1.4959 \times 10^{-17}t^{24} + 1.3207 \times 10^{-18}t^{25} \\ & + 1.4031 \times 10^{-19}t^{26} + 1.2530 \times 10^{-20}t^{27})e^{-2.1806t}. \end{aligned}$$

By Theorem 4.20, we have the final approximation function $g_{27}(t)$ as

$$\begin{aligned} g_{27}(t) &= g_4(t) + g_{27\backslash 4}(t) \\ &= (1 + 1.1806t + 1.1969t^2 + 0.5312t^3 + 0.4109t^4 \\ &\quad + 8.6935 \times 10^{-2}t^5 + 4.0355 \times 10^{-2}t^6 + 1.0132 \times 10^{-2}t^7 \\ &\quad + 2.4110 \times 10^{-3}t^8 + 5.6881 \times 10^{-4}t^9 + 1.1416 \times 10^{-4}t^{10} \\ &\quad + 2.0226 \times 10^{-5}t^{11} + 3.4645 \times 10^{-6}t^{12} + 5.6053 \times 10^{-7}t^{13} \\ &\quad + 8.1200 \times 10^{-8}t^{14} + 1.0834 \times 10^{-8}t^{15} + 1.4143 \times 10^{-9}t^{16} \\ &\quad + 1.7826 \times 10^{-10}t^{17} + 2.0439 \times 10^{-11}t^{18} + 2.1364 \times 10^{-12}t^{19} \\ &\quad + 2.2095 \times 10^{-13}t^{20} + 2.3477 \times 10^{-14}t^{21} + 2.3345 \times 10^{-15}t^{22} \\ &\quad + 1.9716 \times 10^{-16}t^{23} + 1.4959 \times 10^{-17}t^{24} + 1.3207 \times 10^{-18}t^{25} \\ &\quad + 1.4031 \times 10^{-19}t^{26} + 1.2530 \times 10^{-20}t^{27})e^{-2.1806t}. \end{aligned}$$

The corresponding WMAE on $[0,6]$ is 7.8239×10^{-3}, occurring at $t = 1.8463$, between $\frac{4}{\lambda}$ and $\frac{5}{\lambda}$. The corresponding WMSE on $[0,6]$ is 9.6544×10^{-5}.

In addition, the integration function of $g_{27}(t)$ is

$$\begin{aligned} G_{27}(t) &= 2.6729 - (2.6729 + 4.8286t + 4.6744t^2 + 2.9987t^3 \\ &\quad + 1.5019t^4 + 0.5729t^5 + 0.1937t^6 + 5.4577 \times 10^{-2}t^7 \\ &\quad + 1.3610 \times 10^{-2}t^8 + 3.0296 \times 10^{-3}t^9 + 6.0376 \times 10^{-4}t^{10} \\ &\quad + 1.0931 \times 10^{-4}t^{11} + 1.8178 \times 10^{-5}t^{12} + 2.7827 \times 10^{-6}t^{13} \\ &\quad + 3.9338 \times 10^{-7}t^{14} + 5.1774 \times 10^{-8}t^{15} + 6.3791 \times 10^{-9}t^{16} \\ &\quad + 7.3507 \times 10^{-10}t^{17} + 7.9146 \times 10^{-11}t^{18} + 8.0078 \times 10^{-12}t^{19} \\ &\quad + 7.6627 \times 10^{-13}t^{20} + 6.9047 \times 10^{-14}t^{21} + 5.7767 \times 10^{-15}t^{22} \\ &\quad + 4.4618 \times 10^{-16}t^{23} + 3.2325 \times 10^{-17}t^{24} + 2.2211 \times 10^{-18}t^{25} \\ &\quad + 1.3549 \times 10^{-19}t^{26} + 5.7460 \times 10^{-21}t^{27})e^{-2.1806t}. \end{aligned}$$

Figure 11 shows the approximation results on $[0, 18]$, which is three times the length of the main interval $[0, 6]$. There are three function graphs in the figure: the final approximation function $g_{27}(t)$ (blue line), the target function $f(t)$ (red line), and the first critically damped approximation function $g_4(t)$ (green line). The graphs of $g_{27}(t)$ and $f(t)$ almost coincide with each other and are indistinguishable on $[0, 6]$. In addition, $g_{27}(t)$ is nicely bounded on $[6, \infty)$ and vanishes at infinity.

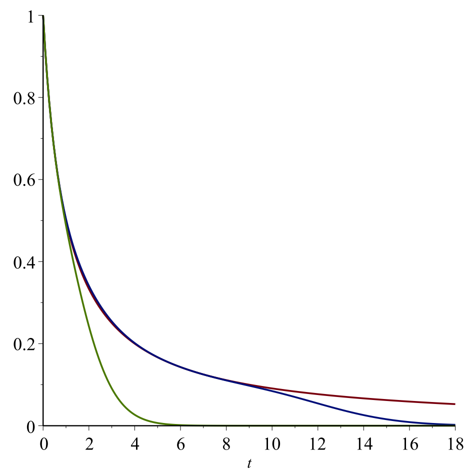

Figure 11. Method Ic overall approximations on $[0, 18]$ ($n = 4, m = 27$): the final approximation function $g_{27}(t)$ (blue line), the target function $f(t)$ (red line), and the first critically damped approximation function $g_4(t)$ (green line).

Figure 12 shows the approximation results on the main interval $[0, 6]$. The left figure shows the final approximation function $g_{27}(t)$ (blue line) and the target function $f(t)$ (red line) almost coincide with each other on $[0, 6]$. It also shows the first critically damped approximation function $g_4(t)$ (green line) only approximates $f(t)$ on $[0, 1]$ and fails to do so on $[1, 6]$. The right figure is the absolute error function $|e_{27}(t)|$ of the approximation error function $e_{27}(t) = f(t) - g_{27}(t)$. $e_{27}(t)$ has no error at the origin and asymptotically increases as t increases about the origin. It also has a very small error at $t = T$ and asymptotically increases as t decreases about $t = T$. The error reaches its maximum at $t = 1.8463$, between $\frac{4}{\lambda}$ and $\frac{5}{\lambda}$.

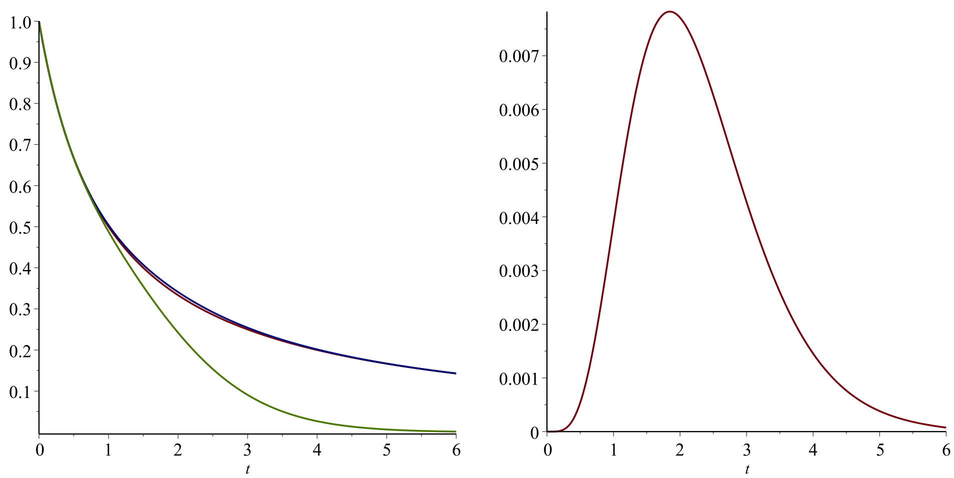

FIGURE 12. Method Ic approximation on the main interval $[0, 6]$ $(n = 4, m = 27)$: Left – the final approximation function $g_{27}(t)$ (blue line), the target function $f(t)$ (red line), and the first critically damped approximation function $g_4(t)$ (green line); Right – the absolute error function $|e_{27}(t)|$ of the approximation error function $e_{27}(t) = f(t) - g_{27}(t)$.

In order to demonstrate the uniform convergence on the main interval for the approximation method Ic, we repeat the above approximation experiment with $n = 10$, 20, 30, and 40, resulting in $m = 49$, 86, 123, and 159, correspondingly. We record the corresponding n, m values, optimal λ values, WMAE's and WMSE's on $[0, 6]$ in Table 5. Figure 13 shows the graphs of the absolute error functions $|e_m(t)|$ for $m = 49$, 86, 123, and 159.

Table 5. Method Ic multiple approximation experiments: the n, m values, optimal λ values, WMAE's and WMSE's on $[0, 6]$ for $n = 4$, 10, 20, 30, and 40, resulting in $m = 27$, 49, 86, 123, and 159, correspondingly.

n	λ	m	$\text{WMAE}_{[0,6]}$	$\text{WMSE}_{[0,6]}$	$\frac{n}{\lambda}$	$\frac{m}{\lambda}$
4	2.1806	27	7.8239×10^{-3}	9.6544×10^{-5}	1.83	12.38
10	3.9055	49	3.3427×10^{-3}	1.5479×10^{-5}	2.56	12.55
20	6.7431	86	1.9936×10^{-3}	4.5702×10^{-6}	2.97	12.75
30	9.5626	123	1.4217×10^{-3}	2.0130×10^{-6}	3.14	12.86
40	12.3733	159	1.1130×10^{-3}	1.1034×10^{-6}	3.23	12.85

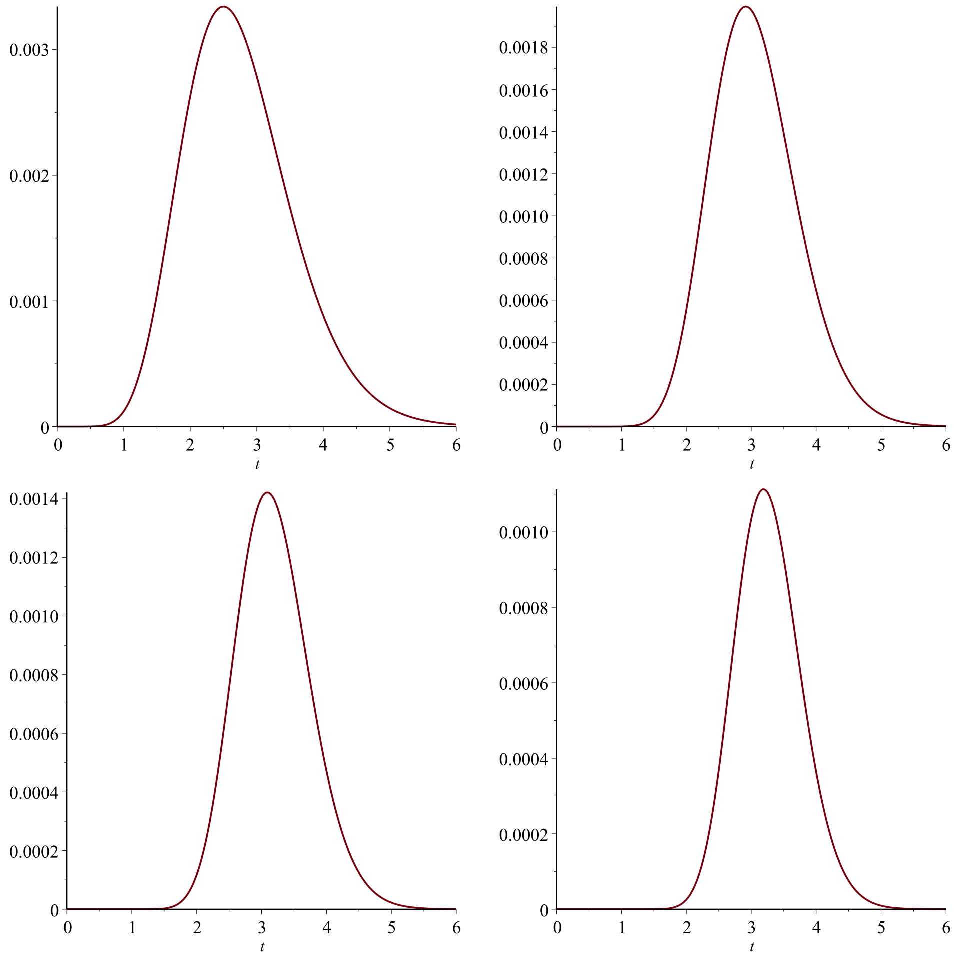

FIGURE 13. Method Ic multiple approximation experiments: The graphs of the absolute error functions $|e_m(t)|$ on the main interval $[0, 6]$ for the approximation function $g_m(t)$ with $(n, m) = (10, 49)$ (top-left), $(n, m) = (20, 86)$ (top-right), $(n, m) = (30, 123)$ (bottom-left), and $(n, m) = (40, 159)$ (bottom-right).

4.3. **Method II: Laplace transform moment matching in $P_n^\lambda(\Omega)$.** In asymptotic series expansion approximation methods Ia, Ib, and Ic, local properties, the derivatives, of the target function are used to find the approximation function. In this section, we shall develop a new method which uses global properties, namely the moment integrals, of the target function to find the approximation function.

In physics, a moment is a quantitative measure describing the distribution of discrete or continuous mass points. The term n-th moment, represented by some integral, is generalized in statistics to describe the distribution of a probability density function about some center. We name our approximation method II as moment matching in the general sense. This way when the method is used to approximate a probability density function, the term moment can be interpreted in its original meaning in statistics.

Moments can also be obtained from the power series expansion of a function's Laplace transform about the origin. Thus, matching the moments is equivalent to matching the power series coefficients of a Laplace transform.

Let $f(t) \in C_0^B(\Omega)$ be a target probability density function. Then

$$f(t) \geq 0 \qquad \text{and} \qquad \int_0^\infty f(t) = 1.$$

The k-th *moment* of $f(t)$ is defined as

$$M_k = \int_0^\infty t^k f(t) dt, \quad k = 0, 1, \cdots,$$

if the improper integral is convergent, and $f(t)$ is said to be the k-th *moment-integral-convergent* on Ω. Otherwise, $f(t)$ is the k-th *moment-integral-divergent*.

Assume $f(t)$ is moment-integral-convergent on Ω for all orders. Then its Laplace transform $\bar{f}(s)$ exists and has a power series expansion about the origin as

$$\bar{f}(s) = \mathscr{L}\{f(t)\} = \sum_{k=0}^\infty b_k s^k = \sum_{k=0}^\infty (-1)^k \frac{M_k}{k!} s^k,$$

where

$$b_k = (-1)^k \frac{M_k}{k!}, \quad k = 0, 1, \cdots,$$

with some positive radius of convergence.

In the moment matching approximation method, we need to find a function $g(t) = g_n(t; \lambda) \in P_n^\lambda(\Omega)$, for some $\lambda > 0$, whose moments match those of the target function $f(t)$ up to the n-th order. Assume such $g(t)$ is found and let

$$\bar{g}(s) = \bar{g}_n(s) = \mathscr{L}\{g(t)\}.$$

By Proposition 3.31, expand $\bar{g}(s)$ into a power series about the origin as

$$\bar{g}(s) = \sum_{k=0}^{\infty} b'_k s^k.$$

Then moment matching is equivalent to matching the power series coefficients of $\bar{g}(s)$ with those of $\bar{f}(s)$, i.e.

$$b'_k = b_k, \quad k = 0, 1, \cdots, n. \tag{4.18}$$

In fact, $\bar{g}(s)$ is found before $g(t)$: we shall invert $\bar{g}(s)$ to get $g(t)$. Generally, one needs to perform a complex contour integration to invert a Laplace transform. However, when the Laplace transform is rational or meromorphic, the computation of contour integration can be simplified as partial fractions by the residue theorem. In $P_n^\lambda(\Omega)$ spaces, inverting a Laplace transform is even simpler. Since both Laplace transformation and inverse Laplace transformation are linear and one-to-one operations, they can be represented by matrices between function spaces. This means a matrix can represent the mapping between the coefficients of power series expansion of $\bar{g}(s)$ about the origin and the coefficients of $g(t)$ in the standard form. This seamlessly integrates Laplace transform inversion into method II.

Theorem 4.25. *(Existence theorem for method* II*). Let $f(t)$ and $\bar{f}(s)$ be defined above. For every $\lambda > 0$ and $n \in \mathbb{N}$, there exists a function $g_n(t;\lambda) \in P_n^\lambda(\Omega)$ such that the moments of $g_n(t;\lambda)$ match those of $f(t)$ up to the n-th order.*

Proof. (Constructive proof). Let $\lambda > 0$ and n fixed. Consider a function $g_n(t;\lambda) \in P_n^\lambda(\Omega)$, denoted by

$$g_n(t;\lambda) = \sum_{k=0}^{n} c_k t^k e^{-\lambda t},$$

whose Laplace transform $\bar{g}(s) = \bar{g}_n(s) = \mathscr{L}\{g_n(t;\lambda)\}$ is written as

$$\begin{aligned}
\bar{g}_n(s) &= \sum_{k=0}^{n} \frac{c_k k!}{(s+\lambda)^{k+1}} && (4.19)\\
&= \sum_{k=0}^{n} \frac{c'_k}{(s+\lambda)^{k+1}} && (4.20)\\
&= \frac{\sum_{k=0}^{n} d_k s^k}{(s+\lambda)^{n+1}} && (4.21)\\
&= \sum_{k=0}^{\infty} b'_k s^k, && (4.22)
\end{aligned}$$

where the c'_k and d_k are intermediate variables in order to simplify the calculations. In addition, by hypothesis,

$$\bar{f}(s) = \sum_{k=0}^{\infty} b_k s^k.$$

We will show there exists a $\bar{g}(s)$ such that $b_k = b'_k$, $k = 0, 1, \cdots, n$. The theorem is proved if we can constructively find the c_k of $g_n(t; \lambda)$ from the b_k of $\bar{f}(s)$ and show the transformation is one-to-one or invertible. However, it is difficult to directly write the relation from the b_k to the c_k. We will instead derive the relation indirectly from Equation (4.22) to (4.21), and finally to Equation (4.20) or (4.19). Clearly, the last two equations are equivalent, so we only need to derive the c'_k from the b_k. We will also show this transformation is simply the multiplication of two invertible matrices and is one-to-one.

Firstly, by matching the moments up to the n-th order, we can calculate the d_k from the b_k by multiplying out

$$\left(\sum_{k=0}^{\infty} b_k s^k \right) (s + \lambda)^{n+1}$$

and associating the like power terms up to the n-th order. This can be described by the following system of equations:

$$\begin{aligned}
d_0 &= b_0 \binom{n+1}{0} \lambda^{n+1}, \\
d_1 &= b_0 \binom{n+1}{1} \lambda^{n} + b_1 \binom{n+1}{0} \lambda^{n+1}, \\
d_2 &= b_0 \binom{n+1}{2} \lambda^{n-1} + b_1 \binom{n+1}{1} \lambda^{n} + b_2 \binom{n+1}{0} \lambda^{n+1}, \\
\vdots &= \quad \vdots \\
d_n &= b_0 \binom{n+1}{n} \lambda^{1} + b_1 \binom{n+1}{n-1} \lambda^{2} + \cdots + b_n \binom{n+1}{0} \lambda^{n+1},
\end{aligned}$$

with the general row equation as

$$d_k = \sum_{i=0}^{k} b_i \binom{n+1}{k-i} \lambda^{n+1-k+i}, \quad k = 0, 1, \cdots, n,$$

or in the matrix form as

$$\begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} \binom{n+1}{0}\lambda^{n+1} & 0 & 0 & \cdots & 0 \\ \binom{n+1}{1}\lambda^{n} & \binom{n+1}{0}\lambda^{n+1} & 0 & \cdots & 0 \\ \binom{n+1}{2}\lambda^{n-1} & \binom{n+1}{1}\lambda^{n} & \binom{n+1}{0}\lambda^{n+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \binom{n+1}{n}\lambda^{1} & \binom{n+1}{n-1}\lambda^{2} & \binom{n+1}{n-2}\lambda^{3} & \cdots & \binom{n+1}{0}\lambda^{n+1} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Let the above matrix be A; then A is $(n+1) \times (n+1)$ and is a matrix in λ. For $n = 4$, we have

$$A = \begin{pmatrix} \lambda^5 & 0 & 0 & 0 & 0 \\ 5\lambda^4 & \lambda^5 & 0 & 0 & 0 \\ 10\lambda^3 & 5\lambda^4 & \lambda^5 & 0 & 0 \\ 10\lambda^2 & 10\lambda^3 & 5\lambda^4 & \lambda^5 & 0 \\ 5\lambda & 10\lambda^2 & 10\lambda^3 & 5\lambda^4 & \lambda^5 \end{pmatrix}.$$

Next, we shall calculate the c'_k from the d_k. Simplifying Equation (4.20) in common denominator and equating the like terms of the resulting numerator with those in (4.21), we derive the d_k from the c'_k in the matrix form as

$$\begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} \binom{n}{0}\lambda^{n} & \binom{n-1}{0}\lambda^{n-1} & \binom{n-2}{0}\lambda^{n-2} & \cdots & \binom{1}{0}\lambda^{1} & \binom{0}{0}\lambda^{0} \\ \binom{n}{1}\lambda^{n-1} & \binom{n-1}{1}\lambda^{n-2} & \binom{n-2}{1}\lambda^{n-3} & \cdots & \binom{1}{1}\lambda^{0} & 0 \\ \binom{n}{2}\lambda^{n-2} & \binom{n-1}{2}\lambda^{n-3} & \binom{n-2}{2}\lambda^{n-4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \binom{n}{n-1}\lambda^{1} & \binom{n-1}{n-1}\lambda^{0} & 0 & \cdots & 0 & 0 \\ \binom{n}{n}\lambda^{0} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} c'_0 \\ c'_1 \\ c'_2 \\ \vdots \\ c'_n \end{pmatrix}.$$

Let this matrix be B; then B is $(n+1)\times(n+1)$ and is invertible since $|B| = 1$. Obtaining B^{-1}, we can calculate the c'_k from the d_k as

$$\begin{pmatrix} c'_0 \\ c'_1 \\ c'_2 \\ \vdots \\ c'_n \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & 0 & \binom{n}{0}(-\lambda)^0 \\ 0 & \cdots & 0 & \binom{n-1}{0}(-\lambda)^0 & \binom{n}{1}(-\lambda)^1 \\ 0 & \cdots & \binom{n-2}{0}(-\lambda)^0 & \binom{n-1}{1}(-\lambda)^1 & \binom{n}{2}(-\lambda)^2 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ \binom{0}{0}(-\lambda)^0 & \cdots & \binom{n-2}{n-2}(-\lambda)^{n-2} & \binom{n-1}{n-1}(-\lambda)^{n-1} & \binom{n}{n}(-\lambda)^n \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}.$$

For $n = 4$, B and B^{-1} are

$$B = \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda^2 & \lambda & 1 \\ 4\lambda^3 & 3\lambda^2 & 2\lambda & 1 & 0 \\ 6\lambda^2 & 3\lambda & 1 & 0 & 0 \\ 4\lambda & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } B^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -4\lambda \\ 0 & 0 & 1 & -3\lambda & 6\lambda^2 \\ 0 & 1 & -2\lambda & 3\lambda^2 & -4\lambda^3 \\ 1 & -\lambda & \lambda^2 & -\lambda^3 & \lambda^4 \end{pmatrix},$$

respectively.

Now, we can transform the b_k into the c'_k using matrix $B^{-1}A$. For $n = 4$, this gives

$$B^{-1}A = \begin{pmatrix} 5\lambda & 10\lambda^2 & 10\lambda^3 & 5\lambda^4 & \lambda^5 \\ -10\lambda^2 & -30\lambda^3 & -35\lambda^4 & -19\lambda^5 & -4\lambda^6 \\ 10\lambda^3 & 35\lambda^4 & 46\lambda^5 & 27\lambda^6 & 6\lambda^7 \\ -5\lambda^4 & -19\lambda^5 & -27\lambda^6 & -17\lambda^7 & -4\lambda^8 \\ \lambda^5 & 4\lambda^6 & 6\lambda^7 & 4\lambda^8 & \lambda^9 \end{pmatrix}$$

and

$$\begin{pmatrix} c'_0 \\ c'_1 \\ c'_2 \\ c'_3 \\ c'_4 \end{pmatrix} = \begin{pmatrix} 5\lambda & 10\lambda^2 & 10\lambda^3 & 5\lambda^4 & \lambda^5 \\ -10\lambda^2 & -30\lambda^3 & -35\lambda^4 & -19\lambda^5 & -4\lambda^6 \\ 10\lambda^3 & 35\lambda^4 & 46\lambda^5 & 27\lambda^6 & 6\lambda^7 \\ -5\lambda^4 & -19\lambda^5 & -27\lambda^6 & -17\lambda^7 & -4\lambda^8 \\ \lambda^5 & 4\lambda^6 & 6\lambda^7 & 4\lambda^8 & \lambda^9 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

The last step is to invert $\bar{g}_n(s)$ to obtain

$$g_n(t) = \mathscr{L}^{-1}\left\{\bar{g}_n(s)\right\} = \sum_{k=0}^{n} \frac{c'_k}{k!} t^k e^{-\lambda t}.$$

Since the above transformation from the b_k to the c'_k is a chain of one-to-one mappings, it is obvious that $g_n(t)$ matches its moments to those of $f(t)$ up to the n-th order. $\square$

It follows that for any fixed $n \in \mathbb{N}$, there is a family of functions $g_n(t;\lambda)$ in $P_n^\lambda(\Omega)$ with a parameter $\lambda > 0$ matching the moments of $f(t)$ up to the n-th order. It is obvious $g_n(t;\lambda)$ is continuous, bounded, integral-convergent, and vanishing at infinity on Ω. In addition, $g_n(t)$ has initial value

$$\lim_{t\to 0^+} g_n(t) = \lim_{s\to\infty} s\bar{g}_n(s) = c_0 = c'_0, \tag{4.23}$$

and final value

$$\lim_{t\to\infty} g_n(t) = \lim_{s\to 0} s\bar{g}_n(s) = 0,$$

i.e. it vanishes at infinity.

Theorem 4.26. *(Uniform convergence theorem for method* II*). In the above problem, for every $\lambda > 0$, the approximation function $g_n(t;\lambda)$ converges to $f(t)$ uniformly on Ω.*

Proof. By hypothesis, both

$$\bar{f}(s) = \sum_{k=0}^{\infty} b_k s^k \quad \text{and} \quad \bar{g}_n(s) = \sum_{k=0}^{\infty} b'_k s^k$$

converge. Assume the radius of convergence for $\bar{f}(s)$ is $R > 0$. Then there exists an $r = \min(R, \lambda) > 0$ such that the two series converges absolutely for any $s \in N(0, r)$. Thus, for any $\epsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that when $n > N_1$ we have

$$\left| \sum_{k=n+1}^{\infty} b_k s^k \right| \le \sum_{k=n+1}^{\infty} \left| b_k s^k \right| < \frac{\epsilon}{2},$$

or when $n > N_2$ we have

$$\left| \sum_{k=n+1}^{\infty} b'_k s^k \right| \le \sum_{k=n+1}^{\infty} \left| b'_k s^k \right| < \frac{\epsilon}{2},$$

for any $s \in N(0, r)$, respectively.

Let

$$N = \max(N_1, N_2).$$

Then, considering the moment matching conditions

$$b'_k = b_k, \quad k = 0, 1, \cdots, n,$$

when $n > N$, we have

$$
\begin{aligned}
\left|\bar{g}_n(s) - \bar{f}(s)\right| &= \left|\bar{g}_n(s) - \sum_{k=0}^{n} b'_k s^k + \sum_{k=0}^{n} b_k s^k - \bar{f}(s)\right| \\
&\leq \left|\bar{g}_n(s) - \sum_{k=0}^{n} b'_k s^k\right| + \left|\sum_{k=0}^{n} b_k s^k - \bar{f}(s)\right| \\
&= \left|\sum_{k=n+1}^{\infty} b_k s^k\right| + \left|\sum_{k=n+1}^{\infty} b'_k s^k\right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

for any $s \in N(0, r)$.

It follows immediately that there exists a $\delta > 0$ such that

$$\|g_n(t) - f(t)\| < \delta$$

on Ω, since the Laplace transform is a linear operator and linear operators on vector spaces preserve continuity. Thus, $g_n(t)$ converges to $f(t)$ on Ω uniformly. $\square$

Let n be fixed and λ vary. Then we have a family of approximation functions $g_n(t;\lambda)$ matching the moments of $f(t)$. Next, we shall find the final approximation function such that it is sufficiently "close" to $f(t)$. This is done by imposing an additional condition on $g_n(t;\lambda)$ to find the optimal value of λ.

One method is to match the function value of $g_n(t;\lambda)$ at the origin to that of $f(t)$, i.e.

$$g_n(0;\lambda) = f(0).$$

By Equation (4.23), we have

$$c'_0(\lambda) = f(0).$$

This is a polynomial equation of degree $(n+1)$. Assume the equation has positive real roots. Then we shall choose the one, for which the corresponding WMSE on $[0, T]$ is minimum.

Note that in the above statements, it is generally difficult to show there is at least one positive real root for the initial value equation. A necessary condition for the existence of the positive roots is that the b_k have alternating or random signs. We shall leave this for future research. In addition, we shall investigate other additional conditions for calculating the optimal λ.

For every $\lambda > 0$ and n fixed, define the approximation error function as

$$e_n(t;\lambda) = f(t) - g_n(t;\lambda).$$

Example 4.27. *Moment matching approximations.*

Let $n = 10$, $T = 8$, and $[0, 8]$ the main interval. Consider a target probability density function

$$f(t) = \frac{t}{3^2} e^{-\frac{t^2}{2\times 3^2}}$$

with Laplace transform

$$\begin{aligned}
\bar{f}(s) &= 1 - 3.7599 s e^{4.5 s^2} \frac{2}{\sqrt{\pi}} \int_{2.1213s}^{\infty} e^{-x^2} dx \\
&= 1.0000 - 3.7599s + 9.0000s^2 - 16.9197s^3 \\
&\quad + 27.0000s^4 + 38.0694s^5 + 48.6000s^6 - 57.1041s^7 \\
&\quad + 62.4857s^8 - 64.2421s^9 + 62.4857s^{10} + \cdots \\
&= \sum_{k=0}^{10} b_k s^k + \sum_{k=11}^{\infty} b_k s^k.
\end{aligned}$$

By Theorem 4.25, we have a family of functions in a parameter λ

$$g_{10}(t; \lambda) = \sum_{k=0}^{10} \frac{c'_k}{k!} t^k e^{-\lambda t},$$

where the c'_k, $k = 0, 1, \cdots, 10$, are transformed from the b_k, $k = 0, 1, \cdots, 10$, and the c_k are functions of λ.

By matching initial values, we have the equation

$$g_{10}(0; \lambda) = f(0),$$

or

$$c'_0(\lambda) = 0.$$

This is a polynomial equation in λ of degree $n + 1 = 11$. Table 6 lists all the roots of this equation and the corresponding WMSE's on $[0, 8]$.

The root $\lambda = 0.9398$ is chosen as the optimal value as the corresponding WMSE on $[0, 8]$ is 4.0459×10^{-6} (minimum). Substituting $\lambda = 0.9398$ into $g_{10}(t; \lambda)$, we have the final approximation function

$$\begin{aligned}
g_{10}(t) &= (1.3894 \times 10^{-1} t - 3.6275 \times 10^{-2} t^2 + 2.5367 \times 10^{-1} t^3 \\
&\quad - 1.2748 \times 10^{-1} t^4 + 4.5828 \times 10^{-2} t^5 - 7.7082 \times 10^{-3} t^6 \\
&\quad + 6.4889 \times 10^{-4} t^7 - 2.8681 \times 10^{-5} t^8 + 6.3626 \times 10^{-7} t^9 \\
&\quad - 5.5724 \times 10^{-9} t^{10}) e^{-0.9398t}.
\end{aligned}$$

The corresponding WMAE on $[0, 8]$ is 1.7865×10^{-3}.

TABLE 6. The eleven positive roots by matching the initial values and the corresponding WMSE's on $[0, 8]$.

i	λ_i	$\mathrm{WMSE}_{[0,8]}$
1	0	1.4732×10^{-1}
2	0.1257	2.4966×10^{-3}
3	0.3002	1.4880×10^{-4}
4	0.4985	1.9534×10^{-5}
5	0.7126	5.9410×10^{-6}
6	0.9398	4.0459×10^{-6}
7	1.1802	6.2194×10^{-6}
8	1.4358	2.2665×10^{-5}
9	1.7117	2.1926×10^{-4}
10	2.0187	7.3120×10^{-3}
11	2.3861	1.6828

We also obtain the integration function of $g_{10}(t)$ as

$$
\begin{aligned}
G_{10}(t) \quad = \quad & 1.0000 - (1.0000 + 9.3981 \times 10^{-1}t + 3.7215 \times 10^{-1}t^2 \\
& + 1.2867 \times 10^{-1}t^3 - 3.3184 \times 10^{-2}t^4 + 1.9259 \times 10^{-2}t^5 \\
& - 4.6213 \times 10^{-3}t^6 + 4.8071 \times 10^{-4}t^7 - 2.4638 \times 10^{-5}t^8 \\
& + 6.1392 \times 10^{-7}t^9 - 5.9293 \times 10^{-9}t^{10})e^{-0.9398t}.
\end{aligned}
$$

Figure 14 shows the approximation results on $[0, 16]$, which is twice the length of the main interval $[0, 8]$. The left figure shows the overall graphs of the approximation function $g_{10}(t)$ (blue line) and the target function $f(t)$ (red line). The right figure shows the overall graphs of $G_{10}(t)$ (blue line) and $F(t)$ (red line), the integration functions of $g_{10}(t)$ and $f(t)$, respectively. In each figure, the graphs of the target function and the approximation function almost coincide with each other and are indistinguishable. In addition, $g_{10}(t)$ is nicely bounded on $[8, \infty)$ and vanishes at infinity.

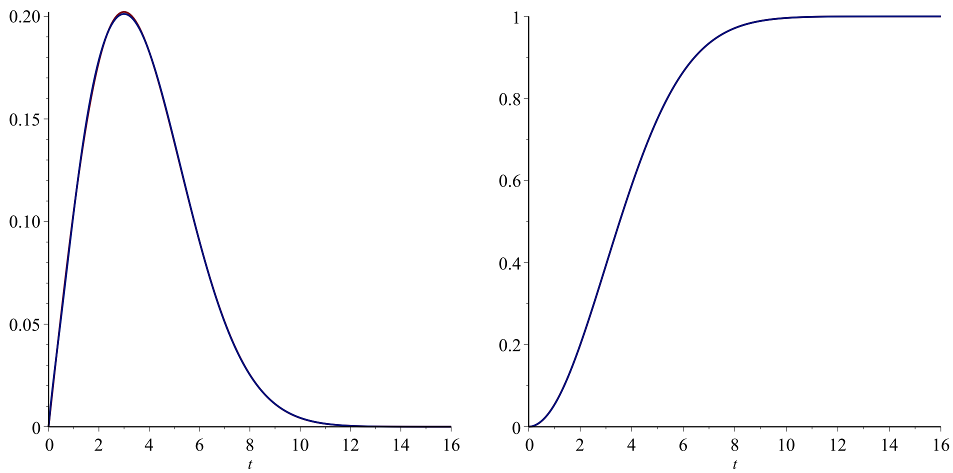

FIGURE 14. Method II overall approximations on $[0, 16]$ ($n = 10$): Left – the overall graphs of the approximation function $g_{10}(t)$ (blue line) and the target function $f(t)$ (red line); Right – the overall graphs of $G_{10}(t)$ (blue line) and $F(t)$ (red line), the integration functions of $g_{10}(t)$ and $f(t)$, respectively.

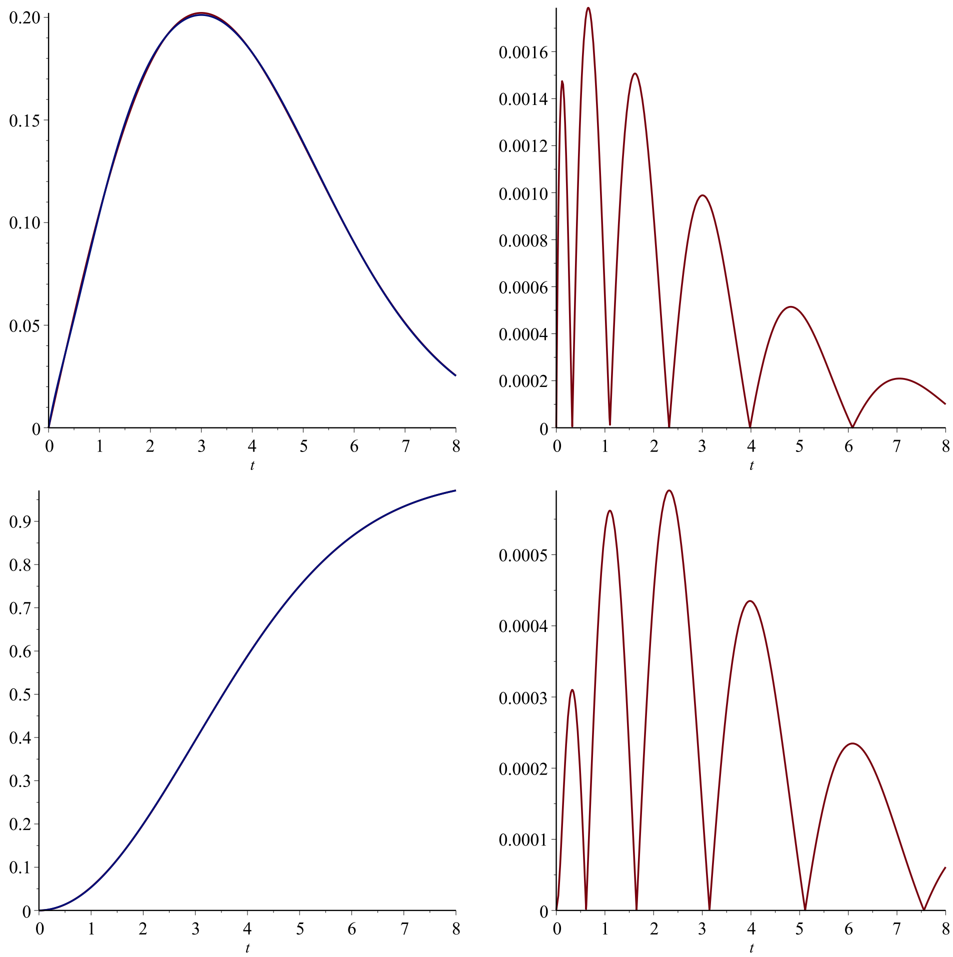

FIGURE 15. Method II approximations on the main interval $[0, 8]$ $(n = 10)$: Top-left – $g_{10}(t)$ (blue line) and $f(t)$ (red line) almost coincide with each other; Top-right – the absolute error function $|e_{10}(t)|$ of the approximation error function $e_{10}(t) = f(t) - g_{10}(t)$; Bottom-left – $G_{10}(t)$ (blue line) and $F(t)$ (red line) almost coincide with each other; Bottom-right – the absolute error function $|F(t) - G_{10}(t)|$.

Figure 15 shows the approximation results on the main interval $[0, 8]$. The top-left figure shows $g_{10}(t)$ (blue line) and $f(t)$ (red line) almost coincide with each other on $[0, 8]$. The top-right figure is the absolute error function $|e_{10}(t)|$ of the approximation error function $e_{10}(t) = f(t) - g_{10}(t)$. $e_{10}(t)$ oscillates between positive and negative values on $[0, 8]$ with not significantly different peak magnitudes. The absolute error function $|e_{10}(t)|$ has a comb shape with the first few peaks having large magnitudes and the rest peaks with declining magnitudes thereafter as t increases inside $[0, 8]$. The relatively flat peak magnitudes and the oscillating patterns of $e_{10}(t)$ on $[0, 8]$ are the characteristics of the Chebyshev type error functions, which are distinctive from those of the asymptotic type ones. The bottom-left figure shows $G_{10}(t)$ (blue line) and $F(t)$ (red line) almost coincide with each other on $[0, 8]$. The bottom-right figure shows the absolute error function $|F(t) - G_{10}(t)|$ on $[0, 8]$, which also has a comb shape.

In order to demonstrate the uniform convergence on the main interval for the approximation method II, we repeat the above approximation experiment with $n = 15$, 20, 25, 30, 35, and 40, and record the corresponding optimal λ values, WMAE's and WMSE's on $[0, 8]$ in Table 7. Figure 16 shows the graphs of the absolute error functions $|e_n(t)|$ for $n = 15$, 20, 25, 30, 35, and 40. It can be seen in Figure 16 that each approximation error function of method II is oscillating with relatively flat peak magnitudes on the main interval, which is the Chebyshev type error.

TABLE 7. Method II multiple approximation experiments: the optimal λ values, WMAE's and WMSE's on $[0, 8]$ for $n = 10$, 15, 20, 25, 30, 35, and 40.

n	λ	$\text{WMAE}_{[0,8]}$	$\text{WMSE}_{[0,8]}$	$\frac{n}{\lambda}$
10	0.9398	1.7865×10^{-3}	4.0459×10^{-6}	10.6
15	1.0645	2.7988×10^{-4}	8.5769×10^{-8}	14.1
20	1.1786	4.4709×10^{-5}	2.0241×10^{-9}	17.0
25	1.2835	7.5856×10^{-6}	5.0732×10^{-11}	19.5
30	1.3809	1.2868×10^{-6}	1.3234×10^{-12}	21.7
35	1.4721	2.0804×10^{-7}	3.5492×10^{-14}	23.8
40	1.5582	3.6243×10^{-8}	9.7155×10^{-16}	25.7

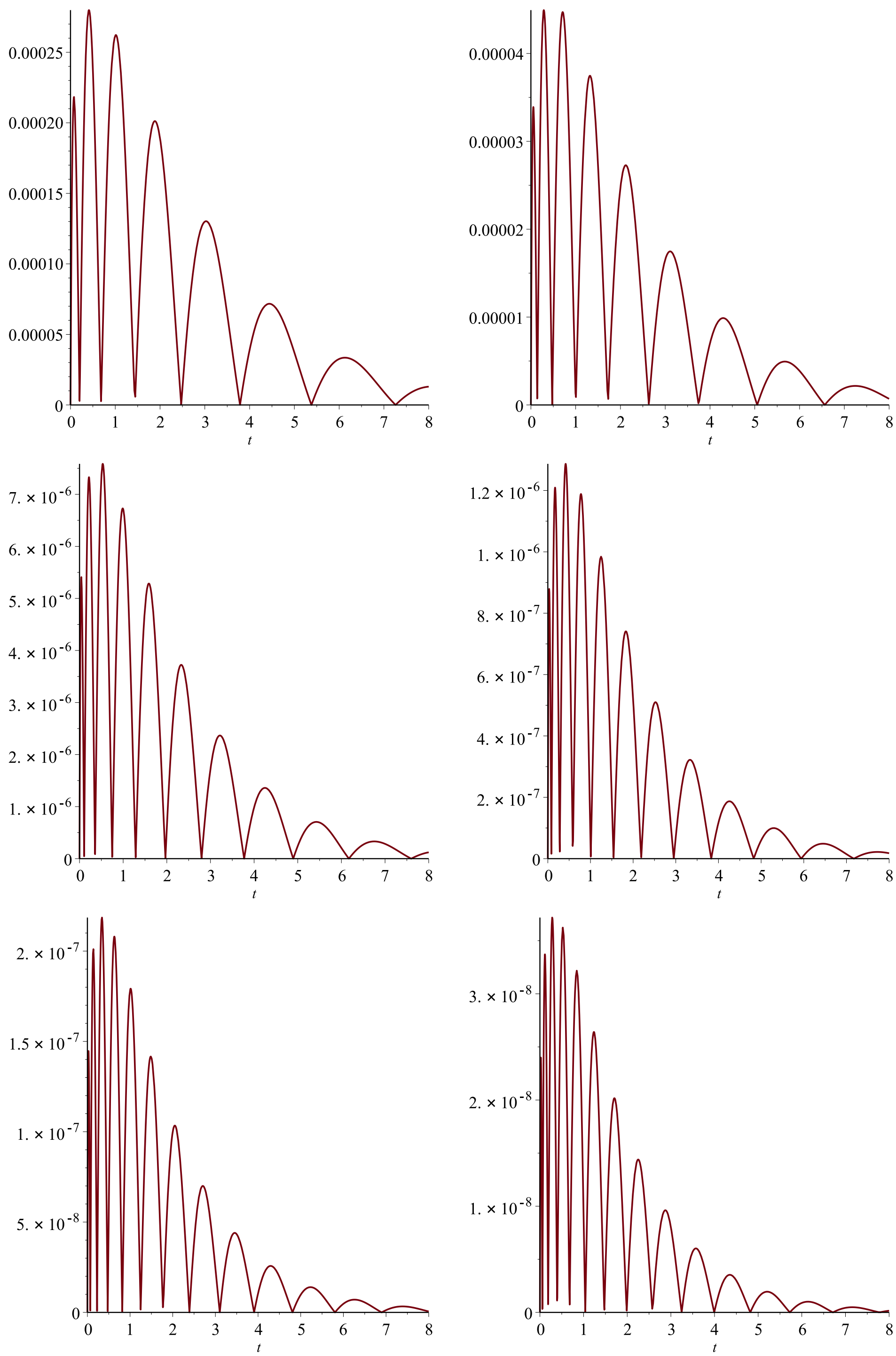

FIGURE 16. Method II multiple approximation experiments: the absolute error functions $|e_n(t)|$ on $[0, 8]$ for $n = 15$ and 20 (top), $n = 25$ and 30 (middle), and $n = 35$ and 40 (bottom).

4.3.1. *Using approximate moments.* The essence of the moment matching method is the power series expansion of the target function's Laplace transform. Whether or not the Laplace transform itself exists as a closed form function is not important. However, this condition may not be satisfied in some situations, such as

(1) the Laplace transform does not exist and some or all of its moments cannot be calculated;
(2) the Laplace transform exists but is difficult to expand as a power series;
(3) the Laplace transform exists but some moments cannot be calculated because the corresponding moment integral diverges; or
(4) the Laplace transform exists but its power series has only one point of convergence and diverges everywhere else.

Even under all the above undesirable situations, the moment matching method can still be applied. Since each moment of the target function is just an improper integral, its value can be numerically approximated and calculated. This is to say an improper integral can be approximated by a definite integral on some closed interval containing the main interval $[0, T]$, which can be calculated as a Riemann sum.

A *partition* of $[0, T]$ is a collection of closed sub-intervals $[t_i, t_{i+1}]$, $i = 0, 1, \cdots, m-1$, such that:

(1) $0 = t_0 < t_1 < t_2 < \cdots < t_m = T$; and
(2) the union $\cup_{i=0}^{m-1}[t_i, t_{i+1}]$ equals $[0, T]$.

A *step function* $f^*(t)$ of a target function $f(t)$ on Ω is a piecewise function whose value is constant on each sub-interval of $[0, T]$ and zero on $[T, \infty)$, i.e.

$$f^*(t) = \begin{cases} f(t_i^*) & , t_i^* \in [t_i, t_{i+1}), \quad i = 0, 1, \cdots, m-1, \\ 0 & , \text{otherwise.} \end{cases}$$

Then the k-th approximate moment of $f(t)$ is

$$M_k = \int_0^\infty t^k f^*(t) dt = \sum_{i=0}^{m-1} \frac{f(t_i^*)}{k+1} \left(t_{i+1}^{k+1} - t_i^{k+1} \right), \quad k = 0, 1, \cdots .$$

This numerical approximate moment calculation method applies to every target function in $C_0^B(\Omega)$, which may or may not have a Laplace transform, regardless of its integrability.

4.3.2. *The Padé rational approximation method.* Our moment matching approximation method belongs to a special class of rational Laplace transform moment matching approximation methods. Rational or meromorphic Laplace transforms are important complex functions with many nice and useful properties. For example, convolutions can be performed in rational Laplace transforms as simple algebraic operations, and inverse rational Laplace transforms can be efficiently done by the method of partial fractions. There are many ideas about approximating a Laplace transform by a rational complex function, and the Padé method is one of the most commonly used.

In the Padé method, we can write the Laplace transform of the approximation function as

$$\bar{g}(s) = \frac{\sum_{k=0}^{m} b_k s^k}{1 + \sum_{k=1}^{n} a_k s^k} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{1 + a_1 s + a_2 s^2 + \cdots + b_n s^n},$$

where the b_k and a_k are undetermined coefficients and can be solved for by equating up to the $(m+n+1)$-st moments of $\bar{g}(s)$ to those of $\bar{f}(s)$, the Laplace transform of the target function $f(t)$. In general, m and n can be chosen arbitrarily. But for a target probability density function, or any function that vanishes at infinity, it must be true that $m < n$. By inverting $\bar{g}(s)$, we obtain $g(t)$ which approximates $f(t)$.

In our numerical experiments using the Padé method, we found the method may give a very poor approximation function $g(t)$ in some occasions but fails most of the time. Even in the rough sense, the approximation does not improve as the number of matching moments increases, let alone uniformly converge. For the same degree n of the denominators for the approximation Laplace transforms, our moment matching method needs to solve a system of $n+1$ equations, while the Padé method needs to solve a system of $2n$ equations, nearly doubled. While the system of equations in our method will always result in solutions, the system of equations in the Padé method may be ill-conditioned and often results in extremely large or small numbers in the solutions. Thus, numerical algorithms for calculating the Padé approximation function need almost infinite precisions. Finally, the Padé method gives a unique approximation function while our method gives a family of approximation functions with a control parameter and various means to impose additional conditions.

In conclusion, the Padé approximation function is poor and is neither uniformly convergent nor convergent. The Padé method is uncontrollable and numerically inefficient.

4.4. **Method III: interpolations in $P_n^\lambda(\Omega)$.** The approximation method I (asymptotic series expansion) or the method II (moment matching) requires either higher order derivatives or higher order integrals of a target function, which may be difficult or impossible to obtain. In this section, we shall develop the approximation method III, the method of interpolation in $P_n^\lambda(\Omega)$ spaces, which is inspired by polynomial interpolations and is generally effective for fitting any continuous (or discontinuous) function curves.

Interpolation is an old technique for function approximations. The basic idea is to construct a continuous function from a set of *data points* or *interpolation nodes* of a target function on an *interpolation interval* containing the interpolation nodes. One of the simplest interpolation methods is the polynomial interpolation, whose error function is bounded and oscillating between positive and negative values on the interpolation interval.

The polynomial interpolation approximation method suffers some major drawbacks for our approximation problem. Firstly, the method is only valid inside the interpolation interval. The *extrapolation* polynomial, the same expression as the interpolation polynomial but defined outside the interpolation interval, is generally not a good approximation to a target function that is bounded on Ω. In other words, an interpolation polynomial is unbounded at infinity and cannot be used to approximate functions in $C_0^B(\Omega)$. Moreover, an interpolation approximation may even be poor inside the interpolation interval for some target functions due to the Runge's phenomenon.

In this section, we shall develop a new interpolation method, interpolation in $P_n^\lambda(\Omega)$, described as follows. Let $y = f(t) \in C_0^B(\Omega)$ be a target function. Consider a set of $n+1$ distinct evaluating points $0 = t_0 < t_1 < \cdots < t_n = T$ on the main interval (or the interpolation interval) $[0, T]$. Let $y_i = f(t_i)$, $i = 0, 1, \cdots, n$. We wish to find a function $g(t)$ passing through all the interpolation nodes $\{(t_i, y_i)\}_{i=0}^n$, which is "close" to $f(t)$ on $[0, T]$ and is nicely bounded on $[T, \infty)$.

Theorem 4.28. *(Interpolation existence theorem). In the above problem, for every $\lambda > 0$, there exists a function $g_n(t; \lambda) \in P_n^\lambda(\Omega)$ passing through all the interpolation nodes $\{(t_i, y_i)\}_{i=0}^n$.*

Proof. (Constructive proof). For every $\lambda > 0$, consider a natural basis of $P_n^\lambda(\Omega)$ $\{\varphi_i(t) = t^i e^{-\lambda t}\}_{i=0}^n$. Let $g_n(t) \in P_n^\lambda(\Omega)$ be expressed as

$$g_n(t) = c_0\varphi_0(t) + c_1\varphi_1(t) + \cdots + c_n\varphi_n(t),$$

for some c_i, $i = 0, 1, \cdots, n$. It is left to prove there exist the c_i such that

$$g_n(t_i) = y_i, \quad i = 0, 1, \cdots, n,$$

in other words, $g_n(t)$ passes through the interpolation nodes $\{(t_i, y_i)\}_{i=0}^n$.

Using the interpolation nodes, a system of equations is set up as

$$\begin{pmatrix} \varphi_0(t_0) & \varphi_1(t_0) & \cdots & \varphi_n(t_0) \\ \varphi_0(t_1) & \varphi_1(t_1) & \cdots & \varphi_n(t_1) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_0(t_n) & \varphi_1(t_n) & \cdots & \varphi_n(t_n) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Since all φ_i are real-valued functions, the above coefficient matrix is a real-entry matrix of size $(n+1)\times(n+1)$. In fact, this is a modified Vandermonde matrix. To prove it, rewrite the above equations as

$$\begin{pmatrix} e^{-\lambda t_0} & t_0 e^{-\lambda t_0} & \cdots & t_0^n e^{-\lambda t_0} \\ e^{-\lambda t_1} & t_1 e^{-\lambda t_1} & \cdots & t_1^n e^{-\lambda t_1} \\ \vdots & \vdots & \vdots & \vdots \\ e^{-\lambda t_n} & t_n e^{-\lambda t_n} & \cdots & t_n^n e^{-\lambda t_n} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Since $e^{-\lambda t_i} > 0,\ i = 0, 1, \cdots, n$, we divide each row by $e^{-\lambda t_i}$ and get

$$\begin{pmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^n \\ 1 & t_1 & t_1^2 & \cdots & t_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 e^{\lambda t_0} \\ y_1 e^{\lambda t_1} \\ \vdots \\ y_n e^{\lambda t_n} \end{pmatrix}.$$

The matrix on the left hand side is a Vandermonde matrix, whose determinant can be calculated as

$$\prod_{i,j=0, i\le j}^{n} (t_i - t_j).$$

Since all t_i are pairwise distinct, i.e. $t_i \neq t_j$ for $i \neq j$, the determinant is non-zero and the Vandermonde matrix is invertible. By Cramer's rule, the coefficients are

$$c_0 = \frac{\begin{vmatrix} y_0 e^{\lambda t_0} & t_0 & \cdots & t_0^n \\ y_1 e^{\lambda t_1} & t_1 & \cdots & t_1^n \\ \vdots & \vdots & \vdots & \vdots \\ y_n e^{\lambda t_n} & t_n & \cdots & t_n^n \end{vmatrix}}{\begin{vmatrix} 1 & t_0 & \cdots & t_0^n \\ 1 & t_1 & \cdots & t_1^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & \cdots & t_n^n \end{vmatrix}}, \qquad \cdots, \qquad c_n = \frac{\begin{vmatrix} 1 & t_0 & \cdots & y_0 e^{\lambda t_0} \\ 1 & t_1 & \cdots & y_1 e^{\lambda t_1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & \cdots & y_n e^{\lambda t_n} \end{vmatrix}}{\begin{vmatrix} 1 & t_0 & \cdots & t_0^n \\ 1 & t_1 & \cdots & t_1^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & \cdots & t_n^n \end{vmatrix}}.$$

Note the c_i are functions of λ.

Thus there exists a $g_n(t) \in P_n^\lambda(\Omega)$ passing through all the interpolation nodes. $\square$

For every $\lambda > 0$, write $g_n(t)$ in the above proof as

$$\begin{aligned} g_n(t;\lambda) &= (c_0 + c_1 t + \cdots + c_n t^n)\, e^{-\lambda t} \\ &= p_n(t) e^{-\lambda t}, \end{aligned}$$

where

$$p_n(t) = c_0 + c_1 t + \cdots + c_n t^n$$

is a polynomial in t of degree up to n with a parameter λ. Let

$$h(t) = f(t) e^{\lambda t}.$$

Then $p_n(t)$ may be viewed as a polynomial interpolation for $h(t)$ with the interpolation nodes $\{(t_i, y_i e^{\lambda t_i})\}_{i=0}^n$ on the interpolation interval. Notice the interpolation coefficients for $p_n(t)$ are the same as those for $g_n(t)$ and the two different interpolation problems are essentially equivalent. Clearly, this is due to the isomorphism between $P_n(\Omega)$ and $P_n^\lambda(\Omega)$.

Theorem 4.29. *(Interpolation uniqueness theorem). In the above two interpolation problems, for every $\lambda > 0$, the interpolation functions $p_n(t;\lambda)$ and $g_n(t;\lambda)$ are unique.*

Proof. By contradiction. Assume $q_n(t;\lambda) \neq p_n(t;\lambda)$ is another polynomial of degree n which interpolates $h(t)$ at t_i, $i = 0, 1, \cdots, n$, in $P_n(\Omega)$. Then the error function $r_n(t;\lambda) = p_n(t;\lambda) - q_n(t;\lambda)$ has at least $n+1$ zeros on $\mathbb{R}$. This is contradictory since $r_n(t;\lambda)$ is at most of degree n. Thus, $q_n(t;\lambda) = p_n(t;\lambda)$ and $p_n(t;\lambda)$ is unique. It follows that $g_n(t;\lambda)$ is also unique. $\square$

Practically, one can directly write $p_n(t;\lambda)$ by the Lagrange interpolation formula for polynomials, i.e.

$$\begin{aligned} p_n(t;\lambda) &= \sum_{i=0}^n y_i e^{\lambda t_i} \prod_{j=0, j\neq i}^n \frac{t - t_j}{t_i - t_j} \\ &= \sum_{i=0}^n y_i e^{\lambda t_i} l_i(t), \end{aligned}$$

where

$$l_i(t) = \prod_{j=0, j\neq i}^n \frac{t - t_j}{t_i - t_j}, \quad i = 0, 1, \cdots, n,$$

are the usual Lagrange interpolation polynomial terms generated by the interpolation nodes. Thus,

$$\begin{aligned} g_n(t;\lambda) &= \sum_{i=0}^{n} y_i e^{\lambda t_i} l_i(t) e^{-\lambda t} \\ &= \sum_{i=0}^{n} y_i \left[e^{-\lambda(t-t_i)} \prod_{j=0, j\neq i}^{n} \frac{t-t_j}{t_i-t_j} \right] \\ &= \sum_{i=0}^{n} y_i l_{i,\lambda}(t), \end{aligned} \tag{4.24}$$

where

$$l_{i,\lambda}(t) = e^{-\lambda(t-t_i)} \prod_{j=0, j\neq i}^{n} \frac{t-t_j}{t_i-t_j}, \quad i = 0, 1 \cdots, n,$$

are the *Lagrange interpolation terms in* $P_n^{\lambda}(\Omega)$.

In order to analyze the error and convergence of the interpolation function on the interpolation interval, we write $g_n(t;\lambda)$ in the Newton's (the Stirling's, or the Bessel's) central divided difference form.

Let $\lambda > 0$ and $[0, T]$ be the interpolation interval with equally spaced interpolation nodes. We shall develop the Newton's central divided difference formula for $h(t)$ about $t_0 = \frac{T}{2}$.

Assume $f(t)$ is analytic on Ω, and its largest singularity is $t = -R$, $R > 0$. Then $h(t) = f(t)e^{\lambda t}$ is analytic about t_0 on the convergence interval $(-R, T + R) \supset [0, T]$. Expand $h(t)$ into a Taylor series about t_0 as

$$\begin{aligned} h(t) &= \sum_{k=0}^{n} \frac{h^{(k)}(t_0)}{k!} (t-t_0)^k + \sum_{k=n+1}^{\infty} \frac{h^{(k)}(t_0)}{k!} (t-t_0)^k \\ &= T_n(t) + R_n(t), \end{aligned}$$

for some $n \in \mathbb{N}$.

Without loss of generality, let n be even and $\delta = \frac{T}{n}$. Denote the interpolation nodes as $\{(t_i, h_i)\}$ for $i = -\frac{n}{2}, -\frac{n-2}{2}, \cdots, +\frac{n-2}{2}, +\frac{n}{2}$, where

$$t_i = t_0 + i\delta$$

and

$$h_i = h(t_i).$$

Define the first order central differences as

$$\Delta h_i = h_{i+1} - h_i$$

and the second order ones as

$$\Delta^2 h_i = \Delta h_i - \Delta h_{i-1}$$

for all possible index values. These slightly different definitions for the first and the second order central differences are purely technical. To define the k-th order central differences for $h(t)$ for $0 \le k \le n$, we use the following formula:

$$\Delta^k h_i = \begin{cases} \Delta^{k-1} h_{i+1} - \Delta^{k-1} h_i & \text{, if } k \text{ is odd,} \\ \\ \Delta^{k-1} h_i - \Delta^{k-1} h_{i-1} & \text{, if } k \text{ is even,} \end{cases}$$

where $\Delta^0 h_i = h_i$ and $\Delta^1 h_i = \Delta h_i$. This process stops when we lastly obtain $\Delta^n h_0$.

Thus, we obtain a set of central differences $\Delta^k h_0$, which approximate the derivatives of $h(t)$ at t_0, i.e.

$$\Delta^k h_0 \simeq h^{(k)}(t_0)$$

for $k = 0, 1, \cdots, n$. Table 8 is an example of a central difference table for $h(t)$ up to the sixth order. As n tends to infinity, δ tends to zero and the k-th order central difference tends to the k-th order derivative of $h(t)$ at t_0, i.e.

$$\lim_{n \to \infty} \Delta^k h_0 = h^{(k)}(t_0)$$

for $k = 0, 1, \cdots$, by the definition of derivatives.

The interpolation polynomial for $h(t)$ in the Newton's form can be written as

$$p_n(t) = \sum_{k=0}^{n} \frac{1}{k!} \Delta^k h_0 \, (t - t_0)^k$$

on $(-R, T+R)$.

Theorem 4.30. *(Interpolation uniform convergence theorem). In the above interpolation problems, for every $\lambda > 0$, $p_n(t)$ and $g_n(t)$ converge to $h(t)$ and $f(t)$ uniformly on $[0, T]$, respectively.*

Proof. Consider $h(t) = T_n(t) + R_n(t)$, $n \in \mathbb{N}$. Then, for every $\epsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that whenever $n > N_1$, we have

$$|h(t) - T_n(t)| = |R_n(t)| < \frac{\epsilon}{2}.$$

Consider the interpolation polynomial for $h(t)$ in the Newton's form as

$$p_n(t) = \sum_{k=0}^{n} \frac{1}{k!} \Delta^k h_0 \, (t - t_0)^k$$

on $(-R, T+R)$. For $m \in \mathbb{N}$ sufficiently large, we have $\delta = \frac{T}{m}$ sufficiently small, and there exists an $N_2 \in \mathbb{N}$ such that whenever $N_2 < n < m$, we have

$$|\Delta^k h_0 - h^{(k)}(t_0)| < \frac{\epsilon}{2} e^{-R-T/2}$$

Table 8. Newton's central difference table of $h(t)$

t_i	h_i	$\Delta^1 h_i$	$\Delta^2 h_i$	$\Delta^3 h_i$	$\Delta^4 h_i$	$\Delta^5 h_i$	$\Delta^6 h_i$
t_{-3}	h_{-3}						
		Δh_{-3}					
t_{-2}	h_{-2}		$\Delta^2 h_{-2}$				
		Δh_{-2}		$\Delta^3 h_{-2}$			
t_{-1}	h_{-1}		$\Delta^2 h_{-1}$		$\Delta^4 h_{-1}$		
		Δh_{-1}		$\Delta^3 h_{-1}$		$\Delta^5 h_{-1}$	
t_0	$\boxed{h_0}$		$\boxed{\Delta^2 h_0}$		$\boxed{\Delta^4 h_0}$		$\boxed{\Delta^6 h_0}$
		$\boxed{\Delta h_0}$		$\boxed{\Delta^3 h_0}$		$\boxed{\Delta^5 h_0}$	
t_1	h_1		$\Delta^2 h_1$		$\Delta^4 h_1$		
		Δh_1		$\Delta^3 h_1$			
t_2	h_2		$\Delta^2 h_2$				
		Δh_2					
t_3	h_3						

for $k = 0, 1, \cdots, n$. Then

$$\begin{aligned} |p_n(t) - T_n(t)| &= \left| \sum_{k=0}^{n} \frac{1}{k!} \left(\Delta^k h_0 - h^{(k)}(t_0) \right) (t - t_0)^k \right| \\ &\leq \sum_{k=0}^{n} \frac{1}{k!} \left| \left(\Delta^k h_0 - h^{(k)}(t_0) \right) \right| |t - t_0|^k \\ &< \frac{\epsilon}{2} e^{-R-T/2} \sum_{k=0}^{n} \frac{1}{k!} |t - t_0|^k \\ &< \frac{\epsilon}{2} e^{-R-T/2} \sum_{k=0}^{n} \frac{1}{k!} \left(R + \frac{T}{2} \right)^k \\ &< \frac{\epsilon}{2} e^{-R-T/2} e^{R+T/2} = \frac{\epsilon}{2}. \end{aligned}$$

Thus, when $n > \max(N_1, N_2)$, we have

$$\begin{aligned} |h(t) - p_n(t)| &= |T_n(t) - p_n(t) + R_n(t)| \\ &\leq |T_n(t) - p_n(t)| + |R_n(t)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for every $t \in (-R, T+R) \supset [0,T]$. This implies $p_n(t)$ converges to $h(t)$ uniformly on $[0,T]$.

It follows immediately that for any $t \in [0,T]$,

$$|f(t) - g_n(t)| = |h(t) - p_n(t)|\, e^{-\lambda t} < \epsilon e^{-\lambda t} < \epsilon.$$

This implies $g_n(t)$ converges to $f(t)$ uniformly on $[0,T]$. $\square$

Let $n \in \mathbb{N}$ be fixed and $\lambda > 0$ vary. Then $g_n(t;\lambda)$ represents a family of interpolation functions in $P_n^{\lambda}(\Omega)$ with a parameter λ passing through a set of $n+1$ interpolation nodes. In addition, for every $\lambda > 0$, $g_n(t;\lambda)$ converges to $f(t)$ uniformly on the interpolation interval. It is obvious $g_n(t;\lambda)$ is continuous, bounded, integral-convergent, and vanishing at infinity on Ω.

For any fixed n and λ, define the interpolation error function as

$$\begin{aligned} e_n(t) &= f(t) - g_n(t;\lambda) \\ &= r_n(t) e^{-\lambda t}, \end{aligned}$$

where

$$r_n(t) = h(t) - p_n(t)$$

on Ω.

Proposition 4.31. *$e_n(t)$ has at least $n+1$ zeros in $[0,T]$.*

Proof. Let $I = \{0, 1, \cdots, n\}$. For each $i, j \in I$, the Lagrange's interpolation term $l_{i,\lambda}(t_j)$ in $P_n^{\lambda}(\Omega)$ has the following properties:

$$l_{i,\lambda}(t_j) = \begin{cases} 1 & , i = j, \\ 0 & , \text{otherwise}. \end{cases}$$

Then

$$e_n(t_i) = f(t_i) - g_n(t_i) = 0, \quad i \in I.$$

This implies $e_n(t)$ has at least $n+1$ zeros in $[0,T]$. $\square$

The Runge's phenomenon can be avoided when using interpolations in $P_n^{\lambda}(\Omega)$. It is well-known that higher order interpolation polynomials may have divergent oscillating approximation errors around both end-points of the interpolation interval, i.e. the Runge's phenomenon. The reason for the Runge's phenomenon is the interpolation polynomial is approximating an analytic function outside its convergence interval. In the new interpolation method, since a target function in $C_0^B(\Omega)$ is assumed to have its largest singularity at $t = -R, R > 0$, the interpolation interval $[0,T]$ is contained inside the convergence interval. Thus, even when the interpolation nodes are arranged in equally spaced manner, the convergence is uniform and the Runge's phenomenon will not occur.

Next, we shall determine the optimal value of λ and obtain the final interpolation approximation function. We shall match the definite integrals as discussed earlier in method Ia in Section 4.2.1. Below is the detailed process.

Firstly, define the error of the two definite integrals as

$$\varepsilon(\lambda) = \int_0^T g_n(t;\lambda)dt - \int_0^T f(t)dt.$$

We wish to find a value of λ such that $\varepsilon^2(\lambda)$ is minimum. This is equivalent to solving the equation

$$\frac{d}{d\lambda}\varepsilon^2(\lambda) = \left(\int_0^T g_n(t;\lambda)dt - \int_0^T f(t)dt\right)\frac{d}{d\lambda}\left(\int_0^T g_n(t;\lambda)dt\right) = 0. \tag{4.25}$$

This is a transcendental equation and has only numerical solutions. If the equation has positive real roots, then we will find the optimal value of λ for which the corresponding WMSE on $[0,T]$ is minimum. Suppose we find the optimal value of λ. Substituting it in $g_n(t;\lambda)$, we obtain the final approximation function. Note we have not strictly proved the existence of solutions for the above equation for the general case, and will investigate this for future research.

Example 4.32. *Interpolations in $P_n^\lambda(\Omega)$ spaces.*

Consider again the finite-radius-convergent target function

$$f(t) = \frac{1}{1+t},$$

whose integration function is

$$F(t) = \int_0^t f(\tau)d\tau = \ln(1+t).$$

Let $T = 8$ and $[0,8]$ be the main interval as well as the interpolation interval. Let $n = 10$ and consider $n+1 = 11$ points equally spaced on $[0,8]$. Then we have a set of interpolation nodes $\{(0,1), (\frac{4}{5}, \frac{5}{9}), (\frac{8}{5}, \frac{5}{13}), \cdots, (8, \frac{1}{9})\}$.

By Theorem 4.28, we have a family of functions $g_{10}(t;\lambda)$ in Equation (4.24) with a parameter λ passing through the above interpolation nodes. Let the integration function of $g_{10}(t;\lambda)$ be

$$G_{10}(t;\lambda) = \int_0^t g_{10}(\tau;\lambda)d\tau.$$

Define the difference of definite integrals of $g_n(t;\lambda)$ and $f(t)$ on $[0,8]$ as

$$\varepsilon(\lambda) = G_{10}(8;\lambda) - \ln 9.$$

Solving Equation (4.25) numerically, we obtain the optimal $\lambda = 0.9650$ with the corresponding WMSE being 5.8927×10^{-9} on $[0,8]$. Substituting this value in $g_{10}(t;\lambda)$, we have the final approximation function

$$\begin{aligned} g_{10}(t) &= (1.0000 - 3.0694 \times 10^{-2}t + 0.4625t^2 \\ &\quad - 0.2245 \times 10^{-2}t^3 + 0.1464t^4 - 5.3939 \times 10^{-2}t^5 \\ &\quad + 1.5016 \times 10^{-2}t^6 - 2.7091 \times 10^{-3}t^7 + 3.2559 \times 10^{-4}t^8 \\ &\quad - 2.2606 \times 10^{-5}t^9 + 7.6288 \times 10^{-7}t^{10})e^{-0.9650t}. \end{aligned}$$

The corresponding WMAE is 1.4644×10^{-4} on $[0, 8]$.

In addition, we obtain $G_{10}(t)$, the integration function of $g_{10}(t)$, as

$$\begin{aligned} G_{10}(t) &= 2.8540 - (2.8540 + 1.7540t + 0.8617t^2 \\ &\quad + 0.1230t^3 + 8.5788 \times 10^{-2}t^4 - 1.2726 \times 10^{-2}t^5 \\ &\quad + 6.9430 \times 10^{-3}t^6 - 1.1880 \times 10^{-3}t^7 + 1.9534 \times 10^{-4}t^8 \\ &\quad - 1.5234 \times 10^{-5}t^9 + 7.9057 \times 10^{-7}t^{10})e^{-0.9650t}. \end{aligned}$$

Figure 17 shows the approximation results on $[0, 32]$, which is four times the length of the main interval $[0, 8]$. The left figure shows the overall graphs of the approximation function $g_{10}(t)$ (blue line) and the target function $f(t)$ (red line). $g_{10}(t)$ and $f(t)$ almost coincide with each other and are indistinguishable on $[0, 8]$. In addition, $g_{10}(t)$ is nicely bounded on $[8, \infty)$ and vanishes at infinity. The right figure shows the overall graphs of $G_{10}(t)$ (blue line) and $F(t)$ (red line), the integration functions of $g_{10}(t)$ and $f(t)$, respectively. $G_{10}(t)$ and $F(t)$ almost coincide with each other and are indistinguishable on $[0, 8]$. $G_{10}(t)$ has a horizontal asymptote $y = 2.8540$ but $F(t)$ does not because it is integral-divergent on Ω.

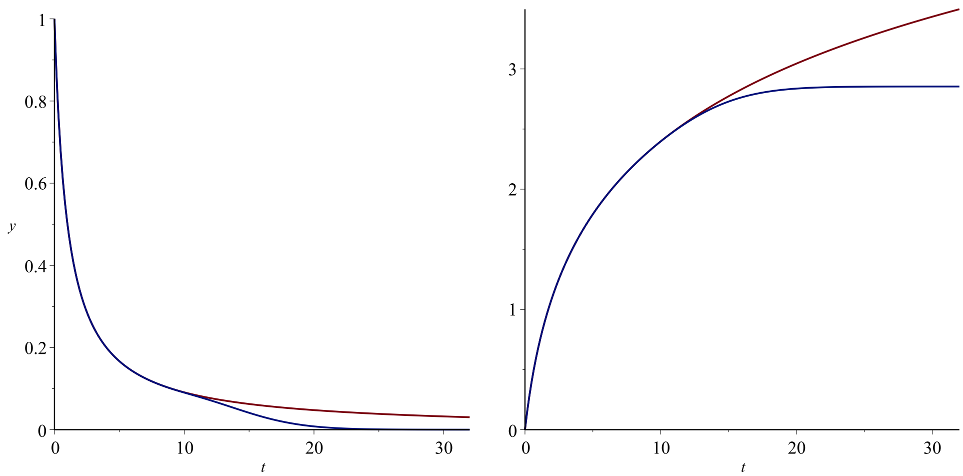

FIGURE 17. Method III overall approximations on $[0, 32]$ ($n = 10$): Left – the overall graphs of the approximation function $g_{10}(t)$ (blue line) and the target function $f(t)$ (red line); Right – the overall graphs of $G_{10}(t)$ (blue line) and $F(t)$ (red line), the integration functions of $g_{10}(t)$ and $f(t)$, respectively.

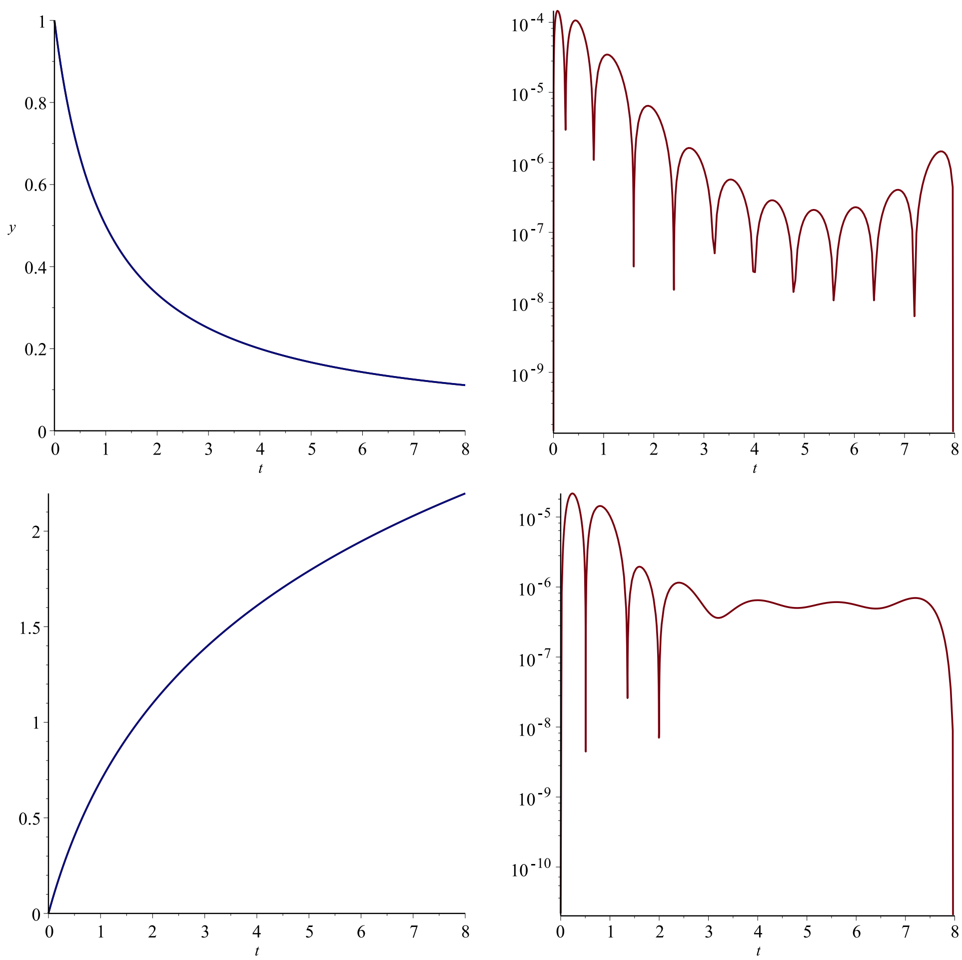

FIGURE 18. Method III approximations on the main interval $[0, 8]$ $(n = 10)$: Top-left – $g_{10}(t)$ (blue line) and $f(t)$ (red line) almost coincide with each other; Top-right – the absolute error function $|e_{10}(t)|$ of the approximation error function $e_{10}(t) = f(t) - g_{10}(t)$; Bottom-left – $G_{10}(t)$ (blue line) and $F(t)$ (red line) almost coincide with each other; Bottom-right – the absolute error function $|F(t) - G_{10}(t)|$.

Figure 18 shows the approximation results on the main interval $[0, 8]$. The top-left figure shows $g_{10}(t)$ (blue line) and $f(t)$ (red line) almost coincide with each other on $[0, 8]$. The top-right figure is the absolute error function $|e_{10}(t)|$ of the approximation error function $e_{10}(t) = f(t) - g_{10}(t)$. $e_{10}(t)$ oscillates between positive and negative values on $[0, 8]$ and have 12 zeros. $|e_{10}(t)|$ has a comb shape of 11 peaks. The first peak from the left has the maximum magnitude. Then, the peak magnitude decreases as the index goes to the right, reaches its minimum at the eighth, and increases as the index goes further to the right. The bottom-left figure shows $G_{10}(t)$ (blue line) and $F(t)$ (red line) almost coincide with each other on $[0, 8]$. The bottom-right figure shows the absolute error function $|F(t) - G_{10}(t)|$.

In order to demonstrate the uniform convergence on the main interval for the approximation method III, we repeat the above interpolation experiment with n = 15, 20, 25, 30, 35, and 40, and record the corresponding optimal λ values, WMAE's and WMSE's on $[0, 8]$ in Table 9. Figure 19 shows the graphs of the absolute error functions $|e_n(t)|$ for n = 15, 20, 25, 30, 35, and 40.

TABLE 9. Method III multiple interpolation experiments: the optimal λ values, WMAE's and WMSE's on $[0, 8]$ for n = 10, 15, 20, 25, 30, 35, and 40.

n	λ	$\text{WMAE}_{[0,8]}$	$\text{WMSE}_{[0,8]}$	$\frac{n}{\lambda}$
10	0.9650	1.4644×10^{-4}	5.8927×10^{-9}	10.4
15	1.1306	1.3653×10^{-4}	3.4860×10^{-9}	13.3
20	1.5872	5.8006×10^{-7}	3.7496×10^{-14}	12.6
25	1.7532	1.0401×10^{-6}	1.1576×10^{-13}	14.3
30	2.2077	3.3063×10^{-9}	7.3167×10^{-19}	13.6
35	2.3738	8.8519×10^{-9}	5.7797×10^{-18}	14.7
40	2.8269	2.2280×10^{-11}	2.3300×10^{-23}	14.1

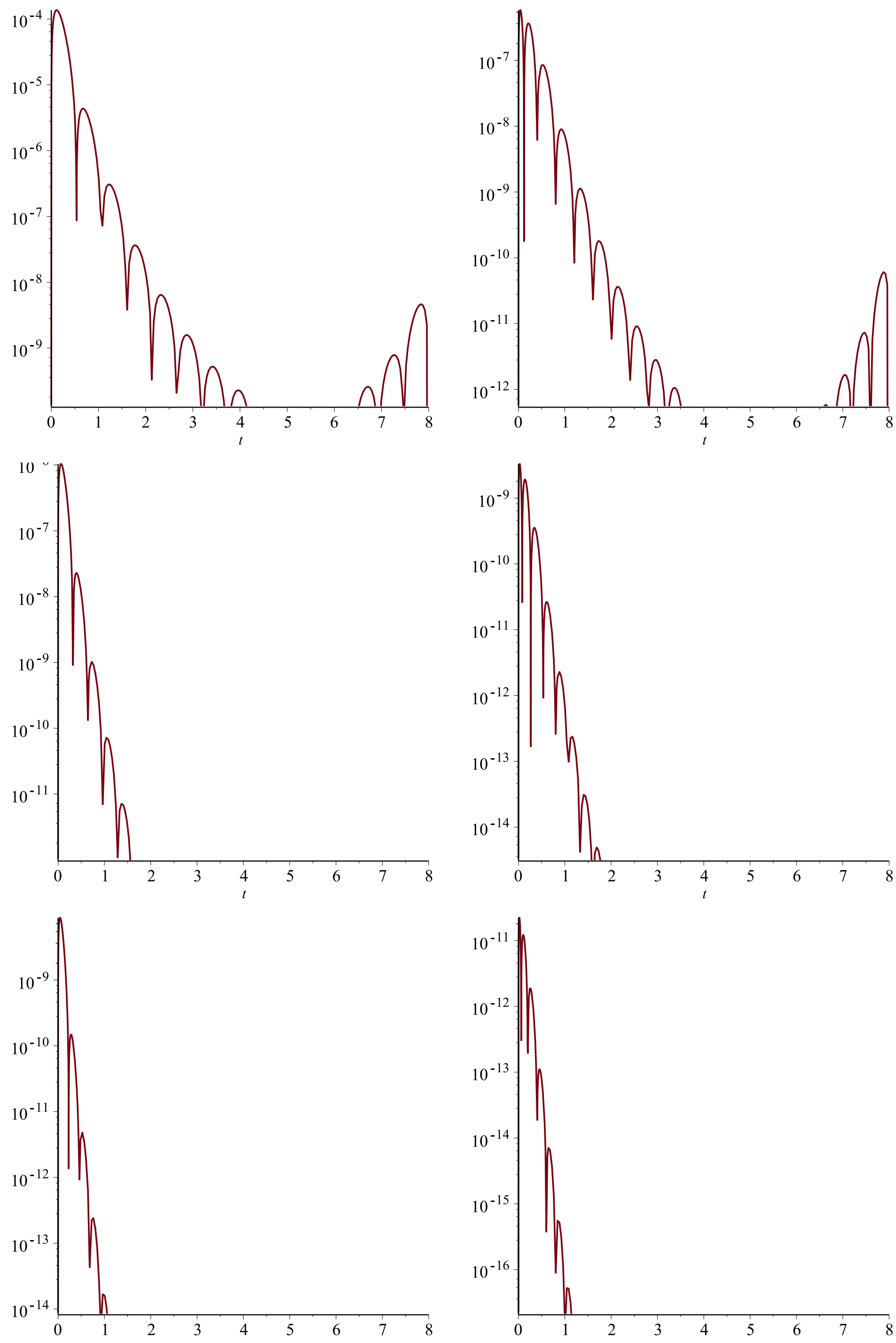

FIGURE 19. Method III multiple interpolation experiments: the absolute error functions $|e_n(t)|$ on $[0, 8]$ for $n = 15$ and 20 (top), $n = 25$ and 30 (middle), and $n = 35$ and 40 (bottom).

5. Conclusions

The main objective of this thesis is to approximate the class of bounded continuous vanishing at infinity functions on Ω, called $C_0^B(\Omega)$. This problem has not been properly addressed within the mathematical communities yet.

In order to solve the above approximation problem, we construct a decaying polynomial space $P_n^\lambda(\Omega)$ which incorporates various mathematical theories and methods on continuous functions, normed vector spaces, infinite function series, and Laplace transforms. Every function in $P_n^\lambda(\Omega)$ is continuous, analytic, bounded, vanishing at infinity, and integral-convergent on Ω.

The new space $P_n^\lambda(\Omega)$ is isomorphic to the polynomial space or the Euclidean space, and has a similar nested subspace structure to the latter. There are various linear structures in $P_n^\lambda(\Omega)$, and they can be transformed into each other. A particular linear structure may be used to build a particular approximation method in $P_n^\lambda(\Omega)$.

We build a new theory of approximations to $C_0^B(\Omega)$ functions based on $P_n^\lambda(\Omega)$ spaces. We first investigate the collective properties and characteristics of $C_0^B(\Omega)$. Then we linearize the target function according to various linear structures of $P_n^\lambda(\Omega)$, such as asymptotic series expansions, Laplace transform moments, or interpolations. We also introduce the concept of convergence in weak norm to deal with issues of non-compact domain and divergence.

The new approximation theories and methods effectively resolve the limitations of some existing approximation methods in the following three perspectives: (1) our asymptotic series expansion method can effectively give a global approximation function while the Taylor series expansion method is restricted locally by its finite convergence interval; (2) our Laplace transform moment matching method is uniformly convergent in the weak norm sense while the Padé method fails as an approximation method although it matches the moments; (3) our higher order interpolation method is uniformly convergent in the weak norm sense while the higher order polynomial interpolation method is unbounded at infinity and may suffer from the Runge's phenomenon.

Using the concept of convergence in weak norm on $P_n^\lambda(\Omega)$, we have the following conclusion about our new approximation theories and methods. Given any bounded continuous vanishing at infinity function on Ω, we can arbitrarily select a compact interval, a subset of Ω, such that the function can be uniformly approximated on the compact interval by a decaying polynomial function in $P_n^\lambda(\Omega)$ and the approximation function is continuous, bounded, integral-convergent, and vanishing at infinity on the complement of the compact interval in Ω. The new approximation method may be modified to approximate other types of bounded continuous or even discontinuous functions on Ω.

In the future, we will explore linear operations on $P_n^\lambda(\Omega)$ and their applications in solving differential, difference, or integral equations. This implies the space may be useful to approximately describe physical systems, such as vibration systems. In addition, we will use $P_n^\lambda(\Omega)$ spaces in various numerical calculations to provide global solutions for some problems which could only be solved locally at the moment.

Lastly, we would like to state the Maple symbolic computing software package plays an important role in developing theories and approximation methods of $P_n^\lambda(\Omega)$ spaces. The software has been used in deriving mathematical expressions, calculating numerical results, and plotting graphs.

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